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Estimates for Nonlinear Harmonic "Measures" on Trees

Venancio Alvarez, José M. Rodríguez, & Dmitry V. Yakubovich

1. Introduction

This paper concerns the asymptotic behavior of nonlinear analogs of harmonic "functions" on trees. Our study was motivated by some open problems for p-harmonic functions on domains in $\mathbf{R}^n$. We hope that our results will suggest correct settings for the continuous case.

Fix $\nu \geq 3$, and let the tree T_ν be a regular directed graph. The set V_ν of its vertices is in one-to-one correspondence with finite words in the alphabet $\mathcal{M} = \{1, 2, \dots, \nu\}$. The vertex $v_\emptyset$ is the origin of the tree. The kth generation is

$$G_k = \{v_I : I \in \mathcal{M}^k\},$$

so that

$$V_\nu = \bigcup_{k \geq 0} G_k.$$

The set of *children* of a vertex $v_I \in V_\nu$ is defined as $H_{v_I} = \{v_{I1}, \dots, v_{I\nu}\}$. We denote by $[v, w]$ the edge that links the vertices v and w. We define the set of edges E_ν of the tree T_ν in the following way: the edge $[v, w] \in E_\nu$ if and only if $w \in H_v$. Observe that if $[v, w] \in E_\nu$ then $[w, v] \notin E_\nu$ (T_ν is a directed graph).

Let $F : \bar{\mathbf{R}}_+^\nu \to \bar{\mathbf{R}}_+$ be a continuous function such that $F(0, 0, \dots, 0) = 0$ and $F(1, 1, \dots, 1) = 1$ (here $\bar{\mathbf{R}}_+ := [0, \infty)$ is the positive closed half-axis and $\bar{\mathbf{R}}_+^\nu := (\bar{\mathbf{R}}_+)^\nu$). We say that such a function F is *admissible.* In what follows, we consider only admissible functions. We understand $F(x_1, x_2, \dots, x_\nu)$ as a kind of nonlinear mean of the arguments $x_1, x_2, \dots, x_\nu$.

Let $n \geq 1$, and let ϕ be a function on $G_0 \cup \dots \cup G_n$. We say that ϕ is F-*harmonic* if

$$\phi(v_I) = F\big(\phi(v_{I1}), \phi(v_{I2}), \dots, \phi(v_{I\nu})\big)$$

for any $v_I \in G_0 \cup \dots \cup G_{n-1}$.

If A is a subset of vertices contained in G_n then we define the F-harmonic "measure" of A, denoted by $\omega_F(v, A)$, as the function defined in $G_0 \cup \dots \cup G_n$

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which takes the value 1 on A, the value 0 on $G_n \setminus A$, and is F-harmonic on $G_0 \cup \cdots \cup G_{n-1}$. We denote $\omega_F(A) = \omega_F(v_\emptyset, A)$.

We study the following problems about ω_F.

THE MARTIO PROBLEM. Does the inequality

$$\omega_F(A \cup B) \le k(\omega_F(A) + \omega_F(B)) \tag{1}$$

hold, where $A, B \subset G_n$ and k does not depend on A, B, n?

THE WEAK MARTIO PROBLEM. Does there exist a continuous function $\psi : \bar{\mathbf{R}}_+^2 \to \bar{\mathbf{R}}_+$, nondecreasing in each argument, such that $\psi(0,0) = 0$ and

$$\omega_F(A \cup B) \le \psi(\omega_F(A), \omega_F(B)) \tag{2}$$

for all n and all $A, B \subset G_n$?

In other words, knowing that $\omega_F(A), \omega_F(B)$ are (very) small, can one conclude that $\omega_F(A \cup B)$ also is small?

Obviously, (2) is much weaker than (1).

In this paper we give the answers to both problems, depending on the function F. We also study the corresponding problems for sets A, B that are contained in certain Cantor-type subsets of T_ν. In fact, Martio asked only about the inequality (1) for special functions F_p (defined hereafter).

These certainly are problems of estimating the iterates of F. Namely, define a sequence of functions $\{F^n\}$ in the following way: F^n is a function of ν^n real variables, with

$$\begin{aligned}
F^1(x_1, \dots, x_\nu) &= F(x_1, \dots, x_\nu), \\
F^2(x_1, \dots, x_{\nu^2}) &= F(F^1(x_1, \dots, x_\nu), \dots, F^1(x_{\nu^2-\nu+1}, \dots, x_{\nu^2})), \\
&\ \ \vdots \\
F^n(x_1, \dots, x_{\nu^n}) &= F(F^{n-1}(x_1, \dots, x_{\nu^{n-1}}), \dots, F^{n-1}(x_{\nu^n-\nu^{n-1}+1}, \dots, x_{\nu^n})).
\end{aligned}$$

In G_n there are ν^n vertices. We can sort the vertices in G_n in alphabetical order:

$$v_{1,\dots,1,1} < v_{1,\dots,1,2} < \cdots < v_{\nu,\dots,\nu,\nu-1} < v_{\nu,\dots,\nu,\nu}.$$

For each subset E of G_n, we define $\delta^E \in \{0,1\}^{\nu^n}$ as follows. The i-coordinate of δ^E, denoted by δ_i^E, is 1 if the ith vertex of G_n is in E and 0 if it is not in E. Then we have that $\omega_F(E) = F^n(\delta^E)$.

Let us introduce a special family of functions F_p, $1 < p < \infty$.

NOTATION. Let $\alpha > 0$. In the following, for simplicity we will use the expression t^α to denote the odd extension of the function t^α defined for $t > 0$:

$$t^\alpha = t|t|^{\alpha-1} \quad \text{for } t \in \mathbf{R}. \tag{3}$$

In particular, $t^2 = t|t|$ is negative if t is negative and so it is different from the usual notation. Everywhere in this note we shall use t^α only with the meaning (3) and no other. We trust this will not lead to any confusion.

With this notation, define $F_p\colon \mathbf{R}^\nu \to \mathbf{R}$ by the implicit rule

$$F_p(a_1, a_2, \ldots, a_\nu) = x \quad \text{if } (x - a_1)^{p-1} + (x - a_2)^{p-1} + \cdots + (x - a_\nu)^{p-1} = 0. \tag{4}$$

The F_p-harmonic functions will be called p-harmonic functions, and the corresponding harmonic measure will be denoted by ω_p.

Elementary properties of p-harmonic functions give that

(a) $\omega_p(\emptyset) = 0$,
(b) $\omega_p(G_n) = 1$, and
(c) $\omega_p(G_n \setminus A) = 1 - \omega_p(A)$, for every $A \subset G_n$.

If $\nu = 1$ or $\nu = 2$, the framework degenerates and every p-harmonic function on a graph is harmonic $(p > 1)$. In the following we consider the case $\nu \geq 3$.

These concepts on graphs have important connections with potential theory on Riemannian manifolds (see e.g. [CFPR; FR; HS; K1; K2; K3; R1; R2; S]).

The main inspiration for our work are p-harmonic functions, on domains in $\mathbf{R}^n$, whose discrete analogs are p-harmonic functions on oriented graphs. A function u on a domain Ω in $\mathbf{R}^n$ is called p-harmonic $(1 < p < \infty)$ if the partial differential equation

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{5}$$

holds in Ω; this equation must be understood in a weak sense (see [HKM, p. 57]). Obviously, 2-harmonic functions are harmonic. Note that p-harmonic functions are not a linear space if $p \neq 2$, but they have many properties that are similar to those of harmonic functions. For instance, they have a comparison principle: If u, v are p-harmonic functions in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω [HKM, p. 133]. It is possible to construct a potential theory for equation (5) because the main tool for developing such theory is the comparison principle [HKM].

There are many reasons to study p-harmonic functions. For instance, if $p \neq 2$ then (5) is a simple example of nonlinear degenerate elliptic equation. Observe that (5) is the Euler equation for the functional

$$J(u) = \int_\Omega |\nabla u(x)|^p \, dx,$$

which is an elementary functional with nonquadratic growing if $p \neq 2$. As a consequence, p-harmonic functions are functions with extremal properties in the Sobolev space $W^{1,p}(\Omega)$.

Moreover, if $p = n$ then p-harmonic functions play an important role in the theory of quasiconformal and quasiregular mappings.

Roughly speaking, we can define the p-harmonic "measure" of the Borel subset $E \subset \partial\Omega$ at a point $x \in \Omega$ as the p-harmonic function in Ω that takes value 1 in E and value 0 in $\partial\Omega \setminus E$, evaluated in x. See [HKM, Chap. 11] for a rigorous definition. Harmonic measure is a main tool in linear potential theory. An important property of harmonic measure is its additivity. If $p \neq 2$ then p-harmonic measure does not have this property, that is, it is not a measure. In spite of this, p-harmonic measure plays an important role in nonlinear potential theory.

OPEN PROBLEM. Is the p-harmonic measure subadditive? That is, does the inequality $\omega(A \cup B) \le k(\omega(A) + \omega(B))$ hold for all Borel subsets $A, B \subset \partial\Omega$ for some constant k? This is an open problem for every domain Ω, even if Ω is the unit ball of $\mathbf{R}^n$ $(n \ge 2)$. (We refer to [B] for some information in the case of the unit disk.)

In view of the difficulty of this problem, Martio asked whether its analog is satisfied for the p-harmonic measure on regular trees. We remark that regular trees are suitable models for the balls in Euclidean spaces.

Let us return to the discrete setting of graphs. In what follows, we will consider admissible functions F satisfying some of the following properties:

- **(i)** $F(x, x, \dots, x) = x$, $x \ge 0$;
- **(ii)** F is nondecreasing with respect to each argument and $F(x_1, x_2, \dots, x_\nu) > 0$ if we have that $(x_1, x_2, \dots, x_\nu) \ne \mathbf{0}$;
- **(iii)** $F(x_1, x_2, \dots, x_\nu) = F(x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(\nu)})$ for any rearrangement τ of the set $\{1, 2, \dots, \nu\}$;
- **(iv)** $F(tx_1, tx_2, \dots, tx_\nu) = tF(x_1, x_2, \dots, x_\nu)$ for $x_1, x_2, \dots, x_\nu, t \in \bar{\mathbf{R}}_+$;
- **(v)** $F(x_1, x_2, \dots, x_\nu) < \max(x_1, x_2, \dots, x_\nu)$ if we do not have $x_1 = x_2 = \dots = x_\nu$;
- **(vi)** F can be defined on the whole $\mathbf{R}^\nu$, and this verifies $F(t + x_1, t + x_2, \dots, t + x_\nu) = t + F(x_1, x_2, \dots, x_\nu)$ for all $x_1, x_2, \dots, x_\nu, t \in \mathbf{R}$;
- **(vii)** $F(1 - x_1, 1 - x_2, \dots, 1 - x_\nu) = 1 - F(x_1, x_2, \dots, x_\nu)$ for $x_1, x_2, \dots, x_\nu \in [0, 1]$.

It is obvious that any admissible function satisfying (**iv**) also satisfies (**i**). If F is strictly increasing with respect to each argument, then (**v**) follows from (**i**). If F is admissible and satisfies (**iv**) and (**vi**) for all $x_1, x_2, \dots, x_\nu, t \in \mathbf{R}$, then it satisfies condition (**vii**).

For each admissible F, ω_F satisfies properties (a) and (b) of ω_p. If F satisfies (**ii**) and (**vii**), then ω_F also satisfies (c).

The function F_p has all properties (**i**)–(**vii**) and is strictly increasing with respect to each argument. In general, we do not assume that all conditions (**i**)–(**vii**) hold. We remark that one can define p-harmonic functions on trees very similarly to the definition (5).

Two subsets A, B of G_n will be called *congruent* if there is an isomorphism of the graph $G_0 \cup G_1 \cup \dots \cup G_n$ onto itself that leaves each G_k invariant and maps A onto B. For such sets, obviously, $\omega_F(A) = \omega_F(B)$ for all admissible functions F satisfying (**iii**).

Theorems 1 and 2 are the key results. They provide conditions to give a negative and a positive answer, respectively, to the weak Martio problem.

From now on, we consider the case $\nu = 3$ in order to simplify notation and the proofs of Theorems 1 and 2; however, we remark that these results are true for any $\nu \ge 3$. Later we will comment on the case of general ν-regular trees (see Remark 3).

THEOREM 1. *Suppose that F satisfies* (**ii**)–(**iv**) *and that* $F(a_0, b_0, c_0) < \sqrt[3]{a_0 b_0 c_0}$ *for some positive* a_0, b_0, c_0. *Then, for every* $n > 0$, *there exist congruent subsets* $B_n^{(0)}, B_n^{(1)}, B_n^{(2)}$ *of* G_{3n} *such that* $G_{3n} = B_n^{(0)} \cup B_n^{(1)} \cup B_n^{(2)}$ *and* $\omega_F(B_n^{(0)}) \to 0$ *as* $n \to \infty$.

It follows that if F satisfies the hypotheses of Theorem 1, then the answers to the weak Martio problem and to the Martio problem are negative. Indeed, (2) and Theorem 1 would imply

$$1 = \omega_F(B_n^{(0)} \cup B_n^{(1)} \cup B_n^{(2)}) \le \psi\big(\omega_F(B_n^{(0)}), \psi(\omega_F(B_n^{(1)}), \omega_F(B_n^{(2)}))\big) \to 0$$

as $n \to \infty$.

COROLLARY 1. *The answer to the weak Martio problem is negative for the p-harmonic measure for all $p \neq 2$.*

Proof. Indeed, put $a_0 = b_0 = 1$ and $c_0 = u^3$. Then

$$(u - a_0)^{p-1} + (u - b_0)^{p-1} + (u - c_0)^{p-1} = (u - 1)^{p-1}(2 - (u + u^2)^{p-1}).$$

For each $p \neq 2$ there exists a u (close to 1) such that $(u - a_0)^{p-1} + (u - b_0)^{p-1} + (u - c_0)^{p-1} > 0$, which gives $F(a_0, b_0, c_0) < u = \sqrt[3]{a_0 b_0 c_0}$ (if $1 < p < 2$ then it is enough to take $u = 1 + \varepsilon$; if $p > 2$, we can take $u = 1 - \varepsilon$ for $\varepsilon = \varepsilon(p)$ small enough). □

Denote by R_3 the triangle $R_3 = \{(x, y, z) \in \bar{\mathbf{R}}^3_+ : x + y + z = 1\}$ and by $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ its center. Let dist denote the usual Euclidean distance.

THEOREM 2. *Suppose that F satisfies* (**iv**) *and* (**v**).

(a) *If for some* $\varepsilon > 0$ *we have* $F(x, y, z) \ge \sqrt[3]{xyz} + \varepsilon\, \mathrm{dist}((x, y, z), q)^2$ *for all* $(x, y, z) \in R_3$, *then there exists an* $N > 0$ *such that* $\omega_F \ge \omega_2^N$.

(b) *If for some* $C > 0$ *we have* $F(x, y, z) \le \sqrt[3]{xyz} + C\, \mathrm{dist}((x, y, z), q)^2$ *for all* $(x, y, z) \in R_3$, *then there exists an* $M > 0$ *such that* $\omega_F \le \omega_2^M$.

Consequently, if

$$\sqrt[3]{xyz} + \varepsilon\, \mathrm{dist}((x, y, z), q)^2 \le F(x, y, z) \le \sqrt[3]{xyz} + C\, \mathrm{dist}((x, y, z), q)^2 \tag{6}$$

on R_3 for some $\varepsilon, C > 0$, then there exist positive constants M, N such that $\omega_2^N \le \omega_F \le \omega_2^M$. In particular, the answer to the weak Martio problem is positive, because

$$\omega_F(A \cup B) \le (\omega_F(A)^{1/N} + \omega_F(B)^{1/N})^M$$

for all sets of vertices A and B.

Observe that (6) and the homogeneity (**iv**) of F in fact imply certain estimates for F on the whole $\bar{\mathbf{R}}^3_+$. In Section 2 we discuss the central role of the goemetric average in Theorems 1 and 2.

REMARKS. 1. Suppose that F is twice differentiable in q, $F(q) \le \frac{1}{3}$, and (**iii**) holds. Then (**iii**) implies that the values of $\frac{\partial F}{\partial x_i}(q)$ are the same for $i = 1, 2, 3$.

Hence the Taylor formula gives that the right-hand inequality in (6) is satisfied on R_3 in a neighborhood of q and thus on the whole R_3 (for a sufficiently large C).

2. If (**iv**) and (**vi**) hold and $F(x,y,z) \ge \sqrt[3]{xyz}$ on $\bar{\mathbf{R}}^3_+$, then $F(x,y,z) = \frac{x+y+z}{3}$ on $\bar{\mathbf{R}}^3_+$. Indeed, put $s = \frac{x+y+z}{3}$, $x = s+\tilde{x}$, $y = s+\tilde{y}$, and $z = s+\tilde{z}$. Then for fixed x,y,z there exist $\varepsilon = \varepsilon(x,y,z) > 0$ and $\delta = \delta(x,y,z) > 0$ such that

$$\begin{aligned} s + tF(\tilde{x},\tilde{y},\tilde{z}) &= F(s+t\tilde{x}, s+t\tilde{y}, s+t\tilde{z}) \\ &\ge \sqrt[3]{(s+t\tilde{x})(s+t\tilde{y})(s+t\tilde{z})} \ge s - \varepsilon|t|^2 \end{aligned}$$

for all $t \in (-\delta,\delta)$. Then $tF(\tilde{x},\tilde{y},\tilde{z}) \ge -\varepsilon|t|^2$ for all $t \in (-\delta,\delta)$; if $t \in (0,\delta)$ we obtain $F(\tilde{x},\tilde{y},\tilde{z}) \ge 0$, and if $t \in (-\delta,0)$ we deduce $F(\tilde{x},\tilde{y},\tilde{z}) \le 0$. Hence $F(\tilde{x},\tilde{y},\tilde{z}) = 0$ and so we deduce that $F(x,y,z) = s$. It follows that (**iv**), (**vi**), and $F(x,y,z) \ge \sqrt[3]{xyz}$ trivially imply $\omega_F = \omega_2$. (Note that this gives an alternative proof of Corollary 1.)

In the sequel we will apply Theorems 1 and 2 to Cantor subsets of G_n; we define these sets following Theorem 4 and explain there why we call them Cantor sets. This will lead to a use of functions F that do not satisfy (**vi**).

3. Analogs of Theorems 1 and 2 hold true for ν-regular trees for any $\nu \ge 3$. Simply replace R_3 with $R_\nu = \{x \in \bar{\mathbf{R}}^\nu_+ : x_1 + \cdots + x_\nu = 1\}$, replace $\sqrt[3]{xyz}$ with $\sqrt[\nu]{x_1 x_2 \dots x_\nu}$, and put $q = (\nu^{-1}, \nu^{-1}, \dots, \nu^{-1}) \in \mathbf{R}^\nu$. We will explain in Section 2 how to change the proofs in order to cover the general case.

As a corollary of Theorem 2 and Remark 1 we obtain the following result, since $F(q) = \frac{1}{3}$ is a consequence of (**iv**).

COROLLARY 2. *Suppose that F is twice differentiable in q and that* (**iii**)–(**v**) *hold. Then there exists an $M > 0$ such that $\omega_F \le \omega_2^M$.*

In order to state the following results, we need some additional definitions.

We define the set of *descendants* of a vertex v, denoted by D_v, as follows:

(a) v is a descendant of v;
(b) if $w \ne v$, then w is a descendant of v if and only if $w \in H_q$ and q is a descendant of v.

If $A \subset G_n$ and $A' \subset G_{n'}$ with $n < n'$, we say that A and A' are *equivalent sets* if A' is the set of all descendants of the vertices in A that are in $G_{n'}$—that is, $A' = \big(\bigcup_{v \in A} D_v\big) \cap G_{n'}$. If A and A' are equivalent then $\omega_F(A) = \omega_F(A')$ for every admissible function F. In the sequel we identify equivalent sets, and then we can write $A' \subset G_n$ and $A \subset G_{n'}$.

Theorems 3–7 can be understood as a study of smoothness properties of the nonlinear measure ω_F.

THEOREM 3. *Consider a fixed set $E \subset G_n$. For each $\nu \ge 3$ and admissible function F satisfying* (**ii**) *and* (**vii**), *there is a positive constant k that depends only on ν, F, and E such that*

$$\omega_F(A \cup B) \le k(\omega_F(A) + \omega_F(B)) \quad \textit{for all } A, B \subset G_r \textit{ with } A \cup B = E \tag{7}$$

for any natural number $r \ge n$.

We denote by $k(E)$ the sharp constant in Theorem 3.

The next corollary gives a partial positive result about subadditivity.

COROLLARY 3. *Consider a fixed natural number n. For each $\nu \geq 3$ and admissible function F satisfying* (**ii**) *and* (**vii**), *there is a positive constant k_n that depends only on ν, F, and n such that*

$$\omega_F(A \cup B) \leq k_n(\omega_F(A) + \omega_F(B)) \quad \textit{for all } A, B \subset G_r \textit{ with } A \cup B \subset G_n$$

for any natural number $r \geq n$.

This corollary gives a partial positive result for the Martio problem. To derive it from Theorem 3, it suffices to set k_n as the maximum of $k(E)$ for $E \subset G_n$.

The next results concern the following question: Given fixed sets $H_n \subset G_n$ with $\omega_F(H_n) \to 0$ as $n \to \infty$, does there exist some ψ verifying $\omega_F(A \cup B) \leq \psi(\omega_F(A), \omega_F(B))$ for all n and all $A, B \subset H_n$?

First we remark that, as so stated, weak Martio inequality (2) is always true in this situation. Indeed, let I_n be a one-point set in G_n. Take any continuous function $\psi(x, y)$ that is increasing with respect to x and y and satisfies $\psi(0, 0) = 0$ and $\psi(0, \omega_F(I_n)) = \psi(\omega_F(I_n), 0) \geq \omega_F(H_n)$ for every natural number n; then (2) trivially holds for this ψ if $A, B \subset H_n$. Instead of (2), we will study the "intermediate" Martio inequality

$$\frac{\omega_F(A \cup B)}{\omega_F(H_n)} \leq \psi\left(\frac{\omega_F(A)}{\omega_F(H_n)}, \frac{\omega_F(B)}{\omega_F(H_n)}\right) \tag{8}$$

for a special class of H_n.

First we need the definition of the product of two sets of vertices. Given $D \subset G_r$ and $E \subset G_s$, we put

$$D \times E = \{v_{IJ} : v_I \in D,\ v_J \in E\} \subset G_{r+s},$$

where IJ is the vector $(i_1, \ldots, i_a, j_1, \ldots, j_b)$, if $I = (i_1, \ldots, i_a)$ and $J = (j_1, \ldots, j_b)$. This product satisfies the distributive laws with respect to the union and the intersection of sets:

$$A \times (B \cup C) = (A \times B) \cup (A \times C), \qquad A \times (B \cap C) = (A \times B) \cap (A \times C).$$

We have $\omega_F(D \times E) = \omega_F(D)\omega_F(E)$ for every admissible function F and sets D and E.

THEOREM 4. *If $\nu \geq 3$ and if F is an admissible function that satisfies* (**ii**), (**iv**), (**v**), *and* (**vii**), *then*

$$k(D \times E) \leq k(D)k(E) \quad \textit{for every } D, E.$$

Let $2 \leq \mu < \nu$, and put $C_{\mu,1}$ to be any fixed subset of G_1 with μ points. We define the Cantor subset $C_{\mu,n}$ of G_n by

$$C_{\mu,n} = \underbrace{C_{\mu,1} \times \cdots \times C_{\mu,1}}_{n}.$$

Obviously, $\omega_F(C_{\mu,n}) = \omega_F(C_{\mu,1})^n \to 0$ as $n \to \infty$. We will study (8) for $H_n = C_{\mu,n}$.

We use the word Cantor to denote the set $C_{\mu,n}$, since the set $C_\mu = \bigcap_n C_{\mu,n}$ contained in the boundary of T_ν (see [GH, Chap. 6] for the definition of the boundary of a tree) is homeomorphic to a Cantor set in the real line.

THEOREM 5. *For each function F_p $(p > 2)$ and each $\nu \geq 3$, we have*

$$k(C_{2,n}) = 1 \quad \text{for all } n,$$

so that the Martio inequality (1) *holds if $A \cup B = C_{2,n}$.*

Theorem 5 is not true for $\nu \geq 3$ and $1 < p < 2$ (see Lemma 6).

COROLLARY 4. *For all $\nu \geq 3$, $n \geq 1$, and all E:*

(i) $k(G_n \times E) = k(E)$ *if* $p > 1$;
(ii) $k(C_{2,n} \times E) \leq k(E)$ *if* $p > 2$.

COROLLARY 5. *Consider a fixed natural number n. For each $\nu \geq 3$ and $p > 1$, there is a positive constant k_n (the same constant as in Corollary 3) depending only on ν, p, and n such that*

$$\omega_p(A \cup B) \leq k_n(\omega_p(A) + \omega_p(B))$$

for all sets A, B satisfying any of the following conditions:

(i) $A \cup B \subset G_n$;
(ii) $A \cup B = G_r \times D$, *with r a natural number and* $D \subset G_n$;
(iii) $A \cup B = G_{r_1} \times C_{2,s_1} \times \cdots \times G_{r_q} \times C_{2,s_q} \times D$, *with $r_1, \dots, r_q, s_1, \dots, s_q$ natural numbers and $D \subset G_n$, if $p > 2$.*

Put $\sigma = F(\underbrace{1, \dots, 1}_{\mu}, 0, \dots, 0)$ and $\tilde{F}(x_1, \dots, x_\mu) = \sigma^{-1}F(x_1, \dots, x_\mu, 0, \dots, 0)$. It is plain that, for $A \subset C_{\mu,n}$, $\omega_F(A)/\omega_F(C_{\mu,n}) = \omega_{\tilde{F}}(A)$. If F satisfies (**i**)–(**iv**), then $\tilde{F}$ also satisfies these properties. If F is strictly increasing in each variable and satisfies (**iv**), then $\tilde{F}$ satisfies (**v**).

THEOREM 6. *Let $2 \leq \mu < \nu$ and $H_n = C_{\mu,n}$.*

(a) *If F satisfies* (**ii**)–(**iv**) *and*

$$F(x_1, \dots, x_\mu, 0, \dots, 0) < \sigma \sqrt[\mu]{x_1 x_2 \dots x_\mu}$$

for some $x_1, \dots, x_\mu \in \bar{\mathbf{R}}_+$, then the intermediate Martio inequality (8) *does not hold.*

(b) *Suppose that F is strictly increasing in each variable and is twice continuously differentiable in $(\underbrace{1, \dots, 1}_{\mu}, 0, \dots, 0)$. If F satisfies* (**iii**) *and* (**iv**) *and if there is an $\varepsilon > 0$ such that*

$$F(x_1, \dots, x_\mu, 0, \dots, 0) \geq \sigma \sqrt[\mu]{x_1 x_2 \dots x_\mu} + \varepsilon \sum_{j=1}^{\mu} (x_j - 1/\mu)^2 \tag{9}$$

for all $(x_1, \dots, x_\mu) \in R_\mu$*, then there exist* $C, \rho > 0$ *such that*

$$\frac{\omega_F(A \cup B)}{\omega_F(C_{\mu,n})} \le C\left(\frac{\omega_F(A) + \omega_F(B)}{\omega_F(C_{\mu,n})}\right)^\rho \tag{10}$$

for all n *and all* $A, B \subset C_{\mu,n}$*.*

THEOREM 7. (a) *Let* $2 \le \mu < \nu$*. Then there are positive* δ, C, ρ *such that* (10) *holds for* $F = F_p$ *for all* $p \in (2 - \delta, 2 + \delta)$*.*

(b) *If* $\mu = 2$ *then there is a* $\delta > 0$ *such that, for any* $p \in (2 - \delta, \infty)$*, there exist* $C > 0$ *and* $\rho > 0$ *such that* (10) *holds for* $F = F_p$*.*

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2. Proofs of Theorems 1 and 2

We denote by ∂R_ν the relative boundary of R_ν as a subset of the affine plane $P = \{x \in \mathbf{R}^\nu : x_1 + \dots + x_\nu = 1\}$. Put $S(x_1, x_2, \dots, x_{\nu-1}, x_\nu) = (x_2, x_3, \dots, x_\nu, x_1)$. Then S is an orthogonal linear transformation of $\mathbf{R}^\nu$ such that $S^\nu = I$. First we consider the case $\nu = 3$.

Let us make the following observation. Let $k \in \mathbb{N}$. Each set $E \subset G_k$ can be represented in a unique way in the form

$$E = \{v_{1I} : v_I \in X\} \cup \{v_{2I} : v_I \in Y\} \cup \{v_{3I} : v_I \in Z\}$$

for some subsets X, Y, Z of G_{k-1}. With the last identity in mind, we will write $E = (X, Y, Z)$; then $\omega_F(E) = F(\omega_F(X), \omega_F(Y), \omega_F(Z))$.

We have a formula

$$(X_1, Y_1, Z_1) \cup (X_2, Y_2, Z_2) = (X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2). \tag{11}$$

Proof of Theorem 1. Let $a_0, b_0, c_0 \in (0, +\infty)$ and $F(a_0, b_0, c_0) < \sqrt[3]{a_0 b_0 c_0}$. The key of this proof is to transform this inequality involving the geometric average in the inequality (17) for some sets $B^{(0)}_{k,l,m}$; the intuition to choose the appropriate sets was inspired by some numerical simulations. By property (**iv**) of F, we can assume that $a_0 b_0 c_0 = 1$. There are closed subsets A_0, $A_1 = SA_0$, and $A_2 = S^2 A_0$ of ∂R_3 such that $A_0 \cup A_1 \cup A_2 = \partial R_3$ and

$$A_0 \subset \{(x, y, z) \in P : x \ln a_0 + y \ln b_0 + z \ln c_0 < 0\}. \tag{12}$$

Indeed, divide P into three equal angles $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ of size $\frac{2\pi}{3}$ with common vertex at q and put $A_j = \mathfrak{A}_j \cap \partial R_3$. The boundary of the half-plane in P involved in (12) contains q. From this, one sees that there is a position of $\mathfrak{A}_j$ that works.

Choose a, b, c slightly larger than a_0, b_0, c_0 (respectively), so that $abc > 1$ but $F(a, b, c) < 1$. We can also still assume that

$$x \ln a + y \ln b + z \ln c < 0 \quad \text{for } (x, y, z) \in A_0. \tag{13}$$

We define $B^{(j)}_{k,l,m}$ subsets of G_{k+l+m} for $j = 0,1,2$ and $k,l,m \in \mathbb{Z}_+$, $k+l+m > 0$, by induction on $n = k+l+m$. The inductive rule is

$$B^{(j)}_{k,l,m} = \big(B^{(j)}_{k-1,l,m}, B^{(j)}_{k,l-1,m}, B^{(j)}_{k,l,m-1}\big) \quad \text{for } k,l,m \ge 1, \tag{14}$$

and it does not depend on j. The "boundary conditions" are

$$B^{(j)}_{k,l,m} = \begin{cases} G_n & \text{if } \big(\frac{k}{n}, \frac{l}{n}, \frac{m}{n}\big) \in A_j, \\ \emptyset & \text{if } \big(\frac{k}{n}, \frac{l}{n}, \frac{m}{n}\big) \in \partial R_3 \setminus A_j; \end{cases} \tag{15}$$

here $n = k+l+m \ge 1$ (k,l,m are natural numbers) and $klm = 0$. This definition is consistent, and by induction on $n = k+l+m$—using (15) (for $n = 1$), (11), and (14)—it is proved that

$$B^{(0)}_{k,l,m} \cup B^{(1)}_{k,l,m} \cup B^{(2)}_{k,l,m} = G_{k+l+m} \tag{16}$$

for all $k,l,m \ge 0$ with $k+l+m \ge 1$.

Next let us apply induction again on $n = k+l+m$ to prove that

$$\omega_F\big(B^{(0)}_{k,l,m}\big) < a^{-k}b^{-l}c^{-m}. \tag{17}$$

If $klm = 0$, then $B^{(0)}_{k,l,m}$ has been formed by the rule (15), and we may assume that $\big(\frac{k}{n}, \frac{l}{n}, \frac{m}{n}\big) \in A_0$ (otherwise $B^{(0)}_{k,l,m} = \emptyset$ and (17) is true). Then $\omega_F\big(B^{(0)}_{k,l,m}\big) = \omega_F(G_n) = 1$, and (17) follows from (13). If $klm > 0$, then the induction hypothesis yields

$$\begin{aligned} \omega_F\big(B^{(0)}_{k,l,m}\big) &= F\big(\omega_F\big(B^{(0)}_{k-1,l,m}\big), \omega_F\big(B^{(0)}_{k,l-1,m}\big), \omega_F\big(B^{(0)}_{k,l,m-1}\big)\big) \\ &\le F(a^{-k+1}b^{-l}c^{-m}, a^{-k}b^{-l+1}c^{-m}, a^{-k}b^{-l}c^{-m+1}) \\ &= a^{-k}b^{-l}c^{-m}F(a,b,c) < a^{-k}b^{-l}c^{-m}. \end{aligned}$$

Now put $B^{(j)}_n = B^{(j)}_{n,n,n}$. Since $SA_0 = A_1$ and $S^2A_0 = A_2$, the sets $B^{(0)}_n$, $B^{(1)}_n$, $B^{(2)}_n$ are congruent. Next, (16) gives $B^{(0)}_n \cup B^{(1)}_n \cup B^{(2)}_n = G_{3n}$. Since $abc > 1$, (17) implies that $\omega_F(B^{(0)}_n) \to 0$ as $n \to \infty$. $\square$

We put

$$G_\alpha(x,y,z) = \left(\frac{x^\alpha + y^\alpha + z^\alpha}{3}\right)^{1/\alpha}, \quad (x,y,z) \in \bar{\mathbf{R}}^3_+.$$

The geometric average plays an important role in Theorem 2. On the one hand, its Taylor formula is quite similar to the Taylor formula of G_α (see (18)); on the other hand, computations involving G_α are quite simple.

LEMMA 1. *The hypothesis* (a) *of Theorem 2 implies that, for some* $\alpha > 0$,

$$G_\alpha(x,y,z) \le F(x,y,z), \quad (x,y,z) \in \bar{\mathbf{R}}^3_+.$$

The hypothesis (b) *of Theorem 2 implies that, for some* $\beta > 0$,

$$F(x,y,z) \le G_\beta(x,y,z), \quad (x,y,z) \in \bar{\mathbf{R}}^3_+.$$

Proof. $G_\alpha(x, y, z)$ is an increasing function of α for $0 < \alpha < \infty$ [HLP, Chap. 2].

Let the hypothesis (a) of Theorem 2 hold. The Taylor formula gives

$$
\begin{aligned}
G_\alpha(x, y, z) &= \frac{1}{3} + \left(\frac{\alpha - 1}{2} + o(1)\right) \operatorname{dist}((x, y, z), q)^2, \\
\sqrt[3]{xyz} &= \frac{1}{3} + \left(-\frac{1}{2} + o(1)\right) \operatorname{dist}((x, y, z), q)^2
\end{aligned}
\tag{18}
$$

if $x + y + z = 1$, $(x, y, z) \to q$. Hence there is an $\alpha_0 > 0$ and an open disc $\mathcal{U}$ in the plane P, centered in q, such that $G_{\alpha_0}(x, y, z) \le F(x, y, z)$ if $(x, y, z) \in \mathcal{U}$. Therefore, $G_\alpha(x, y, z) \le F(x, y, z)$ in $\mathcal{U}$ for all $\alpha \in (0, \alpha_0]$.

Let r be the radius of $\mathcal{U}$. For fixed x, y, z, we have $G_\alpha(x, y, z) \to \sqrt[3]{xyz}$ as $\alpha \to 0$ [HLP, Chap. 2]. By the Dini theorem [Ru, Thm. 7.13], this convergence is uniform for $(x, y, z) \in R_3$. Hence there exists an $\alpha \in (0, \alpha_0]$ such that

$$G_\alpha(x, y, z) < \sqrt[3]{xyz} + \varepsilon r^2 \le F(x, y, z)$$

for $(x, y, z) \in R_3 \setminus \mathcal{U}$. We conclude by (**iv**) that $F(x, y, z) \ge G_\alpha(x, y, z)$ for all $(x, y, z) \in \bar{\mathbf{R}}^3_+$.

If hypothesis (b) holds, then (18) yields that there is a disc $\mathcal{U}$ as before and some large β_0 such that $F(x, y, z) \le G_{\beta_0}(x, y, z)$ in $\mathcal{U}$. It follows from property (**v**) of F that $F(x, y, z) \le \max(x, y, z) - \delta$ on $R_3 \setminus \mathcal{U}$ for some $\delta > 0$. Since $G_\alpha(x, y, z) \to \max(x, y, z)$ uniformly on R_3 as $\alpha \to \infty$, it follows that $F(x, y, z) \le G_\beta(x, y, z)$ on R_3 for some $\beta \ge \beta_0$, and we are done. $\square$

Proof of Theorem 2. An obvious induction argument shows that, if either F or G satisfies (**ii**), then $F \le G$ implies that $\omega_F(X) \le \omega_G(X)$ for all sets X. It is also plain to see that $\omega_{G_\alpha}(X) = \omega_2(X)^{1/\alpha}$ for all sets X. Thus we obtain from Lemma 1 that $\omega_F \ge \omega_2^{1/\alpha}$ if (a) is assumed, and $\omega_F \le \omega_2^{1/\beta}$ if (b) is assumed. These inequalities imply that

$$\omega_F(A \cup B) \le \omega_2(A \cup B)^{1/\beta} \le (\omega_2(A) + \omega_2(B))^{1/\beta} \le (\omega_F(A)^\alpha + \omega_F(B)^\alpha)^{1/\beta},$$

so that the weak Martio inequality (2) holds. $\square$

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LEMMA 2. *There is a dense subset $\mathcal{D}$ of the plane $P_0 = \{x \in \mathbf{R}^\nu : x_1 + \cdots + x_\nu = 0\}$ such that, for every $r \in \mathcal{D}$, vectors $r, Sr, \ldots, S^{\nu-1}r$ span P_0.*

Proof. There is at least one such vector, namely $r = (1, -1, 0, 0, \ldots, 0)$. The determinant criterion of linear dependence shows that the property in question can fail only on an algebraic submanifold of P_0 of codimension 1. $\square$

LEMMA 3. *Let $r \in \mathcal{D}$. Then the sets*

$$A_j = \left\{x \in \partial R_\nu : \langle x - q, S^j r\rangle = \min_{0 \le k \le \nu-1} \langle x - q, S^k r\rangle\right\}$$

are closed and satisfy $S^jA_0 = A_j$ *for* $j = 1, 2, \dots, \nu - 1$. *Moreover,* $A_0 \cup A_1 \cup \dots \cup A_{\nu-1} = \partial R_\nu$ *and*

$$\langle x - q, r \rangle < 0 \quad \text{for } x \in A_0. \tag{19}$$

We remark that, in fact, $\langle q, S^jr \rangle = 0$ for all j and all $r \in P_0$.

Proof. Let us prove (19) (all other properties are plain). Since vectors S^jr span P_0, we have that $\sum_{j=0}^{\nu-1} |\langle x - q, S^jr \rangle| \neq 0$ for $x \in \partial R_\nu$. Suppose $x \in A_0$, and put $t_j = \langle x - q, S^jr \rangle$. Then $\sum_{j=0}^{\nu-1} |t_j| \neq 0$, $\sum_{j=0}^{\nu-1} t_j = \langle x - q, 0 \rangle = 0$, and $t_0 \leq t_j$ for $1 \leq j \leq \nu - 1$. These three facts imply $t_0 < 0$. □

Let $F(x_1, \dots, x_\nu)$ be an admissible function that satisfies properties (**ii**)–(**v**), and consider the corresponding measure ω_F over T_ν. Lemmas 2 and 3 allow one to repeat the construction of Theorem 1 and so obtain congruent sets $B_n^{(0)}, B_n^{(1)}, \dots, B_n^{(\nu-1)}$ of $G_{\nu n}$ whose union is $G_{\nu n}$ and such that $\omega_F(B_n^{(0)}) \to 0$ as $n \to \infty$. One need only choose $(a_1^0, a_2^0, \dots, a_\nu^0)$, which now plays the role of (a_0, b_0, c_0), so that $(\ln a_1^0, \ln a_2^0, \dots, \ln a_\nu^0) \in \mathcal{D}$. The rest of the proof follows the same lines. The proof of the analog of Theorem 2 for ν-regular trees requires no alterations.

We remark that the answer to the weak Martio problem for p-harmonic measure on ν-regular trees still is negative for $p \neq 2$. Indeed, if $p \neq 2$, then there is u close to 1 such that $F(1, 1, \dots, 1, u^\nu) < u = \sqrt[\nu]{u^\nu}$.

3. Proofs of Theorems 3 and 4

LEMMA 4. *If* (**ii**), (**v**), *and* (**vii**) *hold, then the set* $\mathcal{F}$ *of all values* $\omega(A)$ *for all sets* $A \subset G_N$ *for all* $N \geq 0$ *is dense in* $[0, 1]$.

Proof. It is easy to derive from these conditions that

$$\min(x_1, \dots, x_\nu) < F(x_1, \dots, x_\nu) < \max(x_1, \dots, x_\nu) \quad \text{for } x_1, \dots, x_\nu \in [0, 1]$$

if we do not have $x_1 = x_2 = \dots = x_\nu$. Suppose that $\mathcal{F}$ is not dense in [0, 1], and let $(\alpha, \beta) \subset [0, 1]$ be one of the maximal intervals such that $\mathcal{F} \cap (\alpha, \beta) = \emptyset$, where $\alpha < \beta$. Then $\mathcal{F}$ contains points which are arbitrarily close to $F(\alpha, \beta, \dots, \beta) \in (\alpha, \beta)$, a contradiction. □

NOTATION. If $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$, then we put

$$a \cdot b = (a_1b_1, \dots, a_Nb_N) \quad \text{and} \quad \mathbf{1} = (1, \dots, 1), \ \mathbf{0} = (0, \dots, 0).$$

Proof of Theorem 3. The statement is trivial if $E = \emptyset$. Consider $E \subset G_n$ with $E \neq \emptyset$ and fix A and B with $E = A \cup B$. Without loss of generality we can assume that $A \cap B = \emptyset$, since F satisfies (**ii**).

Let $w_1 < w_2 < \dots < w_{\nu^n}$ be the ν^n vertices in G_n. Put $x_i = \omega_F(w_i, A)$ and $y_i = \omega_F(w_i, B)$ for $1 \leq i \leq \nu^n$ with $x = (x_1, \dots, x_{\nu^n})$ and $y = (y_1, \dots, y_{\nu^n})$. We have $\omega_F(A) = F^n(x)$ and $\omega_F(B) = F^n(y)$.

Observe that $x_i + y_i = \delta_i^E$ for $1 \le i \le \nu^n$. This is obvious if $\delta_i^E = 0$; if $\delta_i^E = 1$ then it is a consequence of property (c) of ω_F (this property is true because F satisfies (**ii**) and (**vii**)). Hence $x + y = \delta^E$. Therefore, $\omega_F(A) = F^n(\delta^E \cdot x)$ and $\omega_F(B) = F^n(\delta^E \cdot (\mathbf{1} - x))$.

Consider the function $g(z) = F^n(\delta^E \cdot z) + F^n(\delta^E \cdot (\mathbf{1} - z))$ with $z \in [0,1]^{\nu^n}$. Since $E \neq \emptyset$, we have that $\delta^E \neq \mathbf{0}$; this and (**ii**) imply that $g(z) > 0$ for every $z \in [0,1]^{\nu^n}$. The continuity of g gives that $M = \min\{g(z) : z \in [0,1]^{\nu^n}\} > 0$.

One has

$$\omega_F(E) \le \frac{\omega_F(E)}{M}[F^n(\delta^E \cdot x) + F^n(\delta^E \cdot (\mathbf{1} - x))] = \frac{\omega_F(E)}{M}[\omega_F(A) + \omega_F(B)].$$

Therefore, Theorem 3 is proved with $k = \omega_F(E)/M$. □

Remark. Let (**ii**), (**v**), and (**vii**) hold. If $z^0 \in [0,1]^{\nu^n}$ is such that $g(z^0) = M$ then, by Lemma 4, for every $\varepsilon > 0$ one can choose $A_\varepsilon, B_\varepsilon \subset G_N$ with large N such that $A_\varepsilon \cup B_\varepsilon = E$ and $|\delta^E \cdot z^0 - x^\varepsilon| < \varepsilon$ if $x_i^\varepsilon = \omega_F(w_i, A_\varepsilon)$. This implies that $k(E) = \omega_F(E)/M$.

Proof of Theorem 4. Consider $D \subset G_r$ and $E \subset G_s$. Let $w_1 < \cdots < w_{\nu^r}$ be the ν^r vertices of G_r and $u_1 < \cdots < u_{\nu^s}$ the ν^s vertices of G_s. If $w_i = w_I$ and $u_j = u_J$, then we put $w_i \times u_j = v_{IJ} \in G_{r+s}$. Recall that $D \times E = \{w_i \times u_j : w_i \in D, u_j \in E\}$.

Let $A, B \subset G_n$ for $n \ge r + s$, with $A \cup B = D \times E$. Put $x_j^i = \omega_F(w_i \times u_j, A)$ and $x^i = (x_1^i, \ldots, x_{\nu^s}^i)$ (here $i = 1, 2, \ldots, \nu^r$). By the foregoing remark,

$$F^r(\delta^D \cdot y) + F^r(\delta^D \cdot (\mathbf{1} - y)) \ge \frac{\omega_F(D)}{k(D)} \quad \text{for every} \;\; y \in [0,1]^{\nu^r}. \tag{20}$$

Consider $x^1, \ldots, x^{\nu^r} \in [0,1]^{\nu^s}$ as defined previously. Then

$$F^s(\delta^E \cdot x^i) + F^s(\delta^E \cdot (\mathbf{1} - x^i)) \ge \frac{\omega_F(E)}{k(E)} \quad \text{for} \;\; 1 \le i \le \nu^r.$$

Put

$$y_i = \frac{k(E)}{\omega_F(E)} F^s(\delta^E \cdot x^i) \quad \text{and} \quad z_i = \frac{k(E)}{\omega_F(E)} F^s(\delta^E \cdot (\mathbf{1} - x^i)) \quad \text{for} \;\; 1 \le i \le \nu^r.$$

Then $y_i, z_i \ge 0$ and $y_i + z_i \ge 1$.

Define $y_i^* = \min\{y_i, 1\}$ and $z_i^* = 1 - y_i^*$. We have:

$$0 \le y_i^* \le 1 \quad \text{and} \quad 0 \le z_i^* \le 1; \tag{21}$$

$$y_i^* \le y_i; \tag{22}$$

$$z_i^* = 1 - y_i^* = 1 - \min\{y_i, 1\} = \max\{1 - y_i, 0\} \le \max\{z_i, 0\} = z_i. \tag{23}$$

Therefore,

$$\begin{aligned}
\omega_F(A)+\omega_F(B) &= F^r(\delta_1^D F^s(\delta^E\cdot x^1),\dots,\delta_{\nu^r}^D F^s(\delta^E\cdot x^{\nu^r}))\\
&\quad + F^r(\delta_1^D F^s(\delta^E\cdot(\mathbf{1}-x^1)),\dots,\delta_{\nu^r}^D F^s(\delta^E\cdot(\mathbf{1}-x^{\nu^r})))\\
&= \frac{\omega_F(E)}{k(E)}[F^r(\delta_1^D y_1,\dots,\delta_{\nu^r}^D y_{\nu^r}) + F^r(\delta_1^D z_1,\dots,\delta_{\nu^r}^D z_{\nu^r})]\\
&\ge \frac{\omega_F(E)}{k(E)}[F^r(\delta_1^D y_1^*,\dots,\delta_{\nu^r}^D y_{\nu^r}^*) + F^r(\delta_1^D z_1^*,\dots,\delta_{\nu^r}^D z_{\nu^r}^*)]\\
&= \frac{\omega_F(E)}{k(E)}[F^r(\delta^D\cdot y^*) + F^r(\delta^D\cdot(\mathbf{1}-y^*))]\\
&\ge \frac{\omega_F(E)}{k(E)}\frac{\omega_F(D)}{k(D)} = \frac{\omega_F(D\times E)}{k(D)k(E)}.
\end{aligned}$$

The definition of x^i implies the first equality; (**iv**) gives the second equality; (**ii**), (22), and (23) imply the first inequality; and (20) and (21) give the last inequality. Hence

$$\omega_F(D\times E)\le k(D)k(E)[\omega_F(A)+\omega_F(B)]$$

and thus $k(D\times E)\le k(D)k(E)$. □

4. Proof of Theorem 5

Observe that Theorem 4 will give the statement if we prove that $k(C_{2,1})=1$.

Define the function

$$g(x,y)=F_p(x,y,0,\dots,0)+F_p(1-x,1-y,0,\dots,0).$$

Let $A,B\subset G_m$ be disjoint sets such that $A\cup B=C_{2,1}$. Let w_1,w_2 be the two points of $C_{2,1}$, and put $x=\omega_p(w_1,A)$ and $y=\omega_p(w_2,A)$. Then $\omega_p(A)+\omega_p(B)=g(x,y)$. By the remark before the proof of Theorem 4,

$$k(C_{2,1})=\frac{\omega_p(C_{2,1})}{\min_{x,y\in[0,1]} g(x,y)} \tag{24}$$

$(g$ is continuous on $[0,1]\times[0,1])$. In order to calculate the minimum of g, we need a piece of elementary analysis. Put $c=\nu-2$, $\alpha=p-1$, $\beta=1/(p-1)$, and $\gamma=(p-2)/(p-1)$. Put

$$X_\nu(t)=F_p(t,1,\underbrace{0,\dots,0}_{\nu-2});$$

then $X_\nu\colon[0,\infty)\to[x_0,\infty)$ is a strictly increasing function (here $x_0=X_\nu(0)=(1+(c+1)^\beta)^{-1}$). The inverse function S_ν to X_ν is defined explicitly by

$$S_\nu(x)=x+[(x-1)^{p-1}+cx^{p-1}]^\beta,\quad x\ge x_0. \tag{25}$$

Put $x_1=(1+c^\beta)^{-1}$ and $x_2=1$. Then $x_0<x_1<x_2$, and the points $s_j=S_\nu(x_j)$ are given by $s_0=0$, $s_1=(1+c^\beta)^{-1}$, and $s_2=1+c^\beta$.

LEMMA 5. *Let $\nu \geq 3$. Then X_ν' is continuous on* $[0, +\infty)$.

(1) *If* $p > 2$ $(0 < \beta < 1)$, *then $X_\nu'(t)$ is strictly decreasing on* $[0, s_1]$ *and* $[s_2, \infty)$ *and is strictly increasing on* $[s_1, s_2]$.

(2) *If* $1 < p < 2$ $(\beta > 1)$, *then $X_\nu'(t)$ is strictly increasing on* $[0, s_1]$ *and* $[s_2, \infty)$ *and is strictly decreasing on* $[s_1, s_2]$.

Proof. We have

$$S_\nu'(x) = 1 + (\psi \circ \varphi)(x), \tag{26}$$

where $\varphi(x) = (1 - 1/x)^{p-2}$ and $\psi(\eta) = (|\eta| + c) \cdot |\eta^{1/\gamma} + c|^{-\gamma}$. Calculating ψ', for $1 < p < 2$ we have $\psi' < 0$ on $(-\infty, -c^\gamma)$ and $(0, 1)$ and $\psi' > 0$ on $(-c^\gamma, 0)$ and $(1, \infty)$; all signs are reversed if $p > 2$. Note that $\varphi(x_1) = -c^\gamma$. This implies that S_ν' is monotone on each of the intervals (x_0, x_1), (x_1, x_2), (x_2, ∞) and gives the signs of this monotonicity. Thus we also find the sign of monotonicity of $X_\nu' = 1/(S_\nu' \circ X_\nu)$. We omit trivial details. □

LEMMA 6. *Let $\nu \geq 3$ and $0 < s < 1 < t$.*

(1) *If* $p > 2$ $(0 < \beta < 1)$, *then* $X_\nu'(s) < X_\nu'(1) < X_\nu'(t)$.

(2) *If* $1 < p < 2$ $(\beta > 1)$, *then* $X_\nu'(s) > X_\nu'(1) > X_\nu'(t)$.

Proof. Note that $s_1 < 1 < s_2$ and $X_\nu(1) = 1/(1 + (c/2)^\beta)$. From (26) and the formula $X_\nu' = 1/(S_\nu' \circ X_\nu)$, we have

$$X_\nu'(0) = \frac{(1+c)^{\beta-1}}{(1+c)^\beta + 1}, \quad X_\nu'(1) = \frac{1}{2} X_\nu(1), \quad X_\nu'(\infty) = \frac{1}{(1+c)^\beta + 1} = X_\nu(0). \tag{27}$$

Consider first the case $p > 2$ $(0 < \beta < 1)$. The properties of the function X_ν' appearing in Lemma 5 imply that the statement is true if $X_\nu'(0) \leq X_\nu'(1) < X_\nu'(\infty)$.

The inequality $X_\nu'(1) < X_\nu'(\infty)$ is equivalent to

$$A(c) = 1 + 2^{1-\beta} c^\beta - (1+c)^\beta > 0$$

for every positive integer c. It is easy to check that $A'(c) > 0$ for all positive c. This implies the inequality $A(c) > A(0) = 0$. The inequality $X_\nu'(0) \leq X_\nu'(1)$ is equality if $c = 1$. If $c \geq 2$ then

$$2X_\nu'(0) = \frac{2}{1 + c + (1+c)^{1-\beta}} < \frac{2}{1 + c + 1} = \frac{1}{1 + c/2} \leq \frac{1}{1 + (c/2)^\beta} = 2X_\nu'(1).$$

In the case $1 < p < 2$, the same arguments yield the result. □

Note that $\omega_p(C_{2,1}) = X_\nu(1)$. By (24), the proof of Theorem 5 finishes with the following result.

LEMMA 7. *Let $\nu \geq 3$. The function $g(x, y)$ satisfies, for every* $0 \leq x, y \leq 1$,

$$X_\nu(1) = \frac{1}{1 + ((\nu-2)/2)^\beta} \leq g(x, y) \leq \frac{2}{1 + (\nu-1)^\beta} = 2X_\nu(0) \quad \text{if } p > 2,$$

$$X_\nu(1) \geq g(x, y) \geq 2X_\nu(0) \qquad \text{if } 1 < p < 2.$$

In particular, $k(C_{2,1}) = 1$ *if* $p > 2$ *and* $k(C_{2,1}) = X_\nu(1)/(2X_\nu(0)) > 1$ *if* $1 < p < 2$.

Proof. By the symmetry of the function g, we can assume that $x \geq y$. So let us study the values of g on the set

$$D = \{(x, y) \in [0, 1]^2 : x \geq y\}.$$

First we see that g must attain its maximum and minimum values on D on the boundary of D, because $\nabla g \neq 0$ in the interior of D. Indeed, by property (**iv**) of the function F_p, we have

$$g(x, y) = y\, X_\nu\left(\frac{x}{y}\right) + (1 - y)X_\nu\left(\frac{1 - x}{1 - y}\right).$$

Hence, by Lemma 6,

$$\frac{\partial g}{\partial x}(x, y) = X_\nu'\left(\frac{x}{y}\right) - X_\nu'\left(\frac{1 - x}{1 - y}\right) \neq 0$$

for (x, y) in the interior of D, because $(1 - x)/(1 - y) < 1 < x/y$.

Note that $g(x, x) \equiv X_\nu(1)$ and $g(1, y) = g(1 - y, 0)$. Put

$$B(y) = g(1, y) = X_\nu(y) + (1 - y)X_\nu(0), \quad y \in [0, 1].$$

Since $B(1) = X_\nu(1)$, we obtain that $\min_D g = \min_{[0,1]} B$ and $\max_D g = \max_{[0,1]} B$.

We have $B'(y) = X_\nu'(y) - X_\nu(0)$ and $B(0) = 2X_\nu(0)$. If $p > 2$ then, by Lemma 6 and (27), it follows that $X_\nu'(y) < X_\nu'(1) \leq X_\nu'(\infty) = X_\nu(0)$ for $0 < y < 1$. Hence $B'(y) < 0$ for $0 < y < 1$. Similarly, $B'(y) > 0$ for $0 < y < 1$ if $1 < p < 2$. This finishes the proofs of Lemma 7 and Theorem 5. □

5. Proofs of Theorems 6 and 7

Proof of Theorem 6. Define σ and $\tilde{F}(x_1, \dots, x_\mu)$ as in Section 1 and apply Theorems 1 and 2 and Corollary 2 (in the general case $\nu \geq 3$) to $\tilde{F}$. Observe that $\omega_F(C_{\mu,n}) = \sigma^n$. Part (a) is immediate. If (9) holds, then the differentiability condition on F implies that $\tilde{F}$ also satisfies the (analog of) the right-hand side of (6). This and Theorem 2 yield part (b). □

Proof of Theorem 7. (a) We put $P_a = \{x_1 + \cdots + x_\mu = a\} \subset \mathbf{R}^\mu$, $a \in \mathbf{R}$. The invariance of $\tilde{F}_p$ under rearrangement implies that $\frac{\partial}{\partial x_j}\tilde{F}_p(q)$ does not depend on j (q is the center of R_μ). Therefore $d\tilde{F}_p(q)|P_0 = 0$. Note that there exist $\delta_1 > 0$ and a neighborhood V_1 of q in P_1 such that, for every i, j, $\frac{\partial^2}{\partial x_i \partial x_j}\tilde{F}_p(q')$ is uniformly continuous as a function of $(p, q') \in (2 - \delta_1, 2 + \delta_1) \times V_1$. It is easy to check this fact as follows. Implicit differentiation of equation (4) gives that $\frac{\partial}{\partial x_i}\tilde{F}_p$ is a C^1 function in a neighborhood of the point $(2, q)$ if $\mu < \nu$. Since $\frac{\partial^2}{\partial x_i \partial x_j}\tilde{F}_2 \equiv 0$, the Taylor formula with the Lagrange form of the rest gives that, for any $\varepsilon > 0$, there exist $\delta_0 > 0$ and an open subset V ($q \in V \subset P_1$) such that

$$\left|\tilde{F}_p(q') - \frac{1}{\mu}\right| \leq \varepsilon |q' - q|^2$$

for all $q' \in V$ and all $p \in (2-\delta_0, 2+\delta_0)$. Since $\tilde{F}_p \to \tilde{F}_2$ uniformly on R_μ as $p \to 2$, we conclude that there are positive ε, δ such that (9) holds for $F = F_p$ if $p \in (2-\delta, 2+\delta)$. Theorem 6 then gives the first assertion.

(b) Let $\mu = 2$ and $p > 2$. We have $\sigma = X_\nu(1)$. By Lemma 6 and (27),

$$X_\nu(t) \geq X_\nu(1) + X'_\nu(1)(t-1) = X_\nu(1)\frac{1+t}{2}$$

for $0 \leq t \leq 1$, which implies that

$$F_p(x_1, x_2, \underbrace{0, \ldots, 0}_{\nu-2}) \geq \sigma\frac{x_1+x_2}{2} \quad \text{for } \; x_1, x_2 \geq 0.$$

Hence there is an $\varepsilon > 0$ such that (9) holds for $F = F_p$ for all $p \in (2, \infty)$. The analog of the right-hand inequality in (6) holds for $F = \tilde{F}_p(x, y)$ for any $p \in (1, \infty)$. One can then conclude that assertion (b) holds. □

6. The Case $p = \infty$

In potential theory it is possible to define "∞-harmonic functions". In this section, we consider the "∞-harmonic measure".

The ∞-*harmonic measure* is defined as ω_{F_∞}, where F_∞ is the limit

$$F_\infty(x_1, \ldots, x_\nu) = \lim_{p\to\infty} F_p(x_1, \ldots, x_\nu) = \frac{\min(x_1, \ldots, x_\nu) + \max(x_1, \ldots, x_\nu)}{2}$$

(it is easy to prove). Theorem 1 and Remark 2 give that the answer to the weak Martio problem is negative for the ∞-harmonic measure. Besides, we can now construct sets $A_n, B_n \subset G_n$ with $D_n = A_n \cup B_n$ and

$$\frac{\omega_\infty(D_n)}{\omega_\infty(A_n) + \omega_\infty(B_n)} = \frac{n+1}{2}. \tag{28}$$

Consider $D_1 = G_1$. Given D_n, we construct D_{n+1} in the following way: $D_{n+1} = (D_n, I_n, \ldots, I_n)$, where I_n is any subset of G_n with only one vertex. Then we have $\omega_\infty(D_1) = 1$ and $\omega_\infty(D_{n+1}) = \omega_\infty(D_n)/2 + 2^{-n-1}$, and this implies that $\omega_\infty(D_n) = (n+1)2^{-n}$.

Consider $A_1 = C_{2,1}$. Given A_n, we construct A_{n+1} in the following way: $A_{n+1} = (A_n, I_n, \emptyset, \ldots, \emptyset)$. Consequently, if $B_n = D_n \setminus A_n$ it follows that $B_{n+1} = (B_n, \emptyset, I_n, \ldots, I_n)$. It is immediate by induction that $\omega_\infty(A_n) = \omega_\infty(B_n) = 2^{-n}$. These equalities give (28).

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V. Alvarez
Departamento de Análisis Matemático
Facultad de Ciencias, Campus de Teatinos
Universidad de Málaga
29071 Málaga
Spain

nancho@anamat.cie.uma.es

J. M. Rodríguez
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés, Madrid
Spain

jomaro@math.uc3m.es

D. V. Yakubovich
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés, Madrid
Spain

diakoubo@math.uc3m.es