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Universal Complexes and the Generic Structure of Free Resolutions

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Introduction

An important aspect of modern commutative algebra is the study of the structure of finite free resolutions. The first significant result in this direction goes back to Hilbert [22]; in its most general form, due to Burch [11], it describes the structure of free resolutions of length 2 whose component in degree 0 is a free module of rank 1. This theorem was generalized by Buchsbaum and Eisenbud [10], who obtained structure theorems for arbitrary finite free resolutions. The question of whether these are the "best possible" structure theorems was one of the topics of Hochster's influential CBMS lectures [23]. Hochster's approach to this problem is to describe a generic resolution of a given type from which all other resolutions of the same type are obtained by base change.

To be specific, let R be a commutative algebra over a (fixed) base ring $\Bbbk$, and let

$$\mathbf{F} = 0 \to R^{b_n} \xrightarrow{X^{(n)}} R^{b_{n-1}} \to \cdots \to R^{b_1} \xrightarrow{X^{(1)}} R^{b_0} \to 0 \qquad (\dagger)$$

be a complex, where $X^{(k)} = (x_{ij}^{(k)}) \neq 0$ is the matrix of the kth differential in the standard bases of R^{b_k} and $R^{b_{k-1}}$, $k = 1, \dots, n$. Hochster calls the pair $(R, \mathbf{F})$ a *universal pair* if $\mathbf{F}$ is acyclic and if, for each commutative $\Bbbk$-algebra S and each free resolution

$$\mathbf{G} = 0 \to S^{b_n} \xrightarrow{Z^{(n)}} S^{b_{n-1}} \to \cdots \to S^{b_1} \xrightarrow{Z^{(1)}} S^{b_0} \to 0, \qquad (\ddagger)$$

there exists a unique $\Bbbk$-algebra homomorphism $u\colon R \to S$ such that $u(x_{ij}^{(k)}) = z_{ij}^{(k)}$; thus $\mathbf{G} = \mathbf{F} \otimes_R S$. When it exists, a universal pair $(R, \mathbf{F})$ is determined up to isomorphism by the sequence of its *Betti numbers* $\boldsymbol{b} = (b_0, \dots, b_n)$; we call R the *universal ring of type* $\boldsymbol{b}$ *over* $\Bbbk$, and $\mathbf{F}$ the *universal resolution of type* $\boldsymbol{b}$ *over* $\Bbbk$.

A main step in Hochster's program is to establish the values of $b_0, \dots, b_n$ for which a universal pair exists. Hochster [23] (when $\Bbbk$ is either the ring of integers $\mathbb{Z}$, or a field) and later Bruns [5] (in general) show that, when $n \le 2$, a necessary and sufficient condition for existence is that the "expected ranks" $r_k = \sum_{s=k}^{n} (-1)^{s-k} b_s$ satisfy $r_0 \ge 0$ and $r_k \ge 1$ for $1 \le k \le n$. When $n \ge 3$, Bruns [5] shows that universal pairs do not exist, regardless of the choice of the numbers $b_0, \dots, b_n$.

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These results raise the question of investigating the properties of the universal pair $(R, \mathbf{F})$ for $n \leq 2$. When $n = 1$ it is easy to see that the universal ring R is a polynomial ring $\Bbbk[X^{(1)}]$ over $\Bbbk$ whose variables are the entries of a $k \times m$ matrix of indeterminates $X^{(1)}$ and that the universal resolution $\mathbf{F}$ is given by

$$\mathbf{F} = 0 \to R^m \xrightarrow{X^{(1)}} R^k \to 0.$$

If $n = 2$ and the universal resolution has the form

$$\mathbf{F} = 0 \to R^{m-1} \xrightarrow{X^{(2)}} R^m \xrightarrow{X^{(1)}} R \to 0,$$

then the Hilbert–Burch theorem shows that the universal ring R is a polynomial ring over $\Bbbk$ whose variables are the entries of $X^{(2)}$ and an additional variable v.

When $n = 2$ and $b_0 \geq 2$, the situation is more complicated. Since $\mathbf{F}$ is acyclic, a factorization theorem of Buchsbaum and Eisenbud [10] implies that R contains $\binom{b_0}{r_1}$ special elements called *Buchsbaum–Eisenbud multipliers,* one multiplier for each r_1-element subset of $\{1, \dots, b_0\}$. When $\Bbbk$ is either $\mathbb{Z}$ or a field, results of Hochster [23] and Huneke [27] show that R is a normal domain generated over $\Bbbk$ by the entries of the matrices $X^{(1)}$, $X^{(2)}$ and by the multipliers; under the same assumptions on $\Bbbk$, Bruns [4] proves that R is factorial.

An important application of these results was discovered by Heitmann [21], who used them to construct a counterexample to the rigidity conjecture.

More is known on the structure of R when $\Bbbk$ is a field of characteristic 0: the theorems of Huneke and Bruns just cited, together with an unpublished result of Hochster, yield that R is Gorenstein; Pragacz and Weyman [35] determine the relations of R and show that it has rational singularities.

When $\Bbbk$ is arbitrary, Pragacz and Weyman [35] construct a candidate for R and propose a Hodge algebra structure on it. However, in Example 9.1 we show that this Hodge algebra structure is not well-defined. Further examples in Section 9 suggest that the theory of Hodge algebras in its present form is not appropriate for the study of universal rings.

We prove that the candidate of Pragacz and Weyman is indeed a universal ring, thus obtaining an explicit description of the universal pairs. Using this and the theory of Gröbner bases, we generalize (with new proofs) to arbitrary base rings all those results about universal pairs mentioned in the previous paragraphs, and we perform a detailed investigation of the arithmetic properties and the singularities of the universal ring.

We summarize our results in the following theorem.

MAIN THEOREM. *Let $\Bbbk$ be a commutative ring, let $\boldsymbol{b} = (b_0, b_1, b_2)$ be a sequence of positive integers such that $r_0 = b_0 - b_1 + b_2 \geq 0$ and $r_1 = b_1 - b_2 \geq 1$, and let*

$$\mathbf{F} = 0 \to R^{b_2} \xrightarrow{X^{(2)}} R^{b_1} \xrightarrow{X^{(1)}} R^{b_0} \to 0$$

be the universal resolution of type $\boldsymbol{b}$ over $\Bbbk$.

(i) *The universal ring R is a finitely presented $\Bbbk$-algebra with generators and relations described in* (2.3). *It is a free module over $\Bbbk$, with a free basis described in* (5.8).

(ii) *The ring R has a grading $R = \bigoplus_{j \geq 0} R_j$ with $R_0 = \Bbbk$ such that the entries of $X^{(1)}$ and $X^{(2)}$ are homogeneous of degree $1 + b_2$ and r_1, respectively.*
(iii) *The ring R is reduced if and only if $\Bbbk$ is reduced.*
(iv) *The ring R is a domain if and only if $\Bbbk$ is a domain.*
(v) *The ring R is a Krull domain if and only if $\Bbbk$ is a Krull domain. When $\Bbbk$ is a Krull domain, there is an isomorphism of divisor class groups* $\mathrm{Cl}(R) \cong \mathrm{Cl}(\Bbbk)$.
(vi) *The ring R is factorial if and only if $\Bbbk$ is factorial.*
(vii) *The ring R is Cohen–Macaulay if and only if $\Bbbk$ is Cohen–Macaulay.*
(viii) *The ring R is Gorenstein if and only if $\Bbbk$ is Gorenstein.*
(ix) *The ring R is regular if and only if $\Bbbk$ is regular and $r_1 = 1$.*
(x) *If $\Bbbk$ is a perfect field of positive characteristic then R is F-regular.*
(xi) *If $\Bbbk$ is a field of characteristic 0 then R has rational singularities.*

Universal rings have a particularly nice structure when $r_1 = 1$, that is, when the universal resolution has the form

$$\mathbf{F} = 0 \to R^{m-1} \xrightarrow{X^{(2)}} R^m \xrightarrow{X^{(1)}} R^k \to 0.$$

In this case the defining equations show that R is a polynomial ring over $\Bbbk$ with the entries of $X^{(2)}$ and the Buchsbaum–Eisenbud multipliers as variables. When $k = 1$, this is precisely the Hilbert–Burch theorem.

Another immediate application of the presentation of the universal rings is to a situation "dual" to the Hilbert–Burch theorem—that is, when the universal resolution $\mathbf{F}$ has the form

$$\mathbf{F} = 0 \to R \xrightarrow{X^{(2)}} R^m \xrightarrow{X^{(1)}} R^{m-1} \to 0.$$

The ideal of relations for R is then precisely the generic Herzog ideal of grade m (as defined in [1]), which parametrizes the grade-m Gorenstein ideals two links away from a complete intersection. In this case Kustin and Miller [30] resolve the universal ring and describe a DG-algebra structure on its minimal resolution, and Avramov, Kustin, and Miller [1] prove that all finite R-modules have rational Poincaré series when $\Bbbk$ is a field. It is an interesting open problem whether these properties extend to all universal rings over fields.

Following the approach of Bruns [4], we consider a more general problem and study a wider class of universal objects. For a complex $\mathbf{F}$ as in (†), we say that $(R, \mathbf{F})$ is a 1-*universal pair* if the complex $\mathbf{F}$ is acyclic in depth 1 (i.e., $\mathbf{F}_{\mathfrak{p}}$ is acyclic for each $\mathfrak{p} \in \mathrm{Spec}(R)$ with depth $R_{\mathfrak{p}} \leq 1$) and if, for each commutative $\Bbbk$-algebra S and each acyclic in depth-1 complex (‡), there exists a unique $\Bbbk$-algebra map $u \colon R \to S$ such that $\mathbf{G} = \mathbf{F} \otimes_R S$. It is shown in [5] that a 1-universal pair exists if and only if $r_0 \geq 0$ and $r_i \geq 1$ for $1 \leq i \leq n$ (with no restrictions on n this time).

When $n \leq 2$, the notions of universal pair and 1-universal pair coincide. Thus the various assertions of the Main Theorem are special cases of Remark 2.4 and Theorems 2.5–2.9, which give a detailed description of the 1-universal pairs. These theorems generalize results of Bruns [4] and of Pragacz and Weyman [35] to arbitrary base rings. In particular, we show that the factorization theorem of Buchsbaum and Eisenbud [10, Thm. 3.1] in the version of Eagon and Northcott [16, Thm. 3] determines completely the generic structure of finite free complexes acyclic in depth 1 and hence is—in a precise sense—the best possible structure theorem for such complexes.

1. Buchsbaum–Eisenbud Multipliers

Throughout this paper, rings are commutative with unit, and modules are unitary.

We write $A = \{a_1 < \cdots < a_r\}$ to express in compact form that A is the set of integers $\{a_1, \dots, a_r\}$ arranged in increasing order. When the sets $A = \{a_1 < \cdots < a_r\}$ and $C = \{c_1 < \cdots < c_t\}$ are disjoint, we write $\nabla_{A,C}$ for the sign of the permutation that arranges the elements of the sequence $(a_1, \dots, a_r, c_1, \dots, c_t)$ in increasing order. For a subset $A \subseteq \{1, \dots, b\}$ we denote by $|A|$ the cardinality of A and by $\bar{A}$ (or A^-) the complement of A in $\{1, \dots, b\}$.

If $A = \{a_1 < \cdots < a_r\}$ and $D = \{d_1 < \cdots < d_r\}$ are sets of positive integers and if X is a matrix over a ring, then we write $\langle A|D\rangle_X$ or $\langle A|d_1, \dots, d_r\rangle_X$ or $\langle a_1, \dots, a_r|d_1, \dots, d_r\rangle_X$ for the minor of X on rows $a_1, \dots, a_r$ and columns $d_1, \dots, d_r$.

We recall the notion of the *grade* of an ideal I in a ring R. If I is a proper ideal then set $\operatorname{gr}_R I = \sup\{k \mid I \text{ contains an } R\text{-regular sequence of length } k\}$; else set $\operatorname{gr}_R I = \infty$. The grade of I is defined as

$$\operatorname{grade} I = \lim_{s\to\infty} \operatorname{gr}_{R[X_1,\dots,X_s]} IR[X_1, \dots, X_s],$$

where $R[X_1, \dots, X_s]$ is the polynomial ring over R in the indeterminates $X_1, \dots, X_s$. We refer to [33, Chaps. 5–6] for the properties of this notion of grade (denoted there by $\operatorname{Gr}_R\{I\}$ and called *true grade* or *polynomial grade*). When R has a unique maximal ideal $\mathfrak{m}$ we set depth $R = \operatorname{grade} \mathfrak{m}$.

We say that a complex of free R-modules $\mathbf{F}$ is *acyclic in depth* 1 if $\mathbf{F}_\mathfrak{p}$ is acyclic for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_\mathfrak{p} \le 1$.

For the rest of this section,

$$\mathbf{F} = 0 \longrightarrow R^{b_n} \xrightarrow{X^{(n)}} R^{b_{n-1}} \longrightarrow \cdots \longrightarrow R^{b_1} \xrightarrow{X^{(1)}} R^{b_0} \longrightarrow 0$$

is a complex that is acyclic in depth 1. The *expected ranks* for the sequence $\boldsymbol{b} = (b_0, \dots, b_n)$ are the integers

$$r_i = \begin{cases} \sum_{s=i}^{n} (-1)^{s-i} b_s & \text{if } 0 \le i \le n, \\ 0 & \text{if } i = n+1. \end{cases}$$

Note that $r_i \ge 0$ for $i = 0, \dots, n$ owing to the acyclicity condition on $\mathbf{F}$.

We quote the factorization theorem of Buchsbaum and Eisenbud [10] in the version of Eagon and Northcott [16, Thm. 3]. To simplify notation, we write $\langle A|E\rangle_k$ for the corresponding minor of the matrix $X^{(k)}$.

(1.1) THEOREM [10; 16]. *For every $1 \le k \le n$ and every r_k-element subset $A \subseteq \{1, \dots, b_{k-1}\}$ there exists a uniquely determined element $\langle A\rangle_k \in R$ such that the expressions* (*where* $\langle\varnothing\rangle_{n+1} = 1$)

$$\langle A|E\rangle_k - \nabla_{\bar{E},E}\langle A\rangle_k\langle \bar{E}\rangle_{k+1} \tag{1.2}$$

are equal to 0 *in* R *for any* $E \subseteq \{1, \ldots, b_k\}$ *with* $|E| = r_k$.

The elements $\langle A\rangle_k$ from (1.1) are called *Buchsbaum–Eisenbud multipliers* for the complex $\mathbf{F}$. We derive some further relations in the ring R that involve these multipliers. In somewhat different notation, these formulas appear in [35, Lemma 1.2].

(1.3) PROPOSITION. *The expressions* (*where each sum is over* Γ)

$$\sum_{C\cap A\subset\Gamma\subset C\setminus D}^{|\Gamma|=q} \nabla_{A,C\setminus\Gamma}\ \nabla_{C\setminus\Gamma,\Gamma}\ \nabla_{\Gamma,D}\langle A\cup(C\setminus\Gamma)\rangle_k\langle\Gamma\cup D\rangle_k \tag{1.4}$$

$$\sum_{C\cap A\subset\Gamma\subset C\setminus E}^{|\Gamma|=q} \nabla_{A,C\setminus\Gamma}\ \nabla_{C\setminus\Gamma,\Gamma}\ \nabla_{\Gamma,E}\langle A\cup(C\setminus\Gamma)\rangle_k\langle\Gamma\cup E\mid F\rangle_k \tag{1.5}$$

$$\sum_{\Gamma\subseteq\{1,\ldots,b_k\}\setminus(\Lambda\cup H\cup K)}^{|\Gamma|=t} \nabla_{H,\Gamma}\ \nabla_{\Gamma,K}\langle G\mid H\cup\Gamma\rangle_k\langle\Gamma\cup K\rangle_{k+1} \tag{1.6}$$

are equal to 0 *in the ring* R *for* $1 \le k \le n$, *any* $A, C, D, E, G \subseteq \{1, \ldots, b_{k-1}\}$, *and any* $F, H, K, \Lambda \subseteq \{1, \ldots, b_k\}$ *such that*

$$\begin{array}{lll} |A| = r_k - p, & |F| = s \le r_k, & |G| = m \le r_k, \\ |D| = r_k - q, & |E| = s - q, & |H| = m - t, \\ |C| = p + q \ge r_k + 1, & |K| = r_{k+1} - t, & |\Lambda| < t \le \min(m, r_{k+1}). \end{array}$$

(1.7) REMARK. Since $r_{n+1} = 0$, there are no expressions of the form (1.6) with $k = n$.

Let $N_i \subseteq R^{b_{i-1}}$ be the image of $X^{(i)}$ for $1 \le i \le n$, and write T for the total ring of fractions of R.

(1.8) LEMMA. *Assume that, for* $i = 1, \ldots, n$, *the ideal* $I_{r_i}(X^{(i)})$ *contains an* R-*regular element*.

For each $1 \le i \le n$, *the* T-*module* $N_i \otimes_R T$ *is free of rank* r_i *and has a free* T-*basis such that, for each* $A \subseteq \{1, \ldots, b_{i-1}\}$ *with* $|A| = r_i$, *the Buchsbaum–Eisenbud multiplier* $\langle A\rangle_i \in R$ *is precisely the maximal minor* $\langle A|1, \ldots, r_i\rangle_{M^{(i)}}$ *of the* $b_{i-1} \times r_i$ *matrix* $M^{(i)}$ *of the homomorphism* $N_i \otimes_R T \to R^{b_{i-1}} \otimes_R T$.

Proof. Since $I_{r_i}(X^{(i)})T = T$ for $i = 1, \ldots, n$, the complex $\mathbf{F} \otimes_R T$ is split acyclic and hence $N_i \otimes_R T$ is free of rank r_i.

Because $R \subseteq T$ and the expressions (1.2) determine completely the Buchsbaum–Eisenbud multipliers, after tensoring with T we may assume that $R = T$, that N_i is a free R-module of rank r_i, and that $\mathbf{F}$ is split acyclic. Under these assumptions, the lemma is an immediate consequence of the proof of [6, Thm. (3.2)]. □

Proof of Proposition 1.3. It is immediate from [16, Thm. 2] that grade $I_{r_i}(X^{(i)}) \ge 1$ for $i = 1, \ldots, n$. Thus, by adjoining a variable to R, we may assume that the ideal $I_{r_i}(X^{(i)})$ contains an R-regular element for each $1 \leq i \leq n$. Let $M^{(i)}$ ($i = 1, \dots, n$) be the matrices from Lemma 1.8 whose minors equal the Buchsbaum–Eisenbud multipliers. Note that the relations (1.4) are simply the Plücker relations on the maximal minors of $M^{(k)}$; see [8, (4.4)].

Let $Y^{(i)}$ be the $b_{i-1} \times (b_i + r_i)$ matrix over T whose first b_i columns are the columns of $X^{(i)}$ and whose last r_i columns are the columns of $M^{(i)}$. Since $\operatorname{Im} M^{(i)} = \operatorname{Im} X^{(i)}$, the rank of $Y^{(i)}$ is r_i; thus, the relations (1.5) are among the Plücker relations for $Y^{(i)}$.

Finally, for each $1 \leq i \leq n-1$ we have $X^{(i)}M^{(i+1)} = 0$; hence [15, Cor. 1.2] yields the relations (1.6). □

2. Universal Complexes

For the rest of this paper, $\boldsymbol{b} = (b_0, \dots, b_n)$ is a sequence of positive integers such that the expected ranks satisfy

$$\sum_{s=i}^{n} (-1)^{s-i} b_s = r_i \geq \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } 1 \leq i \leq n. \end{cases}$$

(2.1) DEFINITION. Let $\Bbbk$ be a ring, let R be a $\Bbbk$-algebra, and let $\mathbf{F}$ be a complex of free R-modules as in (†) of the Introduction.

The pair $(R, \mathbf{F})$ is called 1-*universal* (of type $\boldsymbol{b}$ over $\Bbbk$) if $\mathbf{F}$ is acyclic in depth 1 and if, for each $\Bbbk$-algebra S and each acyclic in depth-1 complex (‡), there exists a unique $\Bbbk$-algebra homomorphism $u\colon R \to S$ such that $u(X^{(i)}) = Z^{(i)}$ for $i = 1, \dots, n$.

If $(R, \mathbf{F})$ is 1-universal of type $\boldsymbol{b}$ over $\Bbbk$, then R is the *universal ring of type* $\boldsymbol{b}$ *over* $\Bbbk$ and $\mathbf{F}$ is the *universal complex of type* $\boldsymbol{b}$ *over* $\Bbbk$.

(2.2) REMARKS. (a) It is proved in [5] that a 1-universal pair $(R, \mathbf{F})$ of type $\boldsymbol{b}$ over $\Bbbk$ exists. Clearly it is determined up to a unique isomorphism by $\boldsymbol{b}$.

(b) It is immediate from the acyclicity criterion of Peskine and Szpiro [34] in the version of Northcott [33, Chap. 5, Thm. 21] that, for $n \leq 2$, the notion of 1-universal pair coincides with Hochster's notion of universal pair recalled in the Introduction.

For $k = 1, \dots, n$, let $X^{(k)} = (x_{ij}^{(k)})$ be a $b_{k-1} \times b_k$ matrix of indeterminates and let

$$M_k = \{\langle A \rangle_k \mid A \subset \{1, \dots, b_{k-1}\},\ |A| = r_k\}$$

be a set of indeterminates. Consider the polynomial ring $\Bbbk[X^{(1)}, \dots, X^{(n)}, M_1, \dots, M_n]$ whose variables are the entries of $X^{(k)}$ and the elements of M_k, $1 \leq k \leq n$. Let

$$J' = \sum_{k=1}^{n-1} I_1(X^{(k)} X^{(k+1)}),$$

let J'' be the ideal generated by all expressions from (1.2), (1.4), (1.5), and (1.6), and set

$$R_{\Bbbk}(\boldsymbol{b}) = \Bbbk[X^{(1)}, \ldots, X^{(n)}, M_1, \ldots, M_n]/J_{\Bbbk}(\boldsymbol{b}), \quad \text{where} \quad J_{\Bbbk}(\boldsymbol{b}) = J' + J''. \tag{2.3}$$

By abuse of notation we write $x_{ij}^{(k)}$, $X^{(k)}$, $\langle A \rangle_k$, and M_k also for the images of these objects in $R_{\Bbbk}(\boldsymbol{b})$, and we refer to M_k as the kth *set of multipliers* of $R_{\Bbbk}(\boldsymbol{b})$.

(2.4) REMARKS. (a) By expanding an $(r_k + 1) \times (r_k + 1)$ minor of $X^{(k)}$ along a column, it is easy to see that the minor reduces to 0 modulo the relations (1.2) and (1.5). Thus there are inclusions $I_{r_k+1}(X^{(k)}) \subset J_{\Bbbk}(\boldsymbol{b})$ for $k = 1, \ldots, n-1$.

(b) Define a grading on the ring $\Bbbk[X^{(1)}, \ldots, X^{(n)}, M_1, \ldots, M_n]$ by

$$\deg(x_{ij}^{(k)}) = \begin{cases} 1 + r_2 & \text{if } k = 1, \\ r_{k-1} + r_{k+1} & \text{if } 2 \le k \le n-1, \\ r_{n-1} & \text{if } k = n; \end{cases}$$

$$\deg(\langle A \rangle_k) = \begin{cases} r_1 & \text{if } k = 1, \\ r_{k-1} r_k & \text{if } 2 \le k \le n. \end{cases}$$

Note that the ideal $J_{\Bbbk}(\boldsymbol{b})$ is homogeneous in this grading, thus making $R_{\Bbbk}(\boldsymbol{b})$ a positively graded $\Bbbk$-algebra.

Let $\mathbf{F}_{\Bbbk}(\boldsymbol{b})$ be the complex

$$0 \to R_{\Bbbk}(\boldsymbol{b})^{b_n} \xrightarrow{X^{(n)}} R_{\Bbbk}(\boldsymbol{b})^{b_{n-1}} \to \cdots \to R_{\Bbbk}(\boldsymbol{b})^{b_1} \xrightarrow{X^{(1)}} R_{\Bbbk}(\boldsymbol{b})^{b_0} \to 0.$$

The following theorems are the main results of this paper.

(2.5) THEOREM. *The pair $(R_{\Bbbk}(\boldsymbol{b}), \mathbf{F}_{\Bbbk}(\boldsymbol{b}))$ is 1-universal of type $\boldsymbol{b}$ over $\Bbbk$.*

(2.6) THEOREM. (i) *The ring $R_{\Bbbk}(\boldsymbol{b})$ is reduced (respectively, a domain) if and only if $\Bbbk$ is reduced (respectively, a domain).*

(ii) *The ring $R_{\Bbbk}(\boldsymbol{b})$ is a Krull domain if and only if $\Bbbk$ is a Krull domain. If $\Bbbk$ is a Krull domain then there is an isomorphism of divisor class groups* $\mathrm{Cl}(\Bbbk) \cong \mathrm{Cl}(R_{\Bbbk}(\boldsymbol{b}))$.

(iii) *The ring $R_{\Bbbk}(\boldsymbol{b})$ is factorial if and only if $\Bbbk$ is factorial.*

(2.7) THEOREM. *The homomorphism $\Bbbk \to R_{\Bbbk}(\boldsymbol{b})$ is faithfully flat with Gorenstein fibers. In particular, $R_{\Bbbk}(\boldsymbol{b})$ is Cohen–Macaulay (resp., Gorenstein) if and only if $\Bbbk$ is Cohen–Macaulay (resp., Gorenstein).*

(2.8) THEOREM. *The ring $R_{\Bbbk}(\boldsymbol{b})$ is regular if and only if $\Bbbk$ is regular and $r_i = 1$ for each $1 \le i \le n-1$.*

(2.9) THEOREM. *Let $\Bbbk$ be a perfect field. If* $\operatorname{char}\Bbbk = p > 0$, *then $R_{\Bbbk}(\boldsymbol{b})$ is F-regular; if* $\operatorname{char}\Bbbk = 0$, *then $R_{\Bbbk}(\boldsymbol{b})$ has rational singularities.*

REMARKS. (a) In view of Remarks 2.2(b) and 2.4(b), the Main Theorem is an immediate corollary of (2.5)–(2.9) and (5.8).

(b) When $\Bbbk$ is a field of characteristic 0, Theorems 2.5 and 2.9 are due to Pragacz and Weyman [35]. In the general case they incorrectly proposed (see Example 9.1) a Hodge algebra structure on the ring $R_{\Bbbk}(\boldsymbol{b})$.

In this and the next two sections we present, modulo Lemmas 3.2 and 3.3, the proofs of Theorems 2.5, 2.6, and 2.8. The proofs of Theorems 2.7 and 2.9 are given in Section 8.

The proofs of (2.5) and (2.6) use induction on $|\boldsymbol{b}| = \sum_{s=1}^{n} r_s$. The following lemma provides the basis for these inductive arguments and is also an essential ingredient in the proof of Theorem 2.8.

(2.10) LEMMA. *If $r_i = 1$ for each $1 \le i \le n-1$, then there is a $\Bbbk$-algebra isomorphism*

$$\Bbbk[X^{(n)}, M_1, \ldots, M_{n-1}] \cong R_{\Bbbk}(\boldsymbol{b})$$

induced by the canonical projection $\Bbbk[X^{(1)}, \ldots, X^{(n)}, M_1, \ldots, M_n] \to R_{\Bbbk}(\boldsymbol{b})$.

Proof. For $i = 2, 4, 5, 6$ we set $J_i = \sum_{j=1}^{n} J_{ij}$, where $J_{ij} \subset \Bbbk[X^{(1)}, \ldots, X^{(n)}, M_1, \ldots, M_n]$ is the ideal generated by all expressions from $(1.i)$ with $k = j$; thus

$$J_{\Bbbk}(\boldsymbol{b}) = J' + J_2 + J_4 + J_5 + J_6.$$

Note that $J_{6n} = 0$ by (1.7). Since $r_i = 1$ for $i = 1, \ldots, n-1$, it is straightforward to verify that

$$\begin{aligned} J_{4j} &= 0 && \text{for } j = 1, \ldots, n-1, \\ J_{6j} &\subseteq J_2 && \text{for } j = 1, \ldots, n-1, \\ J_{5j} &\subseteq J_2 + J_{4j} && \text{for } j = 1, \ldots, n-1, \\ I_1(X^{(j)}X^{(j+1)}) &\subseteq J_2 && \text{for } j = 1, \ldots, n-2. \end{aligned}$$

Note also that, modulo J_2, the entries of $X^{(n-1)}X^{(n)}$ (resp., the expressions from (1.4) and (1.5) with $k = n$) are multiples of Plücker relations on the minors of $X^{(n)}$ and hence equal 0. Therefore $J_{\Bbbk}(\boldsymbol{b}) = J_2$, and the conclusion of the lemma is immediate. □

Proof of Theorem 2.8. If $\Bbbk$ is regular and $r_i = 1$ for each $1 \le i \le n-1$, then $R_{\Bbbk}(\boldsymbol{b})$ is regular by (2.10).

Conversely, assume that $R_{\Bbbk}(\boldsymbol{b})$ is regular. Then $\Bbbk$ is regular by Lemma 3.2 and we need only show that $r_i = 1$ for each $1 \le i \le n-1$. Thus we may also assume (after a suitable localization) that $\Bbbk$ is a field.

Let $\mathfrak{m}$ (resp., $\mathfrak{M}$) be the maximal ideal of R (resp., of $\mathcal{Q} = \Bbbk[X^{(1)}, \ldots, X^{(n)}, M_1, \ldots, M_n]$) generated by the entries of $X^{(k)}$ and the elements of M_k, $k = 1, \ldots, n$. Since the ring $R_{\mathfrak{m}} \cong \mathcal{Q}_{\mathfrak{M}}/J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$ is regular and local, each minimal generator of $J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$ is part of a regular system of parameters for $\mathcal{Q}_{\mathfrak{M}}$.

Let i be an integer such that $1 \le i \le n-1$ and $r_i \ge 2$. Set

$$w = \langle 1, \ldots, r_i | r_{i+1} + 1, \ldots, b_i \rangle_i - \langle 1, \ldots, r_i \rangle_i \langle 1, \ldots, r_{i+1} \rangle_{i+1}$$

and note that $w \in J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$ and $w \in \mathfrak{M}^2_{\mathfrak{M}}$. Assume that $w \in \mathfrak{M} J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$ and let $\mathfrak{N} \subset \mathfrak{M}_{\mathfrak{M}}$ be the ideal generated by the set

$$\{x_{pq}^{(j)} \mid j \neq i, \text{ or } p \ge r_i + 1, \text{ or } q \le r_{i+1}\} \cup M_1 \cup \cdots \cup M_n.$$

It is immediate that, in the regular ring $\bar{\mathcal{Q}} = \mathcal{Q}_{\mathfrak{M}}/\mathfrak{N}$, the ideal $J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}} + \mathfrak{N}/\mathfrak{N}$ is generated by $0 \neq \bar{w} = w + \mathfrak{N}$. On the other hand, for $\bar{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}/\mathfrak{N}$ we have $\bar{w} \in \bar{\mathfrak{M}}\bar{w}$ and thus $\bar{w} = 0$. Hence the assumption $w \in \mathfrak{M}J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$ leads to a contradiction and so w is a minimal generator of $J_{\Bbbk}(\boldsymbol{b})_{\mathfrak{M}}$, which is not part of a regular system of parameters for $\mathcal{Q}_{\mathfrak{M}}$. This, however, contradicts the conclusion of the previous paragraph. We therefore have $r_i = 1$ for each $1 \leq i \leq n-1$, which completes the proof of Theorem 2.8. □

3. Proof of Theorem 2.5

(3.1) LEMMA. *If $r_i = 1$ for each $1 \leq i \leq n-1$, then the complex $\mathbf{F}_{\Bbbk}(\boldsymbol{b})$ is acyclic in depth* 1.

Proof. Let $R = R_{\Bbbk}(\boldsymbol{b})$ and $\mathbf{F} = \mathbf{F}_{\Bbbk}(\boldsymbol{b})$. For $j = 1, \ldots, n$, set $I_j = I_{r_j}(X^{(j)})$. By (2.10) we have $R \cong \Bbbk[X^{(n)}, M_1, \ldots, M_{n-1}]$. If $n = 1$ then this implies that $\mathbf{F}$ is acyclic and we are done. Hence we assume $n \geq 2$ and take $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_{\mathfrak{p}} \leq 1$.

We have (see e.g. [8, Thm. (2.5)]) grade $I_n = 2$. Since $|M_1| \geq 1$ and $|M_i| \geq 2$ for $2 \leq i \leq n-1$, we also obtain grade $M_1R \geq 1$ and grade $M_iR \geq 2$ for each $2 \leq i \leq n-1$. Thus $I_nR_{\mathfrak{p}} = R_{\mathfrak{p}}$ and $I_jR_{\mathfrak{p}} = (M_{j+1}R_{\mathfrak{p}})(M_jR_{\mathfrak{p}}) = R_{\mathfrak{p}}$ for each $2 \leq j \leq n-1$. Since grade $I_1R_{\mathfrak{p}} \geq 1$, the desired acyclicity of $\mathbf{F}_{\mathfrak{p}}$ is immediate (e.g., by the acyclicity criterion of [9] in the form of [16, Thm. 2]). □

The following lemma is a direct consequence of a result of Pragacz and Weyman [35, Thm. 1.3] (recalled here as Theorem 5.8).

(3.2) LEMMA. *The homomorphism $\Bbbk \longrightarrow R_{\Bbbk}(\boldsymbol{b})$ is faithfully flat.*

The proof of the next lemma is given in Section 5.

(3.3) LEMMA. *The element $x = x^{(n)}_{b_{n-1}b_n}$ is regular in $R_{\Bbbk}(\boldsymbol{b})$.*

If i is an integer such that $1 \leq i \leq n-1$ and $r_i \geq 2$, then the element $y = x^{(i)}_{b_{i-1}b_i}$ is also regular and (x, y) is a regular sequence in $R_{\Bbbk}(\boldsymbol{b})$.

Let $1 \leq i \leq n$ be an integer such that either $i = n \geq 2$ or $r_i \geq 2$. Set

$$\boldsymbol{b}' = \begin{cases} (b_0, \ldots, b_{n-2}, b_{n-1} - 1) & \text{if } i = n \text{ and } r_n = 1, \\ (b_0, \ldots, b_{i-2}, b_{i-1} - 1, b_i - 1, b_{i+1}, \ldots, b_n) & \text{otherwise;} \end{cases}$$

$$\Bbbk' = \Bbbk[X_{1b_i}, \ldots, X_{b_{i-1}b_i}, X_{b_{i-1}1}, \ldots, X_{b_{i-1}b_i-1}][X^{-1}_{b_{i-1}b_i}];$$

$$\mathbf{E} = 0 \to R_{\Bbbk'}(\boldsymbol{b}') \xrightarrow{X_{b_{i-1}b_i}} R_{\Bbbk'}(\boldsymbol{b}') \to 0,$$

where the nonzero components of the complex $\mathbf{E}$ are in (homological) degrees i and $i-1$. We write $Y^{(k)} = (y^{(k)}_{st})$ for the matrix of the kth differential of $\mathbf{F}_{\Bbbk'}(\boldsymbol{b}')$ and use $'$ to denote multipliers of $R_{\Bbbk'}(\boldsymbol{b}')$.

(3.4) LEMMA. *For $z = x^{(i)}_{b_{i-1}b_i}$ there exists a $\Bbbk$-algebra isomorphism $\varphi\colon R_{\Bbbk}(\boldsymbol{b})_z \to R_{\Bbbk'}(\boldsymbol{b}')$ such that*

$$\varphi(x^{(k)}_{st}) = \begin{cases} -\frac{1}{X_{b_{i-1}b_i}}\sum_{j=1}^{b_i-1} X_{b_{i-1}j}\, y^{(i+1)}_{jt} & \text{if } k = i+1 \text{ and } s = b_{i-1},\\ X_{sb_i} & \text{if } k = i \text{ and } t = b_i,\\ X_{b_{i-1}t} & \text{if } k = i \text{ and } s = b_{i-1},\\ y^{(i)}_{st} + \frac{X_{sb_i}X_{b_{i-1}t}}{X_{b_{i-1}b_i}} & \text{if } k = i,\ s \neq b_{i-1}, \text{ and } t \neq b_i,\\ -\frac{1}{X_{b_{i-1}b_i}}\sum_{j=1}^{b_{i-1}-1} X_{jb_i}\, y^{(i-1)}_{sj} & \text{if } k = i-1 \text{ and } t = b_{i-1},\\ y^{(k)}_{st} & \text{otherwise;} \end{cases}$$

$$\varphi(\langle A\rangle_k) = \begin{cases} \langle A\rangle'_k & \text{if } k > i+1,\\ -\sum_{j\notin A}\frac{X_{b_{i-1}j}}{X_{b_{i-1}b_i}}\,\nabla_{A\setminus b_i,\, j}\langle (A\setminus b_i)\cup j\rangle'_{i+1} & \text{if } k = i+1 \text{ and } b_i \in A,\\ \langle A\rangle'_{i+1} & \text{if } k = i+1 \text{ and } b_i \notin A,\\ X_{b_{i-1}b_i}\langle A\setminus b_{i-1}\rangle'_i & \text{if } k = i \text{ and } b_{i-1}\in A,\\ \sum_{j\in A}\nabla_{A\setminus j,\, j} X_{jb_i}\langle A\setminus j\rangle'_i & \text{if } k = i \text{ and } b_{i-1}\notin A,\\ (-1)^{r_{i-1}}\langle A\rangle'_k & \text{otherwise.} \end{cases}$$

This isomorphism φ induces an isomorphism $\mathbf{F}_{\Bbbk}(\boldsymbol{b})_z \cong \mathbf{F}_{\Bbbk'}(\boldsymbol{b}')\oplus \mathbf{E}$ of complexes over $R_{\Bbbk'}(\boldsymbol{b}')$.

Through long and tedious computations one can verify (3.4) using only the presentation (2.3); it is much more convenient to work out the proofs of (3.4) and (2.5) in parallel.

Proof of Theorem 2.5 and Lemma 3.4. Note that if S is a $\Bbbk$-algebra and if a complex $0 \to S^{b_n} \xrightarrow{Z^{(n)}} \cdots \xrightarrow{Z^{(1)}} S^{b_0} \to 0$ is acyclic in depth 1, then from (1.1), (1.3), and the presentation (2.3) of $R = R_{\Bbbk}(\boldsymbol{b})$ it is clear that there exists a unique $\Bbbk$-algebra homomorphism $u\colon R \to S$ such that $u(X^{(j)}) = Z^{(j)}$ for each $1 \le j \le n$. Thus, to prove (2.5) we need to show that $\mathbf{F} = \mathbf{F}_{\Bbbk}(\boldsymbol{b})$ is acyclic in depth 1.

We argue by induction on $|\boldsymbol{b}| = \sum_{s=1}^n r_s$ that Lemma 3.4 holds and that $\mathbf{F}$ is acyclic in depth 1. If $|\boldsymbol{b}| = 1$ then $n = 1$, hence (3.4) holds for trivial reasons and $\mathbf{F}$ is acyclic in depth 1 by (3.1); so assume that $|\boldsymbol{b}| \ge 2$.

Let $1 \le i \le n$ and z be as in (3.4). Set $S = R_{\Bbbk'}(\boldsymbol{b}')$ and note that $\mathbf{F}_{\Bbbk'}(\boldsymbol{b}')\oplus\mathbf{E}$ has the form

$$\begin{gathered} 0 \longrightarrow S^{b_n} \xrightarrow{Y^{(n)}} S^{b_{n-1}} \longrightarrow \cdots \longrightarrow S^{b_{i+2}} \xrightarrow{Y^{(i+2)}} \\ S^{b_{i+1}} \xrightarrow{\begin{pmatrix} Y^{(i+1)} \\ 0 \end{pmatrix}} S^{b_i} \xrightarrow{\begin{pmatrix} Y^{(i)} & 0 \\ 0 & X_{b_{i-1}b_i} \end{pmatrix}} S^{b_{i-1}} \xrightarrow{(Y^{(i-1)}\ \ 0)} \\ S^{b_{i-2}} \xrightarrow{Y^{(i-2)}} S^{b_{i-3}} \longrightarrow \cdots \longrightarrow S^{b_1} \xrightarrow{Y^{(1)}} S^{b_0} \longrightarrow 0. \end{gathered}$$

Let $\{e_{ij} \mid 1 \le j \le b_i\}$ and $\{e_{i-1j} \mid 1 \le j \le b_{i-1}\}$ be the standard bases of S^{b_i} and $S^{b_{i-1}}$, respectively. Let $\mathbf{G} = 0 \to S^{b_n} \xrightarrow{Z^{(n)}} \cdots \xrightarrow{Z^{(1)}} S^{b_0} \to 0$ be the complex obtained from $\mathbf{F}_{\Bbbk'}(\boldsymbol{b}') \oplus \mathbf{E}$ after choosing for S^{b_i} and $S^{b_{i-1}}$ the new bases $\{e'_{ij} \mid 1 \le j \le b_i\}$ and $\{e'_{i-1j} \mid 1 \le j \le b_{i-1}\}$, where

$$e'_{ij} = \begin{cases} e_{ij} - \frac{X_{b_{i-1}j}}{X_{b_{i-1}b_i}} e_{ib_i} & \text{if } 1 \le j \le b_i - 1, \\ e_{ib_i} & \text{if } j = b_i; \end{cases}$$

$$e'_{i-1,j} = \begin{cases} e_{i-1,j} & \text{if } 1 \le j \le b_{i-1} - 1, \\ e_{i-1,b_{i-1}} + \frac{1}{X_{b_{i-1}b_i}} \sum_{k=1}^{b_{i-1}-1} X_{kb_{i-1}} e_{i-1,k} & \text{if } j = b_{i-1}. \end{cases}$$

Therefore $\mathbf{G} \cong \mathbf{F}_{\Bbbk'}(\boldsymbol{b}') \oplus \mathbf{E}$ and hence is acyclic in depth 1 by the induction hypothesis. Thus there exists a unique $\Bbbk$-algebra homomorphism $u\colon R \to S$ such that $u(X^{(j)}) = Z^{(j)}$ for each $1 \le j \le n$. In particular, $u(z) = X_{b_{i-1}b_i}$ and hence u extends to a map $\varphi\colon R_z \to S$.

A standard calculation shows that φ satisfies the desired formulas. To see that φ is an isomorphism, consider the *surjective* $\Bbbk$-map $\nu\colon \Bbbk'[Y^{(1)}, \dots, Y^{(n')}, M'_1, \dots, M'_{n'}] \to R_z$ (where n' is the length of $\boldsymbol{b}'$) given by

$$\nu(X_{st}) = x^{(i)}_{st},$$

$$\nu(y^{(k)}_{st}) = \begin{cases} x^{(i)}_{st} - \frac{x^{(i)}_{sb_i} x^{(i)}_{b_{i-1}t}}{z} & \text{if } k = i, \\ x^{(k)}_{st} & \text{if } k \ne i; \end{cases}$$

$$\nu(\langle A\rangle'_k) = \begin{cases} \langle A\rangle_k & \text{if } k > i, \\ \frac{\langle A \cup b_{i-1}\rangle_i}{z} & \text{if } k = i, \\ (-1)^{r_{i-1}}\langle A\rangle_k & \text{if } k < i. \end{cases}$$

Since $\nu(\langle A|E\rangle_{Y^{(i)}}) = (1/z)\langle A \cup b_{i-1}|E \cup b_i\rangle_{X^{(i)}}$ and $\nu(\langle A|E\rangle_{Y^{(k)}}) = \langle A|E\rangle_{X^{(k)}}$ for $k \ne i$, it is straightforward to verify that ν factors through S to produce a map $\psi\colon S \to R_z$. An easy calculation shows that $\varphi \circ \psi = \mathrm{id}_S$. Thus ψ is also injective, hence an isomorphism, and φ is its inverse. Therefore φ is an isomorphism and clearly induces the isomorphism of complexes $\mathbf{F}_z \cong \mathbf{G} \cong \mathbf{F}_{\Bbbk'}(\boldsymbol{b}') \oplus \mathbf{E}$. It follows that Lemma 3.4 holds for $\boldsymbol{b}$.

To complete the proofs of (2.5) and (3.4) it remains to show that the complex $\mathbf{F}$ is acyclic in depth 1. When $r_j = 1$ for each $1 \le j \le n-1$ this follows from (3.1). Assume that $r_j \ge 2$ for some $1 \le j \le n-1$, and take $\mathfrak{p} \in \operatorname{Spec}(R)$ such that depth $R_{\mathfrak{p}} \le 1$.

By (3.3) the elements $x = x^{(n)}_{b_{n-1}b_n}$ and $y = x^{(j)}_{b_{j-1}b_j}$ form a regular sequence in R; thus at least one of them, call it z, is not in $\mathfrak{p}$. Since Lemma (3.4) holds for $\boldsymbol{b}$, we obtain the isomorphism of complexes $\mathbf{F}_{\mathfrak{p}} = (\mathbf{F}_z)_{\mathfrak{p}R_z} \cong \mathbf{F}_{\Bbbk'}(\boldsymbol{b}')_{\varphi(\mathfrak{p}R_z)} \oplus \mathbf{E}_{\varphi(\mathfrak{p}R_z)}$, and the desired acyclicity of $\mathbf{F}_{\mathfrak{p}}$ is immediate from our induction hypothesis. □

4. Proof of Theorem 2.6

Proof of Theorem 2.6(i). As in our proof of (2.5), we set $R = R_{\Bbbk}(\boldsymbol{b})$.

By Lemma 3.2, the map $\Bbbk \to R$ is faithfully flat and hence injective. Thus, if R is reduced (resp., a domain) then $\Bbbk$ is reduced (resp., a domain).

Assume next that $\Bbbk$ is reduced (resp., a domain). We show by induction on $|\boldsymbol{b}|$ that R is reduced (resp., a domain). When $|\boldsymbol{b}| = 1$, or more generally when $n = 1$, the assertion is obvious from (2.10). Assume that $|\boldsymbol{b}| \geq 2$ and that $n \geq 2$.

By Lemma 3.3, the element $x = x^{(n)}_{b_{n-1}b_n}$ is regular in R; hence $R \subseteq R_x$. Lemma 3.4 yields $R_x \cong R_{\Bbbk'}(\boldsymbol{b}')$, which is reduced (resp., a domain) by the induction hypothesis. Therefore R is reduced (resp., a domain), which completes the proof of Theorem 2.6(i). □

(4.1) LEMMA. *If $\Bbbk$ is a domain, then $x^{(n)}_{b_{n-1}b_n}$ is a prime element of $R_{\Bbbk}(\boldsymbol{b})$.*

Proof. We proceed by induction on $|\boldsymbol{b}|$. When $|\boldsymbol{b}| = 1$, or more generally when $r_i = 1$ for each $1 \leq i \leq n-1$, the claim is obvious from (2.10). Assume that $|\boldsymbol{b}| \geq 2$ and that $r_i \geq 2$ for some $1 \leq i \leq n-1$.

It follows that $n \geq 2$ and, by (3.3), the elements $x = x^{(n)}_{b_{n-1}b_n}$ and $y = x^{(i)}_{b_{i-1}b_i}$ form a regular sequence in $R = R_{\Bbbk}(\boldsymbol{b})$; in particular, $R/xR \subset (R/xR)_y = R_y/xR_y$. Thus it suffices to show that x is a prime element of R_y. When $i < n-1$, Lemma 3.4 yields an isomorphism $\varphi\colon R_y \cong R_{\Bbbk'}(\boldsymbol{b}')$ with $\varphi(x) = y^{(n)}_{b_{n-1}b_n}$ and we are done by the induction hypothesis.

When $i = n-1$, Lemma 3.4 yields an isomorphism $\varphi\colon R_y \cong R_{\Bbbk'}(\boldsymbol{b}')$ such that

$$\varphi(x) = -\frac{1}{X_{b_{n-2}b_{n-1}}} \sum_{j=1}^{b_{n-1}-1} X_{b_{n-2}j}\, y^{(n)}_{jb_n}.$$

Since $\Bbbk'$ is a domain, our induction hypothesis yields that $z = y^{(n)}_{b_{n-1}-1,b_n}$ is a prime element of the domain (by the already proven part (i) of this theorem) $R' = R_{\Bbbk'}(\boldsymbol{b}')$. By (2.3), the $\Bbbk'$-algebra homomorphism $\varepsilon\colon \Bbbk'[Y^{(1)}, \ldots, Y^{(n)}, M'_1, \ldots, M'_n] \to \Bbbk'$ given by

$$\varepsilon(y^{(k)}_{ij}) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = b_n \text{ and } k = n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varepsilon(\langle A\rangle'_k) = \begin{cases} 1 & \text{if } b_n = 1 \text{ and } A = \{1\} \text{ and } k = n, \\ 0 & \text{otherwise} \end{cases}$$

factors through R', so it follows that $\varphi(x) \notin zR'$ (as $b_{n-1} \geq 3$, we have $\varepsilon(z) = 0$ while $\varepsilon(\varphi(x)) = -(X_{b_{n-2},1}/X_{b_{n-2}b_{n-1}}) \neq 0$). Hence $\varphi(x)$ and z form an R'-regular sequence, and it suffices to show that $R'_z/\varphi(x)R'_z$ is a domain. By using (3.4) again we obtain $R'_z \cong R_{\Bbbk''}(\boldsymbol{b}'')$, where $\boldsymbol{b}'' = (\boldsymbol{b}')'$ and

$$\Bbbk'' = \Bbbk'[Y_{1b_n}, \ldots, Y_{b_{n-1}-1,b_n}, Y_{b_{n-1}-1,1}, \ldots, Y_{b_{n-1}-1,b_n-1}][Y^{-1}_{b_{n-1}-1,b_n}];$$

$\varphi(x)$ is mapped under this isomorphism to

$$w = -\frac{1}{X_{b_{n-2}b_{n-1}}} \sum_{j=1}^{b_{n-1}-1} X_{b_{n-2}j} Y_{jb_n} \in \Bbbk''.$$

Thus $R'_z/\varphi(x)R'_z \cong R_{\bar{\mathbb{k}}}(\boldsymbol{b}'')$, where $\bar{\mathbb{k}} = \mathbb{k}''/w\mathbb{k}'' \cong \mathbb{k}''/X_{b_{n-2},b_{n-1}-1}\mathbb{k}''$ is a domain. Therefore, $R'_z/\varphi(x)R'_z$ is a domain by the already proven part (i) of (2.6). This completes the proof of the lemma. □

Proof of Theorems 2.6(ii) and 2.6(iii). We note that (ii) implies (iii) and proceed with the proof of (ii). Assume $R = R_{\mathbb{k}}(\boldsymbol{b})$ is a Krull domain. By (3.2), R is faithfully flat over $\mathbb{k}$ and hence $\mathbb{k}$ is a Krull domain by a standard argument.

Assume next that $\mathbb{k}$ is a Krull domain, and let $x = x^{(n)}_{b_{n-1}b_n}$. To prove that R is a Krull domain we proceed by induction on $|\boldsymbol{b}|$. When $|\boldsymbol{b}| = 1$, or more generally when $r_i = 1$ for each $1 \le i \le n-1$, the assertion is obvious from (2.10). Assume that $|\boldsymbol{b}| \ge 2$ and that $r_i \ge 2$ for some $1 \le i \le n-1$.

Then $n \ge 2$ and the ring R_x is a Krull domain by (3.4) and the induction hypothesis. Note that, by (4.1), the ideal $\mathfrak{p}' = xR$ is prime; set $\mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{ht}\mathfrak{p} = 1 \text{ and } x \notin \mathfrak{p}\}$. We have that $R_{\mathfrak{p}}$ is a DVR for each $\mathfrak{p} \in \mathcal{P}$ and that $R_x = \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}'}$ is also a DVR, to show that R is a Krull domain it suffices to prove the equality $R = R_{\mathfrak{p}'} \cap R_x$. The inclusion $R \subseteq R_{\mathfrak{p}'} \cap R_x$ is obvious, so we pick an element $u \in R_{\mathfrak{p}'} \cap R_x$. Thus $u = v/x^s = w/r$ with some $v, w, r \in R$ such that $v, r \notin xR$. But then $wx^s = rv \notin xR$, hence $s = 0$ and therefore $u \in R$. This completes the proof that R is a Krull domain if and only if $\mathbb{k}$ is.

To deal with the divisor class groups we use basic facts from [20].

Again, we argue by induction on $|\boldsymbol{b}|$. When $|\boldsymbol{b}| = 1$, or more generally when $r_i = 1$ for each $1 \le i \le n-1$, the assertion follows from Gauss' lemma in view of (2.10). Assume that $|\boldsymbol{b}| \ge 2$ and that $r_i \ge 2$ for some $1 \le i \le n-1$; thus $n \ge 2$. Since by (4.1) the ideal xR is prime, Nagata's theorem gives the first isomorphism of

$$\operatorname{Cl}(R) \cong \operatorname{Cl}(R_x) \cong \operatorname{Cl}(R_{\mathbb{k}'}(\boldsymbol{b}')) \cong \operatorname{Cl}(\mathbb{k}') \cong \operatorname{Cl}(\mathbb{k});$$

the second isomorphism follows from (3.4), the third is our induction hypothesis, and the last isomorphism comes from the definition of $\mathbb{k}'$ by Gauss' lemma and Nagata's theorem.

The proof of Theorem 2.6 is now complete. □

5. Standard Monomials and Monomial Order

The goals of this section are first to describe a set of "standard monomials" that yield a free basis of $R_{\mathbb{k}}(\boldsymbol{b})$ as a module over $\mathbb{k}$ and then to associate with it a certain monomial order. This will allow us, by using the theory of Gröbner bases, to reduce the study of the singularities of the ring $R_{\mathbb{k}}(\boldsymbol{b})$ to the study of the combinatorial structure of a certain simplicial complex.

Young Tableaux

We recall some notions from the theory of Young tableaux.

A *shape* is a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$. It is represented by a sequence of left-justified rows of boxes in the plane, where λ_i is the number of boxes in the ith row of the shape. For example, we have

$$(3,2,2,1) = \ydiagram{3,2,2,1} \qquad \text{and} \qquad (2,3,2) = \ydiagram{2,3,2}\ .$$

The shape given by the empty sequence is called *trivial*.

A shape λ is called *standard* if it is trivial or if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Thus, the first of the shapes just displayed is standard, and the second is not.

A *tableau* of shape λ is a sequence $L = (L_1, \ldots, L_k)$, where $L_i = \{l_{i1} < \cdots < l_{i\lambda_i}\} \subset \mathbb{N}$ for $i = 1, \ldots, k$. A tableau is represented by placing l_{ij} in the jth box of the ith row of λ, so that the entries in each row are strictly increasing from left to right. The unique tableau of trivial shape is called the *trivial tableau* and is denoted as $\varnothing$.

A *Young tableau* is a tableau of standard shape such that $s < t$ implies $l_{sj} \leq l_{tj}$ for each $1 \leq j \leq \lambda_t$. Thus, the entries in each column of a Young tableau are nondecreasing from top to bottom. For example if $L'_1 = \{1, 3, 4\}$, $L'_2 = \{2, 3\}$, $L''_1 = \{2, 4\}$, and $L''_2 = \{5\}$, then $L' = (L'_1, L'_2)$ and $L'' = (L''_1, L''_2)$ are the Young tableaux

$$L' = \begin{ytableau} 1 & 3 & 4 \\ 2 & 3 \end{ytableau} \qquad \text{and} \qquad L'' = \begin{ytableau} 2 & 4 \\ 5 \end{ytableau}\ .$$

Note that the trivial tableau is a Young tableau.

Given tableaux $L' = (L'_1, \ldots, L'_{k'})$ and $L'' = (L''_1, \ldots, L''_{k''})$, we write $\frac{L'}{L''}$ for the tableau $(L'_1, \ldots, L'_{k'}, L''_1, \ldots, L''_{k''})$; it is obtained by placing L' on top of L''. For example, if L' and L'' are the tableaux from the previous display, then $\frac{L'}{L''}$ is the tableau

$$\begin{ytableau} 1 & 3 & 4 \\ 2 & 3 \\ 2 & 4 \\ 5 \end{ytableau}\ .$$

It is clear that there are equalities $\frac{L}{\varnothing} = \frac{\varnothing}{L} = L$.

A *multitableau* is a sequence of tableaux $(L_{\mu_0} \mid \cdots \mid L_{\mu_n})$, where the tableau L_{μ_i} is of shape μ_i for $i = 0, \ldots, n$; it is *standard* if L_{μ_i} is a Young tableau for each $0 \leq i \leq n$.

Orders

We consider a linear order on the set of finite subsets of $\mathbb{N}$ given by

$$\begin{aligned} &\{a_1 < \cdots < a_s\} < \{e_1 < \cdots < e_t\} \\ &\qquad \Longleftrightarrow \begin{cases} s < t \text{ or} \\ s = t \text{ and } a_j > e_j \text{ for } j = \min\{i \mid a_i \neq e_i\}. \end{cases} \end{aligned} \tag{5.1}$$

It induces a linear order on the sets of indeterminates M_k (see Section 2) for $k = 1, \ldots, n$.

Let b', b'', r be positive integers, and set

$$\mathcal{V}(b', b'', r) = \{(A, E) \mid A \subseteq \{1, \dots, b'\},\ E \subseteq \{1, \dots, b''\},\ 1 \le |A| = |E| \le r\}.$$

For $k = 1, \dots, n$, let $V_k = \{\langle A|E\rangle_k \mid (A, E) \in \mathcal{V}(b_{k-1}, b_k, r_k)\}$ be a set of indeterminates. We define for $1 \le k \le n$ a linear order on V_k by

$$\langle A|E\rangle_k < \langle D|F\rangle_k \iff \begin{cases} A < D \text{ or} \\ A = D \text{ and } E < F. \end{cases} \tag{5.2}$$

Let $V(\boldsymbol{b})$ be the disjoint union $V_1 \sqcup M_1 \sqcup \cdots \sqcup V_n \sqcup M_n$. Let $Q_{\Bbbk}(\boldsymbol{b}) = \Bbbk[v \mid v \in V(\boldsymbol{b})]$ be the polynomial ring over $\Bbbk$ on the set of variables $V(\boldsymbol{b})$; it is graded by assigning degree 1 to each variable. Finally, let $\mathbb{N}^{V(\boldsymbol{b})}$ be the set of all monomials in the variables $V(\boldsymbol{b})$. We will use the canonical identification

$$\mathbb{N}^{V(\boldsymbol{b})} = \mathbb{N}^{V_1} \times \mathbb{N}^{M_1} \times \cdots \times \mathbb{N}^{V_n} \times \mathbb{N}^{M_n}.$$

We extend the linear orders on V_i and M_i to monomial orders on $\mathbb{N}^{V_i}$ and $\mathbb{N}^{M_i}$ by using reverse lexicographic ordering; see [17, Sec. 15.2] for the terminology. Thus, if $v_1 \le \cdots \le v_k$ and $u_1 \le \cdots \le u_t$ are variables from V_i (or from M_i), then

$$\begin{aligned} &v_1 \dots v_k < u_1 \dots u_t \\ &\qquad \iff \begin{cases} k < t \text{ or} \\ k = t \text{ and } v_m < u_m \text{ for } m = \min\{j \mid v_j \neq u_j\}. \end{cases} \end{aligned} \tag{5.3}$$

We define a monomial order on $\mathbb{N}^{V(\boldsymbol{b})}$ by taking the lexicographic product of the monomial orders on $\mathbb{N}^{V_1}, \mathbb{N}^{M_1}, \dots, \mathbb{N}^{V_n}$, and $\mathbb{N}^{M_n}$; namely, for $w' = u'_1 \dots u'_{2n}$ and $w'' = u''_1 \dots u''_{2n}$ (where $u'_{2i-1}, u''_{2i-1} \in \mathbb{N}^{V_i}$ and $u'_{2i}, u''_{2i} \in \mathbb{N}^{M_i}$) we set

$$w' < w'' \iff u'_m < u''_m \text{ for } m = \min\{i \mid u'_i \neq u''_i\}. \tag{5.4}$$

Finally, we also introduce a partial order on the set of finite subsets of $\mathbb{N}$ by

$$\{a_1 < \cdots < a_s\} \prec \{e_1 < \cdots < e_t\} \iff s \le t \quad \text{and} \quad a_i \ge e_i \text{ for } i = 1, \dots, s.$$

The sets M_k inherit this order, and we define a partial order on the sets V_k by

$$\langle A|E\rangle_k \preceq \langle D|F\rangle_k \iff A \preceq D \quad \text{and} \quad E \preceq F. \tag{5.5}$$

Standard Monomials

To specify the set of standard monomials, which is a subset of $\mathbb{N}^{V(\boldsymbol{b})}$, we follow Pragacz and Weyman [35, Sec. 1] (where $\langle A|E\rangle_k$ and $\langle A\rangle_k$ are denoted by $(E, A)_k$ and $[A]_k$, respectively).

We associate with each monomial $w \in \mathbb{N}^{V(\boldsymbol{b})}$ a multitableau $L(w)$ as follows.

Let $w = u_1 z_1 \dots u_n z_n$ (where $u_i \in \mathbb{N}^{V_i}$ and $z_i \in \mathbb{N}^{M_i}$ for $1 \le i \le n$). For each $1 \le i \le n$ we write $u_i = u_{i1} \dots u_{is_i}$ (resp., $z_i = z_{i1} \dots z_{it_i}$) so that $u_{ij} = \langle A_{ij}|E_{ij}\rangle_i$ and $u_{i1} \ge \cdots \ge u_{is_i}$ (resp., $z_{ij} = \langle C_{ij}\rangle_i$ and $z_{i1} \ge \cdots \ge z_{it_i}$). For each $1 \le i \le n$, set:

$A_i = (A_{i1}, \ldots, A_{is_i})$;

$C_i = (C_{i1}, \ldots, C_{it_i})$;

$$E_i = \begin{cases} \varnothing & \text{if } i = n \text{ and } E_{nj} = \{1, \ldots, b_n\} \text{ for } j = 1, \ldots, s_n, \\ \big((E_{ns_n})^-, \ldots, (E_{np_n})^-\big) & \text{if } i = n \text{ and } p_n = \min\{j \mid \{1, \ldots, b_n\} \neq E_{nj}\}, \\ \big((E_{is_i})^-, \ldots, (E_{i1})^-\big) & \text{if } i \neq n; \end{cases}$$

note that the sequence of components of E_i is obtained by taking in *reverse* order the sequence of the complements of the sets E_{ij}. Finally, set

$$L(w) = \left(\begin{array}{c|c|c|c|c} & E_1 & & E_{n-1} & E_n \\ C_1 & C_2 & \cdots & C_n & \\ A_1 & A_2 & & A_n & \end{array} \right).$$

(5.6) DEFINITION. A monomial $w \in \mathbb{N}^{V(\boldsymbol{b})}$ is called *standard* if $L(w)$ is a standard multitableau and if the elements of $V^{\max}(\boldsymbol{b}) = \{\langle A|E\rangle_k \mid 1 \le k \le n,\ |A| = |E| = r_k\}$ do not divide w.

EXAMPLE. Let $n = 2$ and let $\boldsymbol{b} = (b_0, b_1, b_2) = (3, 4, 2)$; thus $r_1 = r_2 = 2$. For

$$w = \langle 1, 3|1, 2\rangle_1 \langle 2|3\rangle_1 \langle 2|1\rangle_2 \langle 1, 4\rangle_2 \in \mathbb{N}^{V(\boldsymbol{b})},$$

we have

$$L(w) = \left(\begin{array}{c|c|c} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \cline{1-2} 1 & 4 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array} & \boxed{2} \end{array} \right).$$

This is not a standard multitableau, so w is not a standard monomial.

For the monomials

$$w' = \langle 2|3\rangle_1 \langle 1, 3\rangle_1 \langle 3|1\rangle_2 \langle 1, 3\rangle_2 \langle 1, 4\rangle_2$$

and

$$w'' = \langle 2|3\rangle_1 \langle 1, 3\rangle_1 \langle 1, 4|1, 2\rangle_2 \langle 3|2\rangle_2 \langle 1, 3\rangle_2,$$

we have

$$L(w') = L(w'') = \left(\begin{array}{c|c|c} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 1 & 3 & \\ \cline{1-2} 1 & 4 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array} & \boxed{2} \end{array} \right),$$

which is a standard multitableau. However, whereas w' is indeed a standard monomial, w'' is divisible by $\langle 1, 4|1, 2\rangle_2 \in V^{\max}(\boldsymbol{b})$ and hence is not standard.

(5.7) DEFINITION. Define a homomorphism of $\Bbbk$-algebras

$$\pi : Q_{\Bbbk}(\boldsymbol{b}) \to R_{\Bbbk}(\boldsymbol{b})$$

by mapping the indeterminate $\langle A|C\rangle_k \in V(\boldsymbol{b})$ to the minor $\langle A|C\rangle_k$ of $X^{(k)}$ and by sending the indeterminate $\langle D\rangle_k \in V(\boldsymbol{b})$ to the multiplier $\langle D\rangle_k \in R_{\Bbbk}(\boldsymbol{b})$.

It is clear that π is surjective, and we write $I_{\Bbbk}(\boldsymbol{b})$ for the kernel of π.

We need the following important result of Pragacz and Weyman [35, Thm. 1.3].

(5.8) THEOREM [35]. *The surjection $\pi : Q_{\Bbbk}(\boldsymbol{b}) \to R_{\Bbbk}(\boldsymbol{b})$ maps the set of standard monomials bijectively to a free basis of $R_{\Bbbk}(\boldsymbol{b})$ as a $\Bbbk$-module.*

REMARK. The original proof of (5.8) uses methods from group representation theory. Not being aware of this result, we produced in [37] a proof based on a different use of representation theory combined with Gröbner bases techniques. Theorem 6.1 is one of the main ingredients in that alternative proof.

We now give the proof of (3.3), thus completing the proofs of (2.5), (2.6), and (2.8).

Proof of Lemma 3.3. If $r_n = 1$ then set $u' = \langle b_{n-1}\rangle_n \in Q_{\Bbbk}(\boldsymbol{b})$; else set $u' = \langle b_{n-1}|b_n\rangle_n \in Q_{\Bbbk}(\boldsymbol{b})$. Also set $u'' = \langle b_{i-1}|b_i\rangle_i \in Q_{\Bbbk}(\boldsymbol{b})$. It is immediate from the definition that a monomial $u \in \mathbb{N}^{V(\boldsymbol{b})}$ is standard if and only if uu' is standard, if and only if uu'' is standard. Thus (5.8) implies that $\pi(u') = x$ and $\pi(u'') = y$ are regular in R, and that π maps the set of standard monomials divisible by u' bijectively to a free basis of xR as a $\Bbbk$-module. The regularity of the sequence (x, y) is now clear. $\square$

6. The Initial Ideal of $I_{\Bbbk}(\boldsymbol{b})$

Let $\Sigma(\boldsymbol{b})$ be the set of nonstandard monomials. It follows easily from the definitions that $\Sigma(\boldsymbol{b})$ is a monomial ideal in $\mathbb{N}^{V(\boldsymbol{b})}$.

The next theorem, which is the main result of this section, is a key tool in our study of the singularities of $R_{\Bbbk}(\boldsymbol{b})$.

(6.1) THEOREM. *When $\Bbbk$ is a field, $\Sigma(\boldsymbol{b})$ is the initial ideal of $I_{\Bbbk}(\boldsymbol{b})$ with respect to the monomial order* (5.4).

The proof of Theorem 6.1 requires preparation.

Grassmannians

Let $r \le b$ be positive integers, and let $Q_{\Bbbk}(b, r)$ be the polynomial ring over $\Bbbk$ on the variables $M = \{\langle E\rangle \mid E \subseteq \{1, \dots, b\}, |E| = r\}$, graded by assigning degree 1 to each variable. The set M inherits both the partial and linear orders on the subsets of $\{1, \dots, b\}$; as in (5.3), we extend the linear order on M to a monomial order on $\mathbb{N}^M$ by using reverse lexicographic ordering.

Let $\Sigma(M) \subseteq \mathbb{N}^M$ be the monomial ideal generated by the products of pairs of noncomparable (in the partial order) elements of M, and let $\mathcal{G}_{\Bbbk}(M)$ be the set of Plücker relations (see [8, (4.4)])

$$\sum_{C\cap A\subset\Gamma\subset C\setminus D}^{|\Gamma|=q} \nabla_{A,C\setminus\Gamma}\ \nabla_{C\setminus\Gamma,\Gamma}\ \nabla_{\Gamma,D}\langle A\cup(C\setminus\Gamma)\rangle\langle\Gamma\cup D\rangle,$$

where $A, C, D \subseteq \{1, \dots, b\}$ with $|A| = r - p$, $|D| = r - q$, and $|C| = p + q \ge r + 1$.

(6.2) LEMMA. *For each minimal generator u of $\Sigma(M)$ there exists a monic element $f \in \mathcal{G}_{\Bbbk}(M)$ such that $u = \mathrm{in}(f)$.*

Proof. In view of the monomial order on $\mathbb{N}^M$, the lemma is a direct corollary of [8, Lemma (4.5)]. □

(6.3) REMARK. Let $I_{\Bbbk}(b, r)$ be the ideal generated in $Q_{\Bbbk}(b, r)$ by the set $\mathcal{G}_{\Bbbk}(M)$, let $Y = (Y_{ij})$ be a $b \times r$ matrix of indeterminates, let $\Bbbk[Y]$ be the polynomial ring over $\Bbbk$ whose variables are the entries of Y, and write $G_{\Bbbk}(b, r) \subset \Bbbk[Y]$ for the subring generated over $\Bbbk$ by the $r \times r$ minors of Y. It is well known (cf. [8, (4.7)]) that the map $Q_{\Bbbk}(b, r) \to G_{\Bbbk}(b, r)$ given by $\langle E \rangle \mapsto \langle E|1, \ldots, r\rangle_Y$ induces an isomorphism $Q_{\Bbbk}(b, r)/I_{\Bbbk}(b, r) \cong G_{\Bbbk}(b, r)$. This makes $G_{\Bbbk}(b, r)$ into a graded ordinal Hodge $\Bbbk$-algebra over $(M, \preceq)$ governed by $\Sigma(M)$; see [8, (4.6)]. (Note that the partial order on M is the *reverse* of the order originally considered in [14, Sec. 11].) In particular, the set $\mathbb{N}^M \setminus \Sigma(M)$ is mapped bijectively to a free basis of $G_{\Bbbk}(b, r)$ as a $\Bbbk$-module.

Determinantal Rings

Let b', b'', and r be integers such that $1 \le r \le m$, where $m = \min(b', b'')$. Let $Q_{\Bbbk}(b', b'', r)$ be the polynomial ring over $\Bbbk$ on the variables $V = \{\langle A|E\rangle \mid (A, E) \in \mathcal{V}(b', b'', r)\}$, graded by assigning degree 1 to each variable. The set V is linearly and partially ordered as in (5.2) and (5.5), respectively. As in (5.3), we extend the linear order on V to a monomial order on $\mathbb{N}^V$ by using reverse lexicographic ordering.

Let $\Sigma(V)$ be the monomial ideal generated by the products of pairs of noncomparable (in the partial order) elements of V. The elements of $\mathbb{N}^V \setminus \Sigma(V)$ are called $\Sigma(V)$-standard monomials.

Let $X = (X_{ij})$ be a $b' \times b''$ matrix of indeterminates, and write $\Bbbk[X]$ for the polynomial ring over $\Bbbk$ with variables the entries of X. Let $I_{r+1}(X)$ be the ideal generated by the $(r+1) \times (r+1)$ minors of X, and write $I_{\Bbbk}(b', b'', r)$ for the kernel of the surjection $Q_{\Bbbk}(b', b'', r) \to \Bbbk[X]/I_{r+1}(X)$ given by $\langle A|E\rangle \mapsto \langle A|E\rangle_X$. It is well known (see e.g. [8, proof of (4.11)]) that the $\Sigma(V)$-standard monomials are mapped bijectively to a free $\Bbbk$-basis for $Q_{\Bbbk}(b', b'', r)/I_{\Bbbk}(b', b'', r)$. For $u \in \Sigma(V)$, let

$$\mathrm{st}(u) = \sum r_{uw} w$$

be the unique expression of u mod $I_{\Bbbk}(b', b'', r)$ as a linear combination over $\Bbbk$ of $\Sigma(V)$-standard monomials.

Let $\Sigma^{\max}(V)$ be the monomial ideal generated by $\{\langle A|E\rangle \mid r = |A| = |E|\}$. Set $\mathcal{D}^{-}_{\Bbbk}(V) = \{u - \mathrm{st}(u) \mid u \notin \Sigma^{\max}(V) \text{ is a minimal generator of } \Sigma(V)\}$.

Let $\mathcal{D}^{\mathrm{row}}_{\Bbbk}(V)$ be the set of Plücker relations

$$\sum_{C \cap A \subset \Gamma \subset C \setminus D}^{|\Gamma| = q} \nabla_{A, C\setminus\Gamma}\, \nabla_{C\setminus\Gamma, \Gamma}\, \nabla_{\Gamma, D} \langle A \cup (C \setminus \Gamma) \mid E\rangle \langle \Gamma \cup D \mid F\rangle;$$

here $A, C, D \subseteq \{1, \dots, b'\}$ and $E, F \subseteq \{1, \dots, b''\}$, with $|E| = r$, $|F| = s \leq r$, $|A| = r - p$, $|D| = s - q$, and $|C| = p + q \geq r + 1$. Let $\mathcal{D}_{\Bbbk}^{\text{col}}(V)$ be the set of Plücker relations

$$\sum_{C \cap E \subset \Gamma \subset C \setminus F}^{|\Gamma| = q} \nabla_{E, C\setminus\Gamma}\ \nabla_{C\setminus\Gamma, \Gamma}\ \nabla_{\Gamma, F} \langle A \mid E \cup (C \setminus \Gamma)\rangle \langle D \mid \Gamma \cup F\rangle;$$

here $A, D \subseteq \{1, \dots, b'\}$ and $C, E, F \subseteq \{1, \dots, b''\}$, with $|A| = r$, $|D| = s \leq r$, $|E| = r - p$, $|F| = s - q$, and $|C| = p + q \geq r + 1$.

Finally, set $\mathcal{D}_{\Bbbk}(V) = \mathcal{D}_{\Bbbk}^{-}(V) \cup \mathcal{D}_{\Bbbk}^{\text{row}}(V) \cup \mathcal{D}_{\Bbbk}^{\text{col}}(V)$.

REMARK. Unlike the rings $G(b, r)$, the rings $\Bbbk[X]/I_{r+1}(X)$ are *not* Hodge algebras over $(V, \preceq)$, which complicates the proof of the following lemma.

(6.4) LEMMA. *There is an inclusion $\mathcal{D}_{\Bbbk}(V) \subset I_{\Bbbk}(b', b'', r)$. Furthermore, for each minimal generator u of $\Sigma(V)$, there is a monic element $f \in \mathcal{D}_{\Bbbk}(V)$ with $u = \text{in}(f)$.*

Proof. Since modulo $I_{r+1}(X)$ the rank of the matrix X is at most r, the inclusion $\mathcal{D}_{\Bbbk}(V) \subset I_{\Bbbk}(b', b'', r)$ follows by a standard argument.

If the minimal generator u is divisible by some $\langle A|E\rangle$ with $|A| = |E| = r$, then one shows as in (6.2) that $u = \text{in}(f)$ for some $f \in \mathcal{D}_{\Bbbk}^{\text{row}}(V) \cup \mathcal{D}_{\Bbbk}^{\text{col}}(V)$. Thus, for the rest of the argument we assume $u \notin \Sigma^{\max}(V)$; note that the proof of the lemma will be complete once we show that $u = \text{in}(u - \text{st}(u))$.

We need to prove that $r_{uw} \neq 0$ implies $u > w$. Let $m = \min(b', b'')$. Since the expression of u mod $I_{\Bbbk}(b', b'', r)$ is obtained from the expression of u mod $I_{\Bbbk}(b', b'', m)$ by removing all terms involving monomials divisible by a variable $\langle A|E\rangle$ with $|A| = |E| > r$, we assume for the rest of the proof that $r = m$.

Let P be the $b'' \times b''$ matrix over $\Bbbk$,

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

with ones on the indicated diagonal and zeros elsewhere, and let $Y = (Y_{ij})$ be a $(b' + b'') \times b''$ matrix of indeterminates. Specializing Y to the $(b' + b'') \times b''$ matrix $\left(\begin{smallmatrix} X \\ P \end{smallmatrix}\right)$ (this is X on top of P), we define a map $\Bbbk[Y] \to \Bbbk[X]$. Composing with the inclusion $G_{\Bbbk}(b' + b'', b'') \to \Bbbk[Y]$, we obtain a homomorphism

$$\bar{\varphi}\colon G_{\Bbbk}(b' + b'', b'') \to \Bbbk[X].$$

On the level of generators, $\bar{\varphi}$ is described as follows. For $A = \{a_1 < \cdots < a_{b''}\} \subset \{1, \dots, b' + b''\}$, let $A \cap \{b' + 1, \dots, b' + b''\} = \{a_{s+1}, \dots, a_{b''}\}$ and set $U = \{b' + b'' - a_{b''} + 1, \dots, b' + b'' - a_{s+1} + 1\} \subset \{1, \dots, b''\}$. Then $\bar{\varphi}$ is given by

$$\langle A|1, \dots, b''\rangle_Y \mapsto \begin{cases} (-1)^{b''(b''-1)/2} & \text{if } A = \{b' + 1, \dots, b' + b''\}, \\ (-1)^{b''(b''-1)/2}\ \nabla_{U,E} \langle D|E\rangle_X & \text{if } A \neq \{b' + 1, \dots, b' + b''\}, \end{cases}$$

where $D = A \setminus \{b'+1, \dots, b'+b''\}$ and $E = \{1, \dots, b''\} \setminus U$. By substituting (in the formula just displayed) $\langle A \rangle$ for $\langle A|1, \dots, b''\rangle_Y$ and by deleting X, we lift $\bar{\varphi}$ to a map $\varphi\colon Q_{\Bbbk}(b'+b'', b'') \to Q_{\Bbbk}(b', b'', m)$.

As in [13, Lemma 2.2], one checks that φ induces an isomorphism of the partially ordered sets $M \setminus \{\langle b'+1, \dots, b'+b''\rangle\}$ and V. Furthermore, it is immediate from the definition of φ that there exists a unique $\tilde{u} \in \mathbb{N}^M$ such that $\deg(\tilde{u}) = \deg(u) = 2$ and $\varphi(\tilde{u}) = \pm u$.

Let $\operatorname{st}(\tilde{u}) = \sum r_{\tilde{u}\tilde{w}} \tilde{w}$ be the unique (see Remark 6.3) expression of $\tilde{u}$ modulo $I_{\Bbbk}(b'+b'', b'')$ as a $\Bbbk$-linear combination of $\Sigma(M)$-standard monomials, and note that for each $\tilde{w}$ we have $\varphi(\tilde{w}) = \pm w$ with $w \notin \Sigma(V)$. Applying φ to $\tilde{u} - \operatorname{st}(\tilde{u})$ we see that $u - \sum \pm r_{\tilde{u}\tilde{w}} w \in I_{\Bbbk}(b', b'', m)$, hence $\operatorname{st}(u) = \sum \pm r_{\tilde{u}\tilde{w}} w$.

Pick $\tilde{w}$ with $r_{\tilde{u}\tilde{w}} \neq 0$. Note that $\deg(\tilde{u}) = \deg(\tilde{w})$ (since $I_{\Bbbk}(b'+b'', b'')$ is homogeneous), and let $\langle A_1 \rangle$ be the smallest variable dividing $\tilde{w}$. If $A_1 = \{b'+1, \dots, b'+b''\}$, then $\deg(w) < \deg(\tilde{w}) = \deg(\tilde{u}) = \deg(u)$ and hence $u > w$. Thus we may assume for the rest of the proof that $A_1 \neq \{b'+1, \dots, b'+b''\}$.

Then the image of $\langle A_1 \rangle$ under φ is of the form $\pm\langle D_1|E_1\rangle$, and $\langle D_1|E_1\rangle$ is the smallest variable dividing w. Let $\langle A_2 \rangle$ be any variable dividing $\tilde{u}$. As remarked in (6.3), the ring $G_{\Bbbk}(b'+b'', b'')$ is an ordinal Hodge algebra on $(M, \preceq)$, hence $\langle A_1 \rangle \prec \langle A_2 \rangle$. Thus the image of $\langle A_2 \rangle$ under φ has the form $\pm\langle D_2|E_2\rangle$, and $\langle D_1|E_1\rangle \prec \langle D_2|E_2\rangle$. Therefore $\langle D_1|E_1\rangle$ is smaller than any variable dividing u, and since $\deg(w) \leq \deg(u)$ we obtain $u > w$. $\square$

Proof of Theorem 6.1

For $t = 2, 4, 5, 6$, write $\mathcal{F}_t$ for the set of all elements in $Q_{\Bbbk}(\boldsymbol{b})$ of the form $(1.t)$.

Let $\mathcal{F}_1$ denote the set of elements in $Q_{\Bbbk}(\boldsymbol{b})$ of the form

$$\sum_{\Gamma \subseteq \{1,\dots,b_k\} \setminus (\Lambda \cup H \cup K)}^{|\Gamma| = t} \nabla_{H,\Gamma} \nabla_{\Gamma,K} \langle G \mid H \cup \Gamma \rangle_k \langle \Gamma \cup K \mid F \rangle_{k+1}$$

for $k = 1, \dots, n-1$ and for all subsets $G \subseteq \{1, \dots, b_{k-1}\}$, $H, K, \Lambda \subseteq \{1, \dots, b_k\}$, and $F \subseteq \{1, \dots, b_{k+1}\}$ such that $|G| = m \leq r_k$, $|H| = m - t$, $|F| = s \leq r_{k+1}$, $|K| = s - t$, and $|\Lambda| < t \leq \min(s, m)$.

Set $\mathcal{F}_3 = \bigcup_{i=1}^{n} \mathcal{D}_{\Bbbk}(V_i)$ and note that—by [15, Cor. 1.2], Remark 2.4(a), and (6.4)—there is an inclusion

$$\mathcal{F} = \bigcup_{i=1}^{6} \mathcal{F}_i \subset I_{\Bbbk}(\boldsymbol{b}).$$

Since by (5.8) the standard monomials are linearly independent modulo $I_{\Bbbk}(\boldsymbol{b})$, to complete the proof of (6.1) it suffices to show that, for each minimal generator u of $\Sigma(\boldsymbol{b})$, there exists a monic element f from $\mathcal{F}$ with $u = \operatorname{in}(f)$.

Because $\Sigma(\boldsymbol{b})$ is generated by the elements of $V^{\max}(\boldsymbol{b})$ (see Definition 5.6) and by the nonstandard products of pairs of variables, the minimal generators of $\Sigma(\boldsymbol{b})$ are contained in the union of the sets

$$\Sigma_1 = \{\langle G|H\rangle_{k-1}\langle K|F\rangle_k \mid 2 \le k \le n,\ K \not\le \bar{H}\},$$

$$\Sigma_2 = \{\langle A|E\rangle_k \mid 1 \le k \le n,\ |A| = |E| = r_k\},$$

$$\Sigma_3 = \{\langle A|E\rangle_k\langle D|F\rangle_k \mid 1 \le k \le n,\ \langle A|E\rangle_k \not\le \langle D|F\rangle_k \not\le \langle A|E\rangle_k\},$$

$$\Sigma_4 = \{\langle A\rangle_k\langle D\rangle_k \mid 1 \le k \le n,\ A \not\le D \not\le A\},$$

$$\Sigma_5 = \{\langle A\rangle_k\langle E|F\rangle_k \mid 1 \le k \le n,\ E \not\le A\},$$

$$\Sigma_6 = \{\langle G|H\rangle_{k-1}\langle K\rangle_k \mid 2 \le k \le n,\ K \not\le \bar{H}\}.$$

Thus, the possible cases for u are as follows: (1) $u \in \Sigma_2$; (2) $u \in \Sigma_3$; (3) $u \in \Sigma_4$; (4) $u \in \Sigma_5$; (5) $u \in \Sigma_6$; and (6) $u \in \Sigma_1$.

In case (1), $u = \langle A|E\rangle_k$ with $|A| = |E| = r_k$ for some $1 \le k \le n$, and it is immediate from (5.4) that $u = \mathrm{in}(f)$, where $f \in \mathcal{F}_2$ is the monic polynomial

$$f = \begin{cases} u - \langle A\rangle_n & \text{if } k = n, \\ u - \nabla_{\bar{E},E}\langle A\rangle_k\langle \bar{E}\rangle_{k+1} & \text{if } 1 \le k \le n-1. \end{cases}$$

In case (2), $u \in \Sigma(V_k)$ for some k, and we are done by (6.4). In case (3), $u \in \Sigma(M_k)$ for some k, and we are done by (6.2).

In case (4), $u = \langle A'\rangle_k\langle E'|F\rangle_k$ for some $1 \le k \le n$, where

$$A' = \{a_1 < \cdots < a_{r_k}\}, \quad E' = \{e_1 < \cdots < e_s\}, \quad F = \{f_1 < \cdots < f_s\}$$

satisfy $a_q > e_q$ for some $1 \le q \le s$ and $a_i \le e_i$ for $1 \le i \le q-1$. Using (1.5) with

$$A = \{a_1, \ldots, a_{q-1}\}, \quad C = \{e_1, \ldots, e_q, a_q, \ldots, a_{r_k}\}, \quad E = \{e_{q+1}, \ldots, e_s\}$$

yields $u = \mathrm{in}(f)$, where $f \in \mathcal{F}_5$ is the monic polynomial

$$f = \sum_{C\cap A\subset\Gamma\subset C\setminus E}^{|\Gamma|=q} \nabla_{A,C\setminus\Gamma}\ \nabla_{C\setminus\Gamma,\Gamma}\ \nabla_{\Gamma,E}\langle A\cup(C\setminus\Gamma)\rangle_k\langle \Gamma\cup E \mid F\rangle_k;$$

hence we are done in this case.

In case (5) we have $u = \langle G|H'\rangle_k\langle K'\rangle_{k+1}$ for some $1 \le k \le n-1$, with

$$G = \{g_1 < \cdots < g_m\}, \quad H' = \{h_1 < \cdots < h_m\}, \quad K' = \{k_1 < \cdots < k_{r_{k+1}}\}$$

such that $\bar{h}_p > k_p$ for some $1 \le p \le r_{k+1}$ and $\bar{h}_i \le k_i$ for $1 \le i \le p-1$ (where $\overline{H'} = \{\bar{h}_1 < \cdots < \bar{h}_{b_k-m}\}$). The inequalities $\bar{h}_{p-1} \le k_{p-1} < k_p < \bar{h}_p$ imply $k_p = h_q$ for some $1 \le q \le m$. Set

$$\tilde{H} = \{\bar{h}_1, \ldots, \bar{h}_{p-1}\}, \quad H'' = \{h_1, \ldots, h_q\}, \quad K'' = \{k_1, \ldots, k_p\}.$$

Clearly $H'' \cap \tilde{H} = \varnothing$ and $H'' \cup \tilde{H} = \{1, \ldots, k_p\}$, hence

$$K'' \cap \tilde{H} = K'' \setminus (K'' \cap H'').$$

Set

$$K = K' \setminus (K'' \cap H''), \quad H = H' \setminus (K'' \cap H''), \quad \Lambda = \tilde{H} \setminus (K'' \cap \tilde{H}).$$

We have $(K'' \cap H'') \cap (H \cup K \cup \Lambda) = \varnothing$ and $|K'' \cap H''| = |K''| - |K'' \cap \tilde{H}| = p - |\tilde{H}| + |\Lambda| = |\Lambda| + 1$. Since $\{1, \dots, k_p\} \subset (K \cup \Lambda \cup H) \cup (K'' \cap H'')$, the set $K'' \cap H''$ contains the smallest t elements of $\overline{K \cup \Lambda \cup H}$, where $t = |K'' \cap H''|$. Thus (1.6) with G, K, H, Λ yields $u = \operatorname{in}(f)$, where $f \in \mathcal{F}_6$ is the monic polynomial

$$f = \sum_{\Gamma \subset \{1, \dots, b_k\} \setminus (\Lambda \cup H \cup K)}^{|\Gamma| = t} \nabla_{H,\Gamma} \nabla_{\Gamma,K} \langle G \mid H \cup \Gamma \rangle_k \langle \Gamma \cup K \rangle_{k+1};$$

hence we are done in case (5), too.

Finally, in case (6) we have $u = \langle G | H' \rangle_k \langle K' | F \rangle_{k+1}$ for some $1 \le k \le n-1$, with

$$G = \{g_1 < \cdots < g_m\}, \quad H' = \{h_1 < \cdots < h_m\},$$
$$K' = \{k_1 < \cdots < k_s\}, \quad F = \{f_1 < \cdots < f_s\}$$

such that $\bar{h}_p > k_p$ for some $1 \le p \le s$ and $\bar{h}_i \le k_i$ for $1 \le i \le p-1$ (where $\overline{H'} = \{\bar{h}_1 < \cdots < \bar{h}_{b_k - m}\}$). Define K, H, and Λ as in case (5) and note that the argument given there yields $u = \operatorname{in}(f)$, where $f \in \mathcal{F}_1$ is the monic polynomial

$$f = \sum_{\Gamma \subset \{1, \dots, b_k\} \setminus (\Lambda \cup H \cup K)}^{|\Gamma| = t} \nabla_{H,\Gamma} \nabla_{\Gamma,K} \langle G \mid H \cup \Gamma \rangle_k \langle \Gamma \cup K \mid F \rangle_{k+1}.$$

The proof of Theorem 6.1 is now complete. □

7. Combinatorial Structure

The goal of this section is to prove the combinatorial results needed to study the singularities of $R_{\Bbbk}(\boldsymbol{b})$. For this purpose we consider the simplicial complex $\Delta(\boldsymbol{b})$ that has $V(\boldsymbol{b})$ as vertex set, where $F \subset V(\boldsymbol{b})$ is a face of $\Delta(\boldsymbol{b})$ if and only if the monomial $w_F = \prod_{v \in F} v$ is standard. Our aim is to show that $\Delta(\boldsymbol{b})$ has a good combinatorial structure in the following sense (cf. [7, p. 211]).

(7.1) DEFINITION. A simplicial complex Δ of dimension d is *constructible* if:

(a) Δ is a simplex; or
(b) there exist proper d-dimensional constructible subcomplexes $\Delta_1, \Delta_2 \subset \Delta$ such that $\Delta_1 \cap \Delta_2$ is constructible of dimension $d-1$ and $\Delta_1 \cup \Delta_2 = \Delta$.

(7.2) REMARKS. (a) An easy induction on the number of facets shows that, if Δ_1 and Δ_2 are constructible, then they are pure and their join $\Delta_1 * \Delta_2$ is constructible.

(b) If $(P, \prec)$ is a bounded poset that is a distributive lattice, then the associated simplicial complex of chains in P is shellable [3] and hence constructible.

To show that $\Delta(\boldsymbol{b})$ is constructible, it is more convenient to state and prove the result for a whole class of subcomplexes of $\Delta(\boldsymbol{b})$.

Let $V_1' = \{\langle A|D\rangle_1 \in V_1 \mid r_1 - 1 \geq |A|\}$, set $P(\boldsymbol{b}) = M_1 \cup V_1'$, and extend the partial orders on V_1' and M_1 to a partial order on $P(\boldsymbol{b})$ by setting

$$\langle A|D\rangle_1 \prec \langle C\rangle_1 \iff A \prec C.$$

For $A \subset \{1, \ldots, b_0\}$ with $|A| = r_1$, define

$$P_A(\boldsymbol{b}) = \{v \in P(\boldsymbol{b}) \mid v \preceq \langle A\rangle_1\};$$
$$\Delta_A(\boldsymbol{b}) = \{F \in \Delta(\boldsymbol{b}) \mid F \cap P(\boldsymbol{b}) \subseteq P_A(\boldsymbol{b})\}.$$

REMARK. It is easy to see from the definitions that the assignment $F \mapsto L(w_F)$ is one-to-one and identifies the simplicial complex $\Delta(\boldsymbol{b})$ with a certain set of standard multitableaux. This alternative description of $\Delta(\boldsymbol{b})$ is useful in visualizing the objects we will consider. Thus a face F of $\Delta(\boldsymbol{b})$, with associated standard multitableau

$$L(w_F) = \left(\begin{array}{c|c|c|c|c} & E_1 & & E_{n-1} & E_n \\ C_1 & C_2 & \cdots & C_n & \\ A_1 & A_2 & & A_n & \end{array}\right), \tag{$*$}$$

is a face of the simplicial complex $\Delta_A(\boldsymbol{b})$ precisely when the multitableau

$$\left(\begin{array}{c|c|c|c|c} A & E_1 & & E_{n-1} & E_n \\ C_1 & C_2 & \cdots & C_n & \\ A_1 & A_2 & & A_n & \end{array}\right)$$

is also standard (it is obtained from $(*)$ by placing on top of C_1 the tableau whose only row contains $|A| = r_1$ boxes and whose entries are the elements of A).

(7.3) THEOREM. *For each A, the simplicial complex $\Delta_A(\boldsymbol{b})$ is constructible. In particular, the complex $\Delta(\boldsymbol{b}) = \Delta_{\{1,\ldots,r_1\}}(\boldsymbol{b})$ is constructible.*

The proof of this theorem requires some preparation.

Let $\mathcal{S}$ be the set of $(r_1 - 1)$-element subsets of $\{1, \ldots, b_1\}$. For $E \in \mathcal{S}$ set

$$P_{A,E}(\boldsymbol{b}) = M_1 \cup \{\langle C|D\rangle_1 \in P_A(\boldsymbol{b}) \mid D \preceq E\}.$$

Clearly $P_{A,E}(\boldsymbol{b}) \subseteq P_A(\boldsymbol{b})$, and it is easy to check that both posets are bounded and are distributive lattices. In particular, the corresponding simplicial complexes of chains $\Delta^{(1)}_{A,E}(\boldsymbol{b})$ and $\Delta^{(1)}_A(\boldsymbol{b})$ are constructible by Remark 7.2(b).

REMARK. A face of $\Delta^{(1)}_A(\boldsymbol{b})$ (resp., $\Delta^{(1)}_{A,E}(\boldsymbol{b})$) is a subset F of $P_A(\boldsymbol{b})$ (resp., $P_{A,E}(\boldsymbol{b})$) such that every two elements of F are comparable in the partial order. It follows easily that F is also a face of $\Delta(\boldsymbol{b})$; therefore, $\Delta^{(1)}_A(\boldsymbol{b})$ and $\Delta^{(1)}_{A,E}(\boldsymbol{b})$ are simplicial subcomplexes of $\Delta(\boldsymbol{b})$. Thus a face F of $\Delta(\boldsymbol{b})$ is a face of $\Delta^{(1)}_A(\boldsymbol{b})$ precisely when the standard multitableau associated with F has the form

$$\left(\begin{array}{c|c|c|c|c} & E_1 & & & \\ C_1 & & \varnothing & \cdots & \varnothing \\ A_1 & & & & \end{array}\right) \tag{$**$}$$

(i.e., when the tableaux A_i, C_i, E_i are trivial for $2 \leq i \leq n$) and the multitableau

$$\left(\begin{array}{c|c|c|c|c} A & E_1 & & & \\ C_1 & & \varnothing & \cdots & \varnothing \\ A_1 & & & & \end{array}\right) \qquad (***)$$

is also standard. Similarly, F is a face of $\Delta^{(1)}_{A,E}(\boldsymbol{b})$ precisely when its standard multitableau has the form $(**)$ and the multitableau

$$\left(\begin{array}{c|c|c|c|c} A & E_1 & & & \\ C_1 & \bar{E} & \varnothing & \cdots & \varnothing \\ A_1 & & & & \end{array}\right)$$

is also standard (it is obtained from $(***)$ by placing underneath E_1 the tableau whose only row contains $|\bar{E}| = r_2 + 1$ boxes and whose entries are the elements of $\bar{E}$).

We say that $E \in \mathcal{S}$ *covers* $D \in \mathcal{S}$ if $D \precneqq E$ and $D \preceq D' \precneqq E$ implies $D = D'$.

(7.4) LEMMA. *If E covers the sets $E_1, \ldots, E_k$, then the simplicial complex*

$$\bigcup_{i=1}^{k} \Delta^{(1)}_{A,E_i}(\boldsymbol{b})$$

is constructible.

Proof. We proceed by induction on k. The case $k = 1$ is just the statement that $\Delta^{(1)}_{A,E_1}(\boldsymbol{b})$ is constructible. Assume that $k \geq 2$ and that the assertion is true for $k - 1$ sets. The simplicial complexes $\bigcup_{i=1}^{k-1} \Delta^{(1)}_{A,E_i}(\boldsymbol{b})$ and $\Delta^{(1)}_{A,E_k}(\boldsymbol{b})$ are then constructible by induction. It thus suffices to show that they have the same dimension d and that their intersection is constructible of dimension $d - 1$. We have $\left(\bigcup_{i=1}^{k-1} \Delta^{(1)}_{A,E_i}(\boldsymbol{b})\right) \cap \Delta^{(1)}_{A,E_k}(\boldsymbol{b}) = \bigcup_{i=1}^{k-1} \Delta^{(1)}_{A,E_{ik}}(\boldsymbol{b})$, where E_{ik} is covered by both E_i and E_k. In particular, the intersection is constructible by the induction hypothesis. Furthermore, if D' is covered by D, then it is immediate from the definitions that a maximal simplex of $\Delta^{(1)}_{A,D'}(\boldsymbol{b})$ can be completed to a maximal simplex of $\Delta^{(1)}_{A,D}(\boldsymbol{b})$ by adjoining a single vertex (of the form $\langle C|D\rangle_1$ for a suitable C). Since the complexes $\Delta^{(1)}_{A,E_j}(\boldsymbol{b})$ and $\Delta^{(1)}_{A,E_{ik}}(\boldsymbol{b})$ are pure by Remark 7.2(a), the assertion about the dimensions follows immediately and completes the proof of the lemma. $\square$

Let $E = \{e_1 < \cdots < e_{r_1-1}\} \subset \{1, \ldots, b_1\}$, let $\bar{E} = \{\bar{e}_1 < \cdots < \bar{e}_{r_2} < \bar{e}_{r_2+1}\}$ be its complement, and set $\tilde{E} = \{\bar{e}_1, \ldots, \bar{e}_{r_2}\}$. Recall that $\mathcal{S}$ is linearly ordered, and denote by E_+ the smallest element of $\mathcal{S}$ strictly greater than E.

(7.5) LEMMA. *If E covers a set D such that $\tilde{D} \neq \tilde{E}$ and $\tilde{D} = (D_+)^\sim$, then $\tilde{E} = (E_+)^\sim$.*

Proof. Since E is a cover, $D = \{e_1, \ldots, e_i + 1, \ldots, e_{r_1-1}\}$ for some $1 \leq i \leq r_1 - 1$; hence $e_i + 1 = \bar{e}_k$ for some $1 \leq k \leq r_2 + 1$. Therefore $\bar{D} = \{\bar{e}_1, \ldots, \bar{e}_k - 1, \ldots, \bar{e}_{r_2+1}\}$ and, as $\tilde{D} \neq \tilde{E}$, we must have $1 \leq k \leq r_2$. Since $\tilde{D} = (D_+)^\sim$, we obtain $(D_+)^- = \{\bar{e}_1, \ldots, \bar{e}_k - 1, \ldots, \bar{e}_{r_2}, \bar{e}_{r_2+1} + 1\}$ and hence $D \prec D_+$. Therefore D_+ covers D, and since $D_+ \neq E$ we must have $D_+ = \{e_1, \ldots, e_i + 1, \ldots, e_j - 1, \ldots, e_{r_1-1}\}$ for some $i < j \le r_1 - 1$. But then $E_+ = \{e_1, \dots, e_j - 1, \dots, e_{r_1-1}\}$ and $(E_+)^- = \{\bar{e}_1, \dots, \bar{e}_{r_2}, \bar{e}_{r_2+1} + 1\}$, which yields $\tilde{E} = (E_+)^{\sim}$. $\square$

(7.6) REMARK. If $\tilde{D} = \tilde{E}$, then $\bar{D}$ and $\bar{E}$ coincide except possibly at their greatest element. If in addition $D \le D' \le E$, then necessarily $\tilde{D} = \widetilde{D'} = \tilde{E}$.

Proof of Theorem 7.3. We argue by induction on n. If $n = 1$, then $\Delta_A(\boldsymbol{b}) = \Delta_A^{(1)}(\boldsymbol{b})$ and hence is constructible. Assume $n \ge 2$ and that the assertion holds for $n-1$. Let $\tilde{\boldsymbol{b}} = (\tilde{b}_0, \dots, \tilde{b}_{n-1})$, where $\tilde{b}_i = b_{i+1}$ for $i = 0, \dots, n-1$. We consider $V(\tilde{\boldsymbol{b}})$ as a subset of $V(\boldsymbol{b})$ via the identification $V(\tilde{\boldsymbol{b}}) = V_2 \sqcup M_2 \sqcup \cdots \sqcup V_n \sqcup M_n$. Thus $\Delta(\tilde{\boldsymbol{b}})$ becomes a simplicial subcomplex of $\Delta(\boldsymbol{b})$.

REMARK. A face F of $\Delta(\boldsymbol{b})$ is a face of $\Delta(\tilde{\boldsymbol{b}})$ precisely when its multitableau has the form

$$\left(\begin{array}{c|c|c|c|c|c} & & E_2 & & E_{n-1} & E_n \\ \varnothing & C_2 & C_3 & \cdots & C_n & \\ & A_2 & A_3 & & A_n & \end{array}\right),$$

where A_1, C_1, and E_1 are trivial. If $E \subset \{1, \dots, b_1\}$ with $|E| = r_2$, then $F \in \Delta(\boldsymbol{b})$ is a face of $\Delta_E(\tilde{\boldsymbol{b}})$ precisely when its multitableau is as just displayed, and the multitableau

$$\left(\begin{array}{c|c|c|c|c|c} & E & E_2 & & E_{n-1} & E_n \\ \varnothing & C_2 & C_3 & \cdots & C_n & \\ & A_2 & A_3 & & A_n & \end{array}\right)$$

is also standard.

If $r_1 = 1$, then $\Delta_A(\boldsymbol{b}) = \Delta_A^{(1)}(\boldsymbol{b}) * \Delta(\tilde{\boldsymbol{b}})$ and hence is constructible by Remark 7.2(a) and by the induction hypothesis. Thus we may also assume that $r_1 \ge 2$.

For $E \subset \{1, \dots, b_1\}$ with $|E| = r_1 - 1$, set

$$\Delta_{A,E}(\boldsymbol{b}) = \{F \in \Delta_A(\boldsymbol{b}) \mid F \cap P(\boldsymbol{b}) \subseteq P_{A,E}(\boldsymbol{b}) \text{ and } F \cap P(\tilde{\boldsymbol{b}}) \subseteq P_{\tilde{E}}(\tilde{\boldsymbol{b}})\}.$$

REMARK. A face F of $\Delta(\boldsymbol{b})$ with associated standard multitableau $(*)$ is a face of $\Delta_{A,E}(\boldsymbol{b})$ precisely when the multitableau

$$\left(\begin{array}{c|c|c|c|c|c} & E_1 & & & & \\ A & \bar{E} & E_2 & & E_{n-1} & E_n \\ C_1 & C_2 & C_3 & \cdots & C_n & \\ A_1 & A_2 & A_3 & & A_n & \end{array}\right)$$

is also standard; here, on top of C_1 is placed the tableau whose only row has the elements of A as entries, and between E_1 and C_2 is inserted the tableau whose only row has the elements of $\bar{E}$ as entries.

It follows that

$$\Delta_{A,E}(\boldsymbol{b}) = \Delta_{A,E}^{(1)}(\boldsymbol{b}) * \Delta_{\tilde{E}}(\tilde{\boldsymbol{b}}); \tag{7.7}$$

hence, by Remark 7.2(a) and by the induction hypothesis, $\Delta_{A,E}(\boldsymbol{b})$ is constructible. Furthermore, it is easy to see that if $E = \{e_1 < \cdots < e_{r_1-1}\}$ and $D = \{d_1 < \cdots < d_{r_1-1}\}$ then

$$\Delta_{A,E}(\boldsymbol{b}) \cap \Delta_{A,D}(\boldsymbol{b}) = \Delta^{(1)}_{A,C'}(\boldsymbol{b}) * \Delta_{\widetilde{C''}}(\tilde{\boldsymbol{b}}), \tag{7.8}$$

where $C' = \{c'_1 < \cdots < c'_{r_1-1}\}$ is given by $c'_i = \max(e_i, d_i)$ for $1 \le i \le r_1 - 1$ and $C'' = \{c''_1 < \cdots < c''_{r_1-1}\}$ is given by $c''_i = \min(e_i, d_i)$ for $1 \le i \le r_1 - 1$. It follows from (7.7) and (7.8) that a facet of $\Delta_{A,D}(\boldsymbol{b})$ is in $\Delta_{A,E}(\boldsymbol{b})$ if and only if $\tilde{D} = \tilde{E}$ and $D \le E$. If this is the case, then it is clear that we actually have an inclusion $\Delta_{A,D}(\boldsymbol{b}) \subset \Delta_{A,E}(\boldsymbol{b})$.

Let $\mathcal{M} = \{E \in \mathcal{S} \mid E = \max\{D \in \mathcal{S} \mid \tilde{D} = \tilde{E}\}\}$. For $E \in \mathcal{M}$ define

$$\Upsilon_{A,E}(\boldsymbol{b}) = \bigcup_{D \in \mathcal{M},\, D \le E} \Delta_{A,D}(\boldsymbol{b}).$$

The observations from the previous paragraph imply that F is a facet of $\Upsilon_{A,E}(\boldsymbol{b})$ if and only if F is a facet of $\Delta_{A,D}(\boldsymbol{b})$ for some $D \in \mathcal{M}$ with $D \le E$. Since $\Delta_A(\boldsymbol{b}) = \Upsilon_{A,\{1,\dots,r_1-1\}}(\boldsymbol{b})$, Theorem 7.3 will be proved once we show that $\Upsilon_{A,E}(\boldsymbol{b})$ is a constructible simplicial complex. This we do by induction on the linearly ordered set $\mathcal{M}$.

Let E_0 be the smallest element of $\mathcal{M}$. Then $\Upsilon_{A,E_0}(\boldsymbol{b}) = \Delta_{A,E_0}(\boldsymbol{b})$ and hence is constructible.

Let $E > E_0$ and assume the claim holds for all $D < E$. Let $E_1, \dots, E_k$ be all the elements of $\mathcal{S}$ that are covered by E and have $\widetilde{E_i} \neq \tilde{E}$. Because $E \in \mathcal{M}$, it is immediate from (7.5) and (7.6) that $E_i \in \mathcal{M}$ for $i = 1, \dots, k$. Let $D \in \mathcal{M}$ with $D < E$, and choose C' as in (7.8); thus we have $\tilde{E} \precneqq \widetilde{C'}$. Since $\widetilde{E_1}, \dots \widetilde{E_k}$ are all possible covers of $\tilde{E}$, we have $\tilde{E} \prec \widetilde{E_j} \preceq \widetilde{C'}$ for some j. Then necessarily $C' \preceq E_j \prec E$, and we obtain from (7.8) that

$$\Delta_{A,E}(\boldsymbol{b}) \cap \Delta_{A,D}(\boldsymbol{b}) \subset \Delta^{(1)}_{A,E_j}(\boldsymbol{b}) * \Delta_{\tilde{E}}(\tilde{\boldsymbol{b}}) = \Delta_{A,E}(\boldsymbol{b}) \cap \Delta_{A,E_j}(\boldsymbol{b}).$$

Let $E_- \in \mathcal{M}$ be the element immediately preceding E. By Lemma 7.4 and Remark 7.2(a), the complex $\Upsilon = \Upsilon_{A,E_-}(\boldsymbol{b}) \cap \Delta_{A,E}(\boldsymbol{b}) = \left(\bigcup_{i=1}^{k} \Delta^{(1)}_{A,E_i}(\boldsymbol{b})\right) * \Delta_{\tilde{E}}(\tilde{\boldsymbol{b}})$ is constructible. Let F be a facet of Υ. Since Υ is pure by Remark 7.2(a), $\dim \Upsilon = \dim F$. Furthermore, F is a facet of $\Delta^{(1)}_{A,E_i}(\boldsymbol{b}) * \Delta_{\tilde{E}}(\tilde{\boldsymbol{b}})$ for some $1 \le i \le k$. Thus F can be completed to a facet of $\Delta_{A,E}(\boldsymbol{b})$ by adding a single vertex of the form $\langle C|E\rangle_1$, and F can be completed to a facet of $\Delta_{A,E_i}(\boldsymbol{b})$ (i.e., to a facet of $\Upsilon_{A,E_-}(\boldsymbol{b})$) by adding the single vertex $\langle \widetilde{E_i}\rangle_1$. Since both $\Delta_{A,E}(\boldsymbol{b})$ and $\Upsilon_{A,E_-}(\boldsymbol{b})$ are constructible (and hence pure), we obtain $\dim \Delta_{A,E}(\boldsymbol{b}) = \dim \Upsilon_{A,E_-}(\boldsymbol{b}) = \dim \Upsilon + 1$. Therefore $\Upsilon_{A,E}(\boldsymbol{b}) = \Upsilon_{A,E_-}(\boldsymbol{b}) \cup \Delta_{A,E}(\boldsymbol{b})$ is constructible, which completes the proof of Theorem 7.3. □

8. Singularities

Our goal in this section is to prove Theorems 2.7 and 2.9. As a main step in the proof of Theorem 2.7, we have the following.

(8.1) THEOREM. *If $\Bbbk$ is a field then $R_{\Bbbk}(\boldsymbol{b})$ is Gorenstein.*

Proof. We set $R = R_{\Bbbk}(\boldsymbol{b})$ and $\mathrm{in}(R) = Q_{\Bbbk}(\boldsymbol{b})/\Sigma(\boldsymbol{b})Q_{\Bbbk}(\boldsymbol{b})$. Since $\Sigma(\boldsymbol{b})$ is generated by square-free monomials, $\mathrm{in}(R)$ is the Stanley–Reisner ring of the simplicial complex $\Delta(\boldsymbol{b})$. It is known that a constructible simplicial complex is Cohen–Macaulay (CM for short) over every field (see e.g. the discussion after [7, Cor. 5.1.14]); therefore, $\mathrm{in}(R)$ is a CM ring by (7.3).

By Theorem 6.1 we have a monomial order on $\mathbb{N}^{V(\boldsymbol{b})}$ such that $\Sigma(\boldsymbol{b}) = \mathrm{in}(I_{\Bbbk}(\boldsymbol{b}))$. Thus, by [2] (see also [17, Thm. 15.17]) there exist a finitely generated $\Bbbk$-algebra $\mathcal{R}$ and a flat $\Bbbk$-algebra homomorphism $\Bbbk[t] \to \mathcal{R}$, together with $\Bbbk[t]$-algebra isomorphisms $\mathcal{R}/t\mathcal{R} \cong \mathrm{in}(R)$ and $\psi : \mathcal{R}_t \cong R \otimes_{\Bbbk} \Bbbk[t, t^{-1}]$, such that the following diagram commutes:

$$\begin{array}{ccccc}
\Bbbk[t] & \longrightarrow & \mathcal{R} & \xrightarrow{\subseteq} & \mathcal{R}_t \\
\Big\| & & \Big\downarrow{\scriptstyle \psi|_{\mathcal{R}}} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \psi} \\
\Bbbk[t] & \longrightarrow & R \otimes_{\Bbbk} \Bbbk[t] & \xrightarrow{\subseteq} & R \otimes_{\Bbbk} \Bbbk[t, t^{-1}].
\end{array}$$

The rings $\mathcal{R}$, R, and $S = R \otimes_{\Bbbk} \Bbbk[t, t^{-1}]$ are homomorphic images of CM rings, so well-known results (see e.g. [31, Sec. 24]) imply that their non-CM loci are Zariski-closed. Furthermore, as the extension $R \to S$ is faithfully flat with regular fibers, S is not CM at a prime $\mathfrak{p}$ if and only if R is not CM at the prime $\mathfrak{p} \cap R$. Thus, if $V(J) \subseteq \mathrm{Spec}(R)$ is the non-CM locus of R, then $V(JS) = V(J[t, t^{-1}]) \subseteq \mathrm{Spec}(S)$ is the non-CM locus of S.

Let $V(\mathcal{I}) \subseteq \mathrm{Spec}(\mathcal{R})$ be the non-CM locus of $\mathcal{R}$, and note that the ring $\mathcal{R}/t\mathcal{R} \cong \mathrm{in}(R)$ is CM. By flatness, t is $\mathcal{R}$-regular and so $\mathcal{R}$ is CM at each prime containing t. Therefore $V(\mathcal{I}\mathcal{R}_t) \subseteq \mathrm{Spec}(\mathcal{R}_t)$ is the non-CM locus of $\mathcal{R}_t$, and $1 = u + vt$ for some $u \in \mathcal{I}$ and $v \in \mathcal{R}$. In particular, for some positive integer m we obtain $\psi(u)^m = (1 - \psi(v)t)^m = 1 + v_1 t + \cdots + v_s t^s \in J[t, t^{-1}]$ with $v_j \in R$ for $j = 1, \ldots, s$. Thus $1 \in J$; hence $V(J) = \varnothing$ and so R is CM. Since by Theorem 2.6 the ring R is factorial, it is Gorenstein by [7, Cor. 3.3.19]. $\square$

The proof of (2.7) is now straightforward.

Proof of Theorem 2.7. By (5.8), the ring $R = R_{\Bbbk}(\boldsymbol{b})$ is a free (and hence a faithfully flat) $\Bbbk$-module, and it remains only to show that the extension $\Bbbk \to R$ has Gorenstein fibers. This, however, is immediate from (8.1). $\square$

Unless specified otherwise, for the rest of this section $\Bbbk$ is a field of characteristic $p > 0$.

Our goal now is to prove Theorem 2.9, and we start by recalling some notions from the theory of tight closure created by Hochster and Huneke [24]. For more details, we refer the reader to the excellent exposition in [28].

Let R be a Noetherian $\Bbbk$-algebra, and let $I = (x_1, \ldots, x_s)$ be an ideal. The *tight closure* I^* of I is the ideal consisting of all elements $x \in R$ for which there exists $c = c(x) \in R$ such that $c \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p}$ and $cx^q \in (x_1^q, \ldots, x_s^q)$ for all $q = p^e \gg 0$. The ideal I is *tightly closed* if $I^* = I$. The ring R is *F-regular* if, for each $\mathfrak{p} \in \mathrm{Spec}(R)$, every ideal of $R_{\mathfrak{p}}$ is tightly closed.

The ring R is *F-rational* if every parameter ideal of R is tightly closed. Here we call an ideal a “parameter ideal” if it is generated by a sequence of elements

$x_1, \dots, x_n$ whose images generate an ideal of height n in any localization $R_{\mathfrak{p}}$ of R such that $(x_1, \dots, x_n) \subseteq \mathfrak{p} \in \operatorname{Spec}(R)$.

A local $\mathbb{k}$-algebra R with maximal ideal $\mathfrak{m}$ is called *F-injective* if the natural action $F\colon H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ of the Frobenius endomorphism of R on the local cohomology modules is injective for all i. When R is Cohen–Macaulay, its F-injectivity is characterized by the property that, for any system of parameters $y_1, \dots, y_d$ of R (where $d = \dim R$) and for any $a \in R$, the condition $a^p \in (y_1^p, \dots, y_d^p)$ implies $a \in (y_1, \dots, y_d)$; see [18, Prop. 1.4].

We need the following modification of a criterion of Fedder and Watanabe [19, Prop. 2.13].

(8.2) LEMMA. *Let R be a local Cohen–Macaulay ring, essentially of finite type over $\mathbb{k}$. If there exists a regular element $z \in R$ such that R_z is F-rational and Gorenstein and such that R/zR is F-injective, then R is F-rational.*

Proof. Because R_z is F-rational and Gorenstein, it follows that R_z is F-regular (see e.g. [28, Thm. 1.5]). By [26, Thm. 6.2], there exists a power z^k of z such that z^k is a completely stable test element of R. Let $I = (z, y_2, \dots, y_d)$ be a system of parameters of R. To show that R is F-rational, it suffices by [19, Prop. 2.2] to show that I is tightly closed. Let $w \in I^*$. Then for $q = p^e \gg 0$ we have $z^k w^q \in (z^q, y_2^q, \dots, y_d^q)$. Since $z^q, y_2^q, \dots, y_d^q$ is a regular sequence, it follows that $w^q \in (z^{q-k}, y_2^q, \dots, y_d^q)$. Going modulo z yields $\tilde{w}^q \in (\tilde{y}_2^q, \dots, \tilde{y}_d^q)R/zR$; hence the F-injectivity of R/zR yields $\tilde{w} \in (\tilde{y}_2, \dots, \tilde{y}_d)$. Lifting back to R gives $w \in (z, y_2, \dots, y_d) = I$. Thus $I = I^*$ and so R is F-rational. □

Let $R = R_{\mathbb{k}}(\boldsymbol{b})$, let $\mathfrak{m} \subset R$ be the maximal ideal of R generated by the entries of $X^{(k)}$ and the elements of M_k $(k = 1, \dots, n)$, and let $z = x^{(n)}_{b_{n-1}b_n}$.

(8.3) LEMMA. *The ring $R_{\mathfrak{m}}/zR_{\mathfrak{m}}$ is F-injective and Cohen–Macaulay.*

Proof. Since R is CM by (2.7) and since z is regular by (3.3), the ring $R_{\mathfrak{m}}/zR_{\mathfrak{m}}$ is CM.

We proceed with the proof of F-injectivity. If $r_n = 1$ then set $u = \langle b_{n-1}\rangle_n \in Q_{\mathbb{k}}(\boldsymbol{b})$; else set $u = \langle b_{n-1}|b_n\rangle_n \in Q_{\mathbb{k}}(\boldsymbol{b})$. We have $R/zR \cong Q_{\mathbb{k}}(\boldsymbol{b})/(u, I_{\mathbb{k}}(\boldsymbol{b}))$. Observe that $u = \operatorname{in}(u)$ does not divide the minimal generators of $\Sigma(\boldsymbol{b}) = \operatorname{in}(I_{\mathbb{k}}(\boldsymbol{b}))$. Therefore, $\operatorname{in}(I_{\mathbb{k}}(\boldsymbol{b}), u) = (\Sigma(\boldsymbol{b}), u)$ and thus

$$\operatorname{in}(R/zR) = Q_{\mathbb{k}}(\boldsymbol{b})/(\Sigma(\boldsymbol{b}), u)$$

is a Stanley–Reisner ring. The proof of [12, Cor. 2.2] shows that a Stanley–Reisner ring is F-injective and thus is $\operatorname{in}(R/zR)$. Furthermore, u is regular on the CM ring $\operatorname{in}(R) = Q_{\mathbb{k}}(\boldsymbol{b})/\Sigma(\boldsymbol{b})$; therefore, $\operatorname{in}(R/zR) \cong \operatorname{in}(R)/(u)$ is also CM. It follows that $R_{\mathfrak{m}}/zR_{\mathfrak{m}}$ is F-injective by [12, Thm. 2.1]. □

Proof of Theorem 2.9. Let $\mathbb{k}$ be a perfect field. Assume first that $\operatorname{char}\mathbb{k} = p > 0$. Since R is Gorenstein, by [28, Thm. 1.5] it suffices to show that R is F-rational. We proceed by induction on $|\boldsymbol{b}|$, as the case $|\boldsymbol{b}| = 1$ is obvious. Note that by (3.4) we have the isomorphism $R_z \cong R_{\mathbb{k}'}(\boldsymbol{b}')$. Since $R_{\mathbb{k}'}(\boldsymbol{b}')$ is a localization of a polynomial extension of the positively graded $\Bbbk$-algebra $R_{\Bbbk}(\boldsymbol{b}')$, it is F-rational by the induction hypothesis and [26, Thm. (4.2)]. The same result shows that the localization $(R_{\mathfrak{m}})_z$ is F-rational. Because $(R_{\mathfrak{m}})_z$ is also Gorenstein and the ring $R_{\mathfrak{m}}/zR_{\mathfrak{m}}$ is F-injective by Lemma 8.3, $R_{\mathfrak{m}}$ is F-rational by Lemma 8.2.

Since $\Bbbk$ is perfect by assumption and since R is a positively graded $\Bbbk$-algebra with irrelevant maximal ideal $\mathfrak{m}$ by Remark 2.4(b), the desired F-rationality of R follows by [25, Thm. 1.4]. This completes the proof of the first part of (2.9).

Assume next that $\operatorname{char} \Bbbk = 0$. Since $R_{\Bbbk}(\boldsymbol{b}) = R_{\mathbb{Z}}(\boldsymbol{b}) \otimes_{\mathbb{Z}} \Bbbk$, it is immediate by the first part of (2.9) that $R_{\Bbbk}(\boldsymbol{b})$ has F-rational type; hence it has rational singularities by [36, Thm. 4.3]. □

9. Examples

Throughout this section, $\Bbbk$ is a field.

Let $(H, \preceq)$ be a partially ordered set of indeterminates, and let $\Bbbk[H]$ be the polynomial ring over $\Bbbk$ whose variables are the elements of H. Let $\Sigma \subseteq \mathbb{N}^H$ be a monomial ideal. A monomial $u \in \mathbb{N}^H$ is called *standard* with respect to Σ if $u \notin \Sigma$. Let $\phi\colon \Bbbk[H] \to A$ be a homomorphism of $\Bbbk$-algebras. In [14] the algebra A is called a *Hodge algebra governed by* Σ *and generated by* $\phi(H)$ if the following two axioms are satisfied.

Hodge-1. The map ϕ sends the set of standard monomials (with respect to Σ) bijectively to a basis of A as a module over $\Bbbk$.

Hodge-2. If $u \in \Sigma$ is a generator and if the unique expression of $\phi(u)$ as a linear combination of images of distinct standard monomials v_i (guaranteed by Hodge-1) has the form

$$\phi(u) = \sum_i r_i \phi(v_i) \quad \text{with } r_i \in \Bbbk \setminus \{0\},$$

then for each $x \in H$ that divides u and for each i there exists a $y_i \in H$ that divides v_i and satisfies $y_i \prec x$.

(9.1) EXAMPLE. In [35, p. 6], Pragacz and Weyman describe a structure of Hodge algebra on the ring $R_{\Bbbk}(\boldsymbol{b})$. The following example shows that the partial order they propose on the indeterminates is not well-defined.

Let $n = 2$ and $\boldsymbol{b} = (b_0, b_1, b_2) = (2, 5, 3)$. Let $K = \{1\}$, $L = \{1\}$, $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{1\}$, and $D = \{2\}$. Then (in the notation and definitions of [35, p. 6]) the minors $(K, L)_1$, $(A, B)_2$, and $(C, D)_2$ satisfy

$$(K, L)_1 < (C, D)_2, \quad (C, D)_2 < (A, B)_2, \quad \text{and} \quad (K, L)_1 \not< (A, B)_2.$$

This violates the transitivity axiom.

The structure of the universal rings is closely related to the structure of the coordinate rings of the varieties of complexes over $\Bbbk$, which have been extensively studied. Kempf [29] determines their main properties in characteristic 0. The papers of De Concini and Strickland [15] and of De Concini, Eisenbud, and Procesi [14] adopt the characteristic-free approach of the theory of Hodge algebras.

In the examples that follow, we demonstrate that the structures proposed in [15] and [14] do not satisfy the axioms for Hodge algebras. This suggests that the theory of Hodge algebras is not suitable for the study of the generic structure of complexes of free modules. A characteristic-free proof of the Cohen–Macaulayness of the varieties of complexes that does *not* use the theory of Hodge algebras is given in [32].

(9.2) EXAMPLE. The following example shows that the construction of De Concini and Strickland [15] does not satisfy the straightening axiom Hodge-2.

Let $X^{(1)} = (x_{ij}^{(1)})$ and $X^{(2)} = (x_{ij}^{(2)})$ be matrices of indeterminates of sizes 2×3 and 3×1, respectively. Let $\Bbbk[X^{(1)}, X^{(2)}]$ be the polynomial ring over $\Bbbk$ with variables the entries of $X^{(1)}$ and $X^{(2)}$, and set $R = \Bbbk[X^{(1)}, X^{(2)}]/I_1(X^{(1)}X^{(2)})$. Thus (in the terminology of [15]), R is the coordinate ring of the variety of complexes W of $\Bbbk$-vector spaces of ranks $n_0 = 2$, $n_1 = 3$, and $n_2 = 1$.

In R we have the relation $x_{11}^{(1)}x_{11}^{(2)} = -x_{12}^{(1)}x_{21}^{(2)} - x_{13}^{(1)}x_{31}^{(2)}$, which in the notation of [15] has the form

$$[1 \mid 2, 3]_1[1 \mid \varnothing]_2 = [1 \mid 1, 3]_1[2 \mid \varnothing]_2 - [1 \mid 1, 2]_1[3 \mid \varnothing]_2.$$

According to the definitions of [15, p. 71], the monomial on the left side of the equality is not standard whereas the two monomials on the right side are standard. In the partial order imposed in [15, p. 71], the minors $[2 \mid \varnothing]_2$ and $[3 \mid \varnothing]_2$ are not comparable to $[1 \mid 2, 3]_1$. Furthermore we have $[1 \mid 2, 3]_1 < [1 \mid 1, 3]_1 < [1 \mid 1, 2]_1$.

This violates the axiom Hodge-2. The displayed relation also violates the straightening condition (4) of [15, p. 71].

(9.3) EXAMPLE. De Concini, Eisenbud, and Procesi propose in [14] a different Hodge algebra structure on the coordinate ring of the variety of complexes. The following example shows that their standard monomials are not linearly independent.

With R as in the previous example and notation as in [14, Sec. 16], consider the monomials

$$M_1 = (1 \mid 2)^{(1)}(1, 2 \mid 1, 3)^{(1)}(2 \mid 1)^{(2)},$$

$$M_2 = (1 \mid 3)^{(1)}(1, 2 \mid 1, 2)^{(1)}(2 \mid 1)^{(2)},$$

$$M_3 = (1 \mid 1)^{(1)}(1, 2 \mid 1, 3)^{(1)}(1 \mid 1)^{(2)}.$$

These are standard monomials in the sense of the definitions of [14, pp. 70–71]. A simple calculation shows that in R one has

$$M_1 - M_2 + M_3 = 0.$$

This violates the axiom Hodge-1.

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