Interval Orders and Reverse Mathematics

Alberto Marcone

Abstract   We study the reverse mathematics of interval orders. We establish the logical strength of the implications among various definitions of the notion of interval order. We also consider the strength of different versions of the characterization theorem for interval orders: a partial order is an interval order if and only if it does not contain $2 \oplus 2$. We also study proper interval orders and their characterization theorem: a partial order is a proper interval order if and only if it contains neither $2 \oplus 2$ nor $3 \oplus 1$.

1 Introduction

Interval orders are a particular kind of partial orders occurring quite naturally in many different areas and are widely studied. A partial order $\mathbf{P} = (P, \leq_P)$ is an *interval order* if the elements of P can be mapped to nonempty intervals of a linear order $\mathbf{L}$ so that $p <_P q$ holds if and only if every element of the interval associated to p precedes every element of the interval associated to q. The linear order $\mathbf{L}$ and the map from P to intervals are called the *interval representation of* $\mathbf{P}$. The basic reference on interval orders is Fishburn's monograph [9].

The name "interval order" was introduced by Fishburn [8] although the notion was studied much earlier by Wiener [23] who used the terminology "relation of complete sequence." Interval orders model many phenomena occurring in the applied sciences: [9], Section 2.1, includes examples such as chronological dating in archaeology and paleontology, scheduling of manufacturing processes, and psychophysical perception of sounds. Notice that if $\mathbf{P}$ is a countable interval order then we can assume that $\mathbf{L}$ is the rational or (as usual in applications) the real line (a *real representation*, in the terminology of [9]).

Most recent research on interval orders (see, e.g., the survey [22] and Chapter 8 of [18]) focuses on finite partial orders, whereas in this paper we consider mostly infinite ones (although a careful analysis of the finite case is instrumental in obtaining

results in the infinite case). A recent result about infinite interval orders shows that every interval order which is a well quasi order is a better quasi order [15].

The basic characterization for interval orders is given by the following theorem proved independently by Fishburn [8] and Mirkin [13].

Characterization Theorem 1 A partial order is an interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$.

Here "$\mathbf{P}$ does not contain $\mathbf{2} \oplus \mathbf{2}$" means that for no $P' \subseteq P$ the restriction of $\leq_P$ to P' is the partial order with Hasse diagram $\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\ \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}$. It is easy to see that $\mathbf{P}$ does not contain $\mathbf{2} \oplus \mathbf{2}$ if and only if

$$\forall p_0, q_0, p_1, q_1 \in P(p_0 \leq_P q_0 \wedge p_1 \leq_P q_1 \implies p_0 \leq_P q_1 \vee p_1 \leq_P q_0).$$

Two natural ways of strengthening the notion of interval order lead to the definitions of unit interval order and proper interval order. An interval order with a real representation such that all intervals have the same positive length (which can be assumed to be 1) is called a *unit interval order*. If an interval order $\mathbf{P}$ has an interval representation such that an interval associated to an element of P is never a proper subset of another such interval, then we say that $\mathbf{P}$ is a *proper interval order*. An interval representation with the above property is called a *proper interval representation*.

It is immediate that every unit interval order is a proper interval order. If the partial order is finite then the reverse implication is also true ([16], see [2] for a short proof). On the other hand, there exist infinite proper interval orders which are not unit interval orders: a simple example is provided by the ordinal $\omega + 1$. Notice, however, that the fact that $\omega + 1$ is not a unit interval order has more to do with the real line (which in this context appears to be "too short") than with structural properties of the partial order. Therefore, when dealing with infinite partial orders the notion of proper interval order appears to be more natural, as witnessed also by the following characterization theorem.

Characterization Theorem 2 A partial order is a proper interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.

"$\mathbf{P}$ does not contain $\mathbf{3} \oplus \mathbf{1}$" means that for no $P' \subseteq P$ the restriction of $\leq_P$ to P' is the partial order with Hasse diagram $\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}\ \bullet$. It is easy to see that $\mathbf{P}$ does not contain $\mathbf{3} \oplus \mathbf{1}$ if and only if

$$\forall p_0, p_1, p_2, q \in P(p_0 <_P p_1 <_P p_2 \implies p_0 \leq_P q \vee q \leq_P p_2).$$

Characterization Theorem 2 is usually known as the Scott-Suppes Theorem. Scott and Suppes [19] proved the theorem in the finite case for unit interval orders (see [1] for a simple proof in this setting). Fishburn's monograph includes a proof of this theorem with no restrictions on cardinality ([9], Theorem 2.7).

In this paper we study interval orders and proper interval orders from the viewpoint of *reverse mathematics*. The basic reference for reverse mathematics is Simpson's book [21], which contains all background material needed for this paper (and much more). A sample of recent research in the area is contained in [20].

In reverse mathematics, one formalizes theorems of ordinary mathematics and attempts to discover the set theoretic axioms required to prove these theorems. This project is usually carried out in the context of subsystems of second-order arithmetic, taking RCA_0 as the base system. RCA_0 is the subsystem obtained from full second-order arithmetic by restricting the comprehension scheme to $\mathbf{\Delta}_1^0$ formulas and adding

a formula induction scheme for Σ_1^0 formulas. In this paper, we will be concerned only with RCA_0 and its fairly weak extension known as WKL_0 (WKL_0 is strictly weaker than the subsystem ACA_0 obtained by extending the comprehension scheme in RCA_0 to all arithmetic formulas). WKL_0 is obtained by adjoining Weak König's Lemma (i.e., König's Lemma for trees of sequences of 0s and 1s) to RCA_0.

Many results about partial and linear orders have been studied from the viewpoint of reverse mathematics: recent papers include [6], [5], [4], [3], [10], [11], [12], and [14]. Moreover, [17], Section 3, includes a couple of results about interval graphs, which are strictly connected to interval orders.
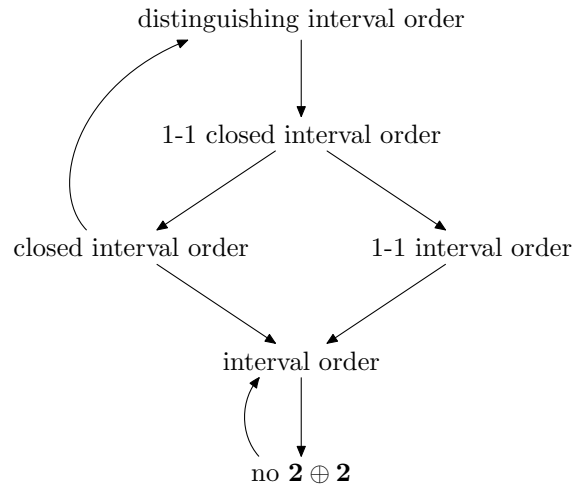
2 Overview of Results and Plan of the Paper

The first step in the study of a new topic in the context of reverse mathematics is finding appropriate formalizations of the relevant notions. Often this requires making choices between classically equivalent definitions for the mathematical concepts appearing in the definitions. In this paper, we consider a number of equivalent definitions for the notions of interval order and of proper interval order, and we examine how difficult it is to prove the equivalences of these definitions.

There is no particular difficulty in coding a countable partial order in the weak base theory RCA_0. The only point to note is that we consider only countable partial orders. However, the notion of interval order hinges on the notion of interval of a linear order, and the latter can be interpreted in different ways, leading to notions that are not necessarily equivalent in the weak base theory RCA_0. We can define an interval of the linear order $\mathbf{L} = (L, \leq_L)$ to be a set $I \subseteq L$ which satisfies $\forall x, y \in I \, \forall z \in L(x \leq_L z \leq_L y \implies z \in I)$. Another possibility is to restrict our attention to closed intervals (this is often done in the literature about interval orders; for example, in [22] this is done from the outset) and code them by pairs (a, b) of elements of L such that $a \leq_L b$ (obviously in this case $x \in L$ belongs to the interval if and only if $a \leq_L x \leq_L b$). If we apply the latter concept of interval we speak of a *closed interval representation* of the partial order. In defining interval orders there is a further subtlety, which turns out to be important in our study of the proof theoretic strength of various statements; that is, we may require the map of the interval representation to be injective. Combining the two possible choices in each of the two cases we obtain four notions of interval order: interval order, 1-1 interval order, closed interval order, and 1-1 closed interval order. Another notion is obtained by further strengthening the definition of 1-1 closed interval order: a closed interval representation is a *distinguishing representation* if all endpoints of the closed intervals are distinct (see, e.g., [22]). This leads to the notion of distinguishing interval order. In Section 3 we will give the precise definitions of these notions in RCA_0.

The five notions introduced above are all equivalent, and we establish the axioms needed to show the equivalences among them and with the characterization provided by Characterization Theorem 1. (Notice that the proofs of the latter theorem in [9] and [22] can be easily carried out in ACA_0; see Remark 4.8 below.)

We show that RCA_0 proves exactly the implications appearing in Figure 1 (where an arrow with origin in the node labeled A pointing toward the node labeled B represents the statement "every partial order which satisfies A satisfies B") or that can be obtained by composing arrows appearing in that diagram. In particular, we obtain the following result about Characterization Theorem 1.

distinguishing interval order

1-1 closed interval order

closed interval order   1-1 interval order

interval order

no $\mathbf{2} \oplus \mathbf{2}$

Figure 1 Implications about interval orders provable in RCA_0.

Theorem 2.1 RCA_0 *proves that a partial order is an interval order if and only if it does not contain* $\mathbf{2} \oplus \mathbf{2}$.

The arrows pointing downward (possibly diagonally) in Figure 1 either follow from the definitions or are straightforward to prove (these implications are collected in Theorem 3.13), while the two arrows pointing upward will be proved in Section 5.

   Figure 1 implies that in RCA_0 there are at most three distinct notions of interval order. In order of decreasing strength these are closed interval order, 1-1 interval order, and interval order. In Section 6 we show that each of the missing implications is equivalent to WKL_0. For the stronger notions of interval order we obtain the following reverse mathematics results about Characterization Theorem 1.
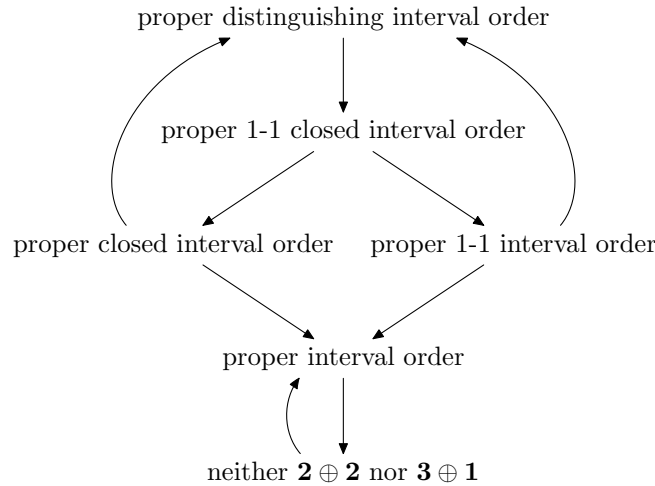
Theorem 2.2   *In* RCA_0 *the following are equivalent:*
1. WKL_0;
2. *a partial order is a 1-1 interval order if and only if it does not contain* $\mathbf{2} \oplus \mathbf{2}$;
3. *a partial order is a closed interval order if and only if it does not contain* $\mathbf{2} \oplus \mathbf{2}$;
4. *a partial order is a 1-1 closed interval order if and only if it does not contain* $\mathbf{2} \oplus \mathbf{2}$;
5. *a partial order is a distinguishing interval order if and only if it does not contain* $\mathbf{2} \oplus \mathbf{2}$.

In particular, this implies that RCA_0 does not prove the equivalence among the three notions of interval order mentioned above.

   Section 4 is devoted to a detailed analysis of the equivalences for finite partial orders. This analysis will be used in the proofs of Sections 5, 6, and 7.

   When defining proper interval orders the same choices about intervals and injectivity are possible: we thus have five different notions of proper interval order, plus the characterization provided by Characterization Theorem 2. We show that RCA_0 proves exactly the implications appearing in Figure 2 or that can be obtained by composing arrows appearing in that diagram. In particular, we obtain the following result

proper distinguishing interval order

proper 1-1 closed interval order

proper closed interval order   proper 1-1 interval order

proper interval order

neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$

Figure 2 Implications about proper interval orders provable in RCA_0.

about Characterization Theorem 2.

Theorem 2.3 RCA_0 *proves that a partial order is a proper interval order if and only if it contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$.

Figures 1 and 2 are similar, except that the latter includes one arrow whose analogous is missing from the former. Indeed within RCA_0, a 1-1 interval order is necessarily a distinguishing interval order if we have a proper representation, but not in general.

Figure 2 implies that in RCA_0 there are at most two distinct notions of proper interval order, that is, proper closed interval order and proper interval order. We show that the missing implication is equivalent to WKL_0 even if we restrict ourselves to closed interval orders. For the stronger notions of interval order we obtain the following reverse mathematics results about Characterization Theorem 2.

Theorem 2.4 *In* RCA_0 *the following are equivalent:*
1. WKL_0;
2. *a partial order is a proper 1-1 interval order if and only if it contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$;
3. *a partial order is a proper closed interval order if and only if it contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$;
4. *a partial order is a proper 1-1 closed interval order if and only if it contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$;
5. *a partial order is a proper distinguishing interval order if and only if it contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$.

In Section 7 the definitions and the arguments of Sections 3 through 6 are adapted to the case of proper interval orders and all results about proper interval orders are proved. Some of the proofs are straightforward translations of the corresponding proofs for interval orders, while others exploit the properties of proper interval orders.

Our results are stated in terms of subsystems of second-order arithmetic but have corollaries that can be viewed as examples of computable mathematics in the style

of [7]. Samples of these corollaries are the following, where we use standard terminology from computability theory.

Corollary 2.5   *For every computable partial order* **P** *not containing* $\mathbf{2} \oplus \mathbf{2}$ *there exist a computable linear order* **L** *and a computable function from* P *to intervals of* **L** *witnessing that* **P** *is an interval order.*

Corollary 2.6   *There exists a computable partial order* **P** *not containing* $\mathbf{2} \oplus \mathbf{2}$ *such that for every computable linear order* **L** *there is no computable function from* P *to closed intervals of* **L** *witnessing that* **P** *is a closed interval order.*

Corollary 2.7   *For every computable partial order* **P** *not containing* $\mathbf{2} \oplus \mathbf{2}$ *there exist a low (respectively, almost recursive) linear order* **L** *and a low (respectively, almost recursive) function from* P *to closed intervals of* **L** *witnessing that* **P** *is a distinguishing interval order.*

(Corollary 2.7 follows from our results by the properties of ω-models of WKL$_0$ which appear in [21], Section VIII.2.)

   We assume some familiarity of the reader with subsystems of second-order arithmetic, but the paper is self-contained as far as interval order theory is concerned. From now on, when a definition or the statement of a result starts with the name of a subsystem of second-order arithmetic in parenthesis, it means that the definition is given, or the statement provable, in that subsystem.

3 Definitions and Elementary Facts

Definition 3.1 (RCA$_0$)   A *partial order* **P** is a pair $(P, \leq_P)$ where P is a set and $\leq_P \subseteq P \times P$ is reflexive, transitive, and antisymmetric. The partial order **P** is a *linear order* if we have also $\forall p, q \in P (p \leq_P q \lor q \leq_P p)$.

Remark 3.2   If **P** is a partial order then $P \subseteq \mathbb{N}$ and hence on P we have also the restriction of the usual order on the natural numbers. When there is danger of confusion we denote the latter by $\leq_\mathbb{N}$.

Definition 3.3 (RCA$_0$)   If **P** is a partial order we define the relations $<_P$ and $\perp_P$ as follows:

$$p <_P q \iff p \leq_P q \land p \neq q,$$
$$p \perp_P q \iff p \not\leq_P q \land q \not\leq_P p.$$

Sometimes it is convenient to use quasi orders, which are defined by dropping the requirement of antisymmetry from the definition of partial order. In particular, we will be interested in linear quasi orders.

Definition 3.4 (RCA$_0$) $\mathbf{P} = (P, \leq_P)$ is a *quasi order* if $\leq_P \subseteq P \times P$ is reflexive and transitive. If we have also $\forall p, q \in P (p \leq_P q \lor q \leq_P p)$ we say that **P** is a *linear quasi order*.

Definition 3.5 (RCA$_0$)   If **P** is a quasi order we define $<_P$ by

$$p <_P q \iff p \leq_P q \land p \not\leq_P q,$$

while no changes are needed in the definition of $\perp_P$. Furthermore, we define $\equiv_P$ by

$$p \equiv_P q \iff p \leq_P q \land p \leq_P q.$$

It is immediate to check in RCA_0 that if $\mathbf{P}$ is a quasi order then $\equiv_P$ is an equivalence relation.

In our setting using (linear) quasi orders in place of partial (respectively, linear) orders is just a matter of convenience, as the following easy lemma shows.

Lemma 3.6 (RCA_0)   *Let* $\mathbf{P}$ *be a quasi order. Then there exist* $P' \subseteq P$ *and* $f : P \rightarrow P'$ *such that* $\mathbf{P}' = (P', \leq_P)$ *is a partial order and* f *is a surjective order-preserving function satisfying* $f(p) = p$ *for every* $p \in P'$. *Furthermore, if* $\mathbf{P}$ *is a linear quasi order then* $\mathbf{P}'$ *is a linear order.*

Proof   Since $P \subseteq \mathbb{N}$ we can let

$$P' = \{ p \in P \mid \forall q <_\mathbb{N} p \; q \not\equiv_P p \};$$
$$f(p) = \text{the } <_\mathbb{N}\text{-least } q \text{ such that } q \equiv_P p. \qquad \square$$

We can now introduce the different notions of interval order.

Definition 3.7 (RCA_0)   A partial order $\mathbf{P}$ is an *interval order* if there exist a linear order $\mathbf{L}$ and a set $F \subseteq P \times L$ such that, abbreviating $\{ x \in L \mid (p, x) \in F \}$ by $F(p)$ for every $p \in P$, we have

(i1) $F(p) \neq \varnothing$ and $\forall x, y \in F(p) \, \forall z \in L(x <_L z <_L y \implies z \in F(p))$ for all $p \in P$;
(i2) $p <_P q \iff \forall x \in F(p) \, \forall y \in F(q) \, x <_L y$ for all $p, q \in P$.

$\mathbf{P}$ is a *1-1 interval order* if we have also

(i3) $F(p) \neq F(q)$ whenever $p \neq q$.

$\mathbf{P}$ is a *closed interval order* if there exist a linear order $\mathbf{L}$ and two functions $f_0, f_1 : P \rightarrow L$ such that

(c1) $f_0(p) \leq_L f_1(p)$ for all $p \in P$;
(c2) $p <_P q \iff f_1(p) <_L f_0(q)$ for all $p, q \in P$.

$\mathbf{P}$ is a *1-1 closed interval order* if we have also

(c3) $f_0(p) \neq f_0(q)$ or $f_1(p) \neq f_1(q)$ whenever $p \neq q$.

$\mathbf{P}$ is a *distinguishing interval order* if besides (c1) and (c2) we have also

(c4) $f_i(p) \neq f_j(q)$ whenever $p \neq q$ or $i \neq j$.

It is immediate that if we set $F(p) = \{ x \in L \mid f_0(p) \leq_L x \leq_L f_1(p) \}$, conditions (c1)–(c3) are the translations of conditions (i1)–(i3).

Remark 3.8   Lemma 3.6 implies that in the preceding definitions we can use linear quasi orders in place of linear orders. Whenever it is convenient for the clarity of the exposition, we will use this fact without mentioning it explicitly.

Definition 3.9 (RCA_0)   A partial order $\mathbf{P}$ *does not contain* $\mathbf{2} \oplus \mathbf{2}$ if

$$\forall p_0, q_0, p_1, q_1 \in P(p_0 <_P q_0 \wedge p_1 <_P q_1 \implies p_0 \leq_P q_1 \vee p_1 \leq_P q_0).$$

Definition 3.10 (RCA_0)   If $\mathbf{P}$ is a partial order and $p \in P$, the *strict downward and upward closures of* p *in* P are the sets

$$p \uparrow^\mathbf{P} = \{ q \in P \mid p <_P q \} \quad \text{and} \quad p \downarrow^\mathbf{P} = \{ q \in P \mid q <_P p \}.$$

When $\mathbf{P}$ is clear from the context we write $p \uparrow$ and $p \downarrow$.

The next lemma is a basic observation about partial orders not containing $\mathbf{2} \oplus \mathbf{2}$.

Lemma 3.11 (RCA$_0$)   *If **P** does not contain **2** ⊕ **2** then for every $p, q \in P$ we have either $p\uparrow \subseteq q\uparrow$ or $q\uparrow \subseteq p\uparrow$, and similarly either $p\downarrow \subseteq q\downarrow$ or $q\downarrow \subseteq p\downarrow$.*

Proof If $p\uparrow \nsubseteq q\uparrow$ and $q\uparrow \nsubseteq p\uparrow$, let $p_1 \in p\uparrow \setminus q\uparrow$ and $q_1 \in q\uparrow \setminus p\uparrow$. Then p, p_1, q, q_1 show that **P** contains **2** ⊕ **2**. The argument for the strict downward closures is similar. □

The following lemma is useful to show that an interval order is actually a 1-1 interval order.

Lemma 3.12 (RCA$_0$)   *Suppose **P** is an interval order such that*

$$\forall p, q \in P(p \neq q \implies p\uparrow \neq q\uparrow \lor p\downarrow \neq q\downarrow).$$

*Then **P** is a 1-1 interval order.*

Proof Let **L** and F satisfy conditions (i1) and (i2). We claim that F satisfies also (i3). Fix $p, q \in P$ with $p \neq q$. We have either $p\uparrow \neq q\uparrow$ or $p\downarrow \neq q\downarrow$. Without loss of generality, we may assume the former inequality holds and there exists $r \in p\uparrow \setminus q\uparrow$. Then $q \nless_P r$ and for some $x \in F(r)$ and $y \in F(q)$ we have $x \leq_L y$. On the other hand, $p <_P r$ so that $z <_L x$ for all $z \in F(p)$. Hence $y \notin F(p)$ and $F(p) \neq F(q)$. □

We now prove the "easy" arrows appearing in Figure 1.

Theorem 3.13 (RCA$_0$)

1. *Every distinguishing interval order is a 1-1 closed interval order.*
2. *Every 1-1 (closed) interval order is a (closed) interval order.*
3. *Every (1-1) closed interval order is a (1-1) interval order.*
4. *Every interval order does not contain **2** ⊕ **2**.*

Proof The statements in (1) and (2) follow immediately from the definitions (since condition (c4) implies condition (c3)). For the statements in (3), given **L**, f_0, and f_1 as in the definition of closed interval order, let

$$F = \{ (p, x) \in P \times L \mid f_0(p) \leq_L x \leq_L f_1(p) \}.$$

To prove (4), let L and F witness that **P** is an interval order. Suppose toward a contradiction that $p_0, q_0, p_1, q_1 \in P$ are such that $p_0 <_P q_0$, $p_1 <_P q_1$, $p_0 \nless_P q_1$, and $p_1 \nless_P q_0$. The third condition implies the existence of $x, y \in L$ such that $x \in F(p_0)$, $y \in F(q_1)$, and $y \leq_L x$. Similarly, by the fourth condition, there exist x', y' such that $x' \in F(p_1)$, $y' \in F(q_0)$, and $y' \leq_L x'$. The first two conditions imply, respectively, $x <_L y'$ and $x' <_L y$: using transitivity we have $x <_L x$, which is impossible. □

4 Finite Interval Orders

We start by introducing one of the basic tools in the analysis of partial orders not containing **2** ⊕ **2**. Within RCA$_0$ we can define it only for finite partial orders.

Definition 4.1 (RCA$_0$)   Given a finite partial order **P**, let $P^+ = \{ p^+ \mid p \in P \}$, $P^- = \{ p^- \mid p \in P \}$, and $P^* = P^+ \cup P^-$. Define a binary relation $\leq^*_{\mathbf{P}}$ on P^* as

follows:

$$p^+ \leq_{\mathbf{P}}^* q^+ \iff p\uparrow^{\mathbf{P}} \supseteq q\uparrow^{\mathbf{P}};$$
$$p^- \leq_{\mathbf{P}}^* q^- \iff p\downarrow^{\mathbf{P}} \subseteq q\downarrow^{\mathbf{P}};$$
$$p^+ \leq_{\mathbf{P}}^* q^- \iff p <_P q;$$
$$p^- \leq_{\mathbf{P}}^* q^+ \iff q \not<_P p.$$

$\mathbf{P}^* = (P^*, \leq_{\mathbf{P}}^*)$ is the *conjoint linear quasi order associated to* $\mathbf{P}$. When $\mathbf{P}$ is clear from the context, we write $\leq^*$ in place of $\leq_{\mathbf{P}}^*$.

The following lemma justifies the use of the words "linear quasi order" in Definition 4.1.

Lemma 4.2 (RCA$_0$)   *If* $\mathbf{P}$ *is a finite partial order which does not contain* $\mathbf{2} \oplus \mathbf{2}$ *then* $\leq^*$ *is a linear quasi order. Moreover,* $\mathbf{P}^*$ *and the functions* $p \mapsto p^-$, $p \mapsto p^+$ *show that* $\mathbf{P}$ *is a closed interval order.*

Proof   Reflexivity of $\leq^*$ follows immediately from the definition. Using Lemma 3.11 it is also immediate that for every $x, y \in P^*$ we have $x \leq^* y$ or $y \leq^* x$. It remains to show that $\leq^*$ is transitive and to this end we need to consider eight cases. We tackle three of them, the others being trivial or similar to one of these:

Case 1   If $p^+ \leq^* q^+ \leq^* r^-$ then $p\uparrow \supseteq q\uparrow$ and $q <_P r$, that is, $r \in q\uparrow$; therefore, $r \in p\uparrow$ which means $p <_P r$ and hence $p^+ \leq^* r^-$;

Case 2   If $p^+ \leq^* q^- \leq^* r^+$, then $p <_P q$ and $r \not<_P q$; hence, $q \in p\uparrow \setminus r\uparrow$ and, by Lemma 3.11, $p\uparrow \supseteq r\uparrow$ holds, so that $p^+ \leq^* r^+$;

Case 3   If $p^+ \leq^* q^- \leq^* r^-$, then $p <_P q$ and $q\downarrow \subseteq r\downarrow$, which imply $p <_P r$ and hence, $p^+ \leq^* r^-$.

Since for every p we have $p^- \leq^* p^+$ (in fact, $p^- <^* p^+$) condition (c1) of Definition 3.7 is satisfied. Condition (c2) follows immediately from the definition.   □

Remark 4.3   Notice that for all $p, q \in P$ we have $p^+ \not\equiv^* q^-$. In other words, each $\equiv^*$-equivalence class is contained in either P^+ or P^-.

Lemma 4.2 does not prove that $\mathbf{P}$ is a distinguishing interval order, or even a 1-1 closed interval order: if $p, q \in P$ are distinct and such that $p\downarrow = q\downarrow$ and $p\uparrow = q\uparrow$ we have $p^- \equiv^* q^-$ and $p^+ \equiv^* q^+$. To obtain the stronger conclusions we can proceed as follows.

Definition 4.4 (RCA$_0$)   Given a finite partial order $\mathbf{P}$ which does not contain $\mathbf{2} \oplus \mathbf{2}$, let $\mathbf{P}^*$ be the conjoint linear quasi order associated to $\mathbf{P}$. A linear order $(P^*, \leq_L)$ is *compatible with* $\mathbf{P}^*$ if

$$\forall x, y \in P^* (x <^* y \implies x <_L y).$$

Remark 4.5   Each $(P^*, \leq_L)$ compatible with $\mathbf{P}^*$ is defined by giving a linear order on each $\equiv^*$-equivalence class and keeping the order between $\equiv^*$-inequivalent elements unchanged.

Lemma 4.6 (RCA$_0$)   *If* $\mathbf{P}$ *is a finite partial order which does not contain* $\mathbf{2} \oplus \mathbf{2}$ *then there exists a linear order compatible with* $\mathbf{P}^*$.

Proof For example, let

$$x \leq_L y \iff x <^* y \vee (x \equiv^* y \wedge x \leq_\mathbb{N} y).$$

$\leq_L$ is a linear order compatible with $\mathbf{P}^*$. $\square$

Lemma 4.7 (RCA$_0$) *Any finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$ is a distinguishing interval order.*

Proof Let $\mathbf{P}$ be a finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$, and, by Lemma 4.6, $\leq_L$ a linear order compatible with $\mathbf{P}^*$. By Lemma 4.2 and Remark 4.3 $(P^*, \leq_L)$ and the functions $p \mapsto p^-$, $p \mapsto p^+$ witness that $\mathbf{P}$ is a distinguishing interval order. $\square$

Combining Lemma 4.7 with Theorem 3.13 we obtain that RCA$_0$ proves the equivalence of the six characterizations of interval orders restricted in the case of finite partial orders.

Remark 4.8 The reader should notice that we carried out the discussion in this section only for finite partial orders, but the constructions and arguments apply also for infinite ones. However, in the infinite case RCA$_0$ does not suffice to define $\leq^*$ and we need to use ACA$_0$. Indeed, arithmetical comprehension guarantees the existence of, say, the set of all pairs (p, q) such that $p \uparrow \supseteq q \uparrow$. Therefore, we showed that ACA$_0$ proves the equivalence of the six characterizations of interval orders for countable partial orders.

Our goal is to obtain sharper results, in particular, showing that all equivalences can be proved in WKL$_0$ (which is strictly weaker than ACA$_0$). We will in fact use the results of this section about finite partial orders to prove results about infinite partial orders without resorting to the full power of ACA$_0$.

The following fact about the conjoint linear quasi order will be useful in the proof of Theorem 5.2.

Lemma 4.9 *Let $\mathbf{P}^*$ be the conjoint linear quasi order associated to the finite partial order $\mathbf{P}$ and let $p \in P$. Then*

1. *either p^- is a minimum in $\mathbf{P}^*$ (i.e., $\forall x \in P^* \ p^- \leq^* x$) or there exists $q \in P$, $q \neq p$ such that q^+ is an immediate predecessor of p^- in $\mathbf{P}^*$ (i.e., $x <^* p^-$ implies $x \leq^* q^+$ for all $x \in P^*$);*
2. *either p^+ is a maximum in $\mathbf{P}^*$ (i.e., $\forall x \in P^* \ x \leq^* p^+$) or there exists $q \in P$, $q \neq p$ such that q^- is an immediate successor of p^+ in $\mathbf{P}^*$ (i.e., $p^+ <^* x$ implies $q^- \leq^* x$ for all $x \in P^*$).*

Proof We prove the first statement (the second is proved similarly). Since $\mathbf{P}$ and $\mathbf{P}^*$ are finite, if p^- is not minimal in $\mathbf{P}^*$ there exists $x \in P^*$ which is an immediate predecessor of p^-.

To show that $x = q^+$ for some q, it suffices to show that for every $r \in P$ with $r^- <^* p^-$ there exists $q \in P$ with $r^- \leq^* q^+ <^* p^-$. Indeed, $r^- <^* p^-$ means $r \downarrow \subsetneqq p \downarrow$ and there exists $q \in p \downarrow \setminus r \downarrow$. Then $q \not<_P r$ and $q <_P p$ which imply $r^- \leq^* q^+$ and $q^+ <^* p^-$. It is obvious that $q \neq p$, since $p^- <^* p^+$. $\square$

5   Proofs in RCA$_0$

We start this section with the quite simple proof that the upper upward-pointing arrow of Figure 1 is provable in RCA$_0$.

Theorem 5.1 (RCA$_0$)   *Every closed interval order is a distinguishing interval order.*

Proof   Let **P** be a closed interval order and let **L**, f_0 and f_1 witness this.   Let $P^* = \{\, p^+, p^- \mid p \in P \,\}$ and $L' = L \cup P^*$ (we are assuming $L \cap P^* = \varnothing$).

We would like to define a linear order $\leq_{L'}$ on L' so that the maps $p \mapsto p^-$ and $p \mapsto p^+$ witness that **P** is a distinguishing interval order. We first describe $\leq_{L'}$ informally: the restriction of $\leq_{L'}$ to L coincides with $\leq_L$, and p^+ and p^- are placed, respectively, "just above $f_1(p)$" and "just below $f_0(p)$"; if distinct p and q are such that $f_1(p) = f_1(q)$ then p^+ and q^+ are placed according to $\leq_{\mathbb{N}}$, and similarly for p^- and q^- when $f_0(p) = f_0(q)$; if $f_0(p) = f_1(q)$ then p^- is below q^+.

To simplify the explicit definition of $\leq_{L'}$, we can exclude the elements not belonging to the range of the functions we have in mind and, therefore, consider only the restriction of $\leq_{L'}$ to P^*. Thus we set, for every $p, q \in P$,

$$p^+ \leq_{L'} q^+ \iff f_1(p) <_L f_1(q) \vee (f_1(p) = f_1(q) \wedge p \leq_{\mathbb{N}} q);$$
$$p^- \leq_{L'} q^- \iff f_0(p) <_L f_0(q) \vee (f_0(p) = f_0(q) \wedge p \leq_{\mathbb{N}} q);$$
$$p^+ \leq_{L'} q^- \iff f_1(p) <_L f_0(q);$$
$$p^- \leq_{L'} q^+ \iff f_0(p) \leq_L f_1(q).$$

It is left to the reader checking that $\mathbf{L}' = (P^*, \leq_{L'})$ is a linear order. We define $f_0', f_1' : P \to P^*$ by $f_0'(p) = p^-$ and $f_1'(p) = p^+$, and again we leave to the reader checking that conditions (c1), (c2), and (c4) of Definition 3.7 hold. Therefore, **P** is a distinguishing interval order.   □

We now show that also the bottom upward-pointing arrow of Figure 1 is provable in RCA$_0$.

Theorem 5.2 (RCA$_0$)   *Every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is an interval order.*

Proof   Let **P** be a partial order not containing $\mathbf{2} \oplus \mathbf{2}$. Let $\{\, p_n \mid n > 0 \,\}$ be an enumeration of P (notice that for notational convenience we start our enumeration from 1). If $s \in \mathbb{N}$, let $\mathbf{P}_s = (\{\, p_n \mid 0 < n \leq s \,\}, \leq_P)$ and let $\mathbf{P}_s^*$ be the conjoint linear quasi order associated to the finite partial order $\mathbf{P}_s$. We have $P_{s-1}^* \subset P_s^*$ and we can investigate which relations are preserved from $\mathbf{P}_{s-1}^*$ to $\mathbf{P}_s^*$.

Claim 5.2.1 $x <_{s-1}^* y$ implies $x <_s^* y$ for every $x, y \in P_{s-1}^*$.

Proof   If exactly one of x and y is in P_{s-1}^+ (and the other is in P_{s-1}^-) the claim follows immediately from the definition of conjoint linear quasi order. If $x, y \in P_{s-1}^+$, say $x = p_n^+$ and $y = p_m^+$, then $x <_{s-1}^* y$ means that $p_n \uparrow^{\mathbf{P}_{s-1}} \supsetneq p_m \uparrow^{\mathbf{P}_{s-1}}$. Since $p_i \uparrow^{\mathbf{P}_s} \cap P_{s-1}^* = p_i \uparrow^{\mathbf{P}_{s-1}}$, $p_n \uparrow^{\mathbf{P}_s} \subseteq p_m \uparrow^{\mathbf{P}_s}$ cannot hold and, by Lemma 3.11 (which uses the hypothesis that **P** does not contain $\mathbf{2} \oplus \mathbf{2}$), $p_n \uparrow^{\mathbf{P}_s} \supsetneq p_m \uparrow^{\mathbf{P}_s}$, that is, $x <_s^* y$. The argument for the case $x, y \in P_{s-1}^-$ is similar.   □

On the other hand, it is obvious that $x \equiv_{s-1}^* y$ does not imply $x \equiv_s^* y$; for example, if $x = p_n^+$, $y = p_m^+$, $p_n \uparrow^{\mathbf{P}_{s-1}} = p_m \uparrow^{\mathbf{P}_{s-1}}$, $p_n <_{\mathbf{P}} p_s$, and $p_m \nless_{\mathbf{P}} p_s$. We say that x

is separated below at s if for some y we have $x \equiv^*_{s-1} y$ and $x <^*_s y$. Analogously, x *is separated above at s* if for some y we have $x \equiv^*_{s-1} y$ and $y <^*_s x$.

Claim 5.2.2   At most one $\equiv^*_{s-1}$-equivalence class contained in P^+_{s-1} (recall Remark 4.3) contains elements separated at s (and the same for $\equiv^*_{s-1}$-equivalence classes contained in P^-_{s-1}).

Proof   Notice that, by Lemma 4.9, p^+_n can be separated at s only if $x <^*_s p^-_s <^*_s y$ for some $x, y \equiv^*_{s-1} p^+_n$. By the previous claim, this can happen for the elements of at most one $\equiv^*_{s-1}$-equivalence class.   □

We define a linear quasi order $\mathbf{L} = (L, \leq_L)$ where

$$ L = \left\{ x^k_n \mid n \in \mathbb{N} \wedge n > 0 \wedge k \in \mathbb{Z} \wedge n \leq |k| \right\}. $$

If $s \in \mathbb{N}$, let $L_s = \left\{ x^k_n \in L \mid n \leq |k| \leq s \right\}$. We define $\leq_L$ by stages so that, at stage s, $\leq_L$ is defined on the finite set L_s and satisfies the following conditions:

  (i)   the set $\left\{ x^s_n, x^{-s}_n \mid n \leq s \right\} \subseteq L_s$ is ordered by $\leq_L$ according to $\mathbf{P}^*_s$, where x^s_n and x^{-s}_n replace, respectively, p^+_n and p^-_n;
  (ii) if $n < s$ then $x^{-s}_n <_L x^{-s+1}_n$ and $x^s_n >_L x^{s-1}_n$;
  (iii) if $n < s$ and $y \in L_{s-1}$ then neither $x^{-s}_n \leq_L y <_L x^{-s+1}_n$ nor $x^{s-1}_n <_L y \leq_L x^s_n$ hold.

An easy induction using (i) and (ii) yields $x^k_n <_L x^h_n$ if and only if $k <_{\mathbb{Z}} h$. Notice also that (i) and (iii) imply $x^k_n \not\equiv_L x^h_m$ whenever $k \neq h$.

Since $L_0 = \varnothing$ at stage 0 there is nothing to do. Let $s > 0$ and suppose we have defined $\leq_L$ on L_{s-1} satisfying (i)–(iii). To define $\leq_L$ on L_s it suffices to describe the position of the x^s_ns and x^{-s}_ns for $n \leq s$.

First consider x^s_n for $n < s$. If p^+_n is not separated above at s then x^s_n is an immediate successor (among the elements of L_s) of x^{s-1}_n. If p^+_n is separated above at s, fix p^+_m which is separated below at s. By Claim 5.2.2 we have $p^+_m \equiv^*_{s-1} p^+_n$, and hence by (i) $x^{s-1}_m \equiv_L x^{s-1}_n$. Let x^s_n be an immediate successor of x^{-s}_s, which is an immediate successor of x^s_m (which, by the first clause of the present definition, is an immediate successor of $x^{s-1}_m \equiv_L x^{s-1}_n$). The position of x^{-s}_n for $n < s$ is established similarly: if p^-_n is not separated below at s then x^{-s}_n is an immediate predecessor of x^{-s+1}_n, otherwise fix p^-_m which is separated above at s and let x^{-s}_n be an immediate predecessor of x^s_s, which is an immediate predecessor of x^{-s}_m.

If $p^+_s \equiv_{P^*} p^+_n$ for some $n < s$, then set $x^s_s \equiv_L x^s_n$, and similarly if $p^-_s \equiv_{P^*} p^-_n$ for some $n < s$, set $x^{-s}_s \equiv_L x^{-s}_n$. If the previous case does not hold and p^+_s is the maximum in $\mathbf{P}^*_s$ then x^s_s is the maximum in L_s. Similarly, if p^-_s is the minimum in $\mathbf{P}^*_s$ then x^{-s}_s is the minimum in L_s. If the position of x^s_s is not yet determined, by Lemma 4.9, p^+_s is the immediate predecessor in $\mathbf{P}^*_s$ of some p^-_n with $n < s$: let x^s_s be the immediate predecessor of x^{-s}_n in L_s. Similarly, if p^-_s is the immediate successor in $\mathbf{P}^*_s$ of some p^+_n with $n < s$, let x^{-s}_s be the immediate successor of x^s_n in L_s.

Notice that the latter part of the definition is compatible with the positions of x^s_s and x^{-s}_s given earlier in some cases (i.e., if some p^-_n or p^+_n is separated at s) above: in fact if p^+_m and p^+_n are separated below and above, respectively, at s then p^-_s is an immediate successor in $\mathbf{P}^*_s$ of p^+_m (and similarly for the other case). It is straightforward to check that $\leq_L$ restricted to L_s satisfies (i)–(iii).

The definition of **L** is thus complete. We need to define $F \subseteq P \times L$, and we would like to set

$$F = \left\{ (p_n, x_m^k) \mid \exists s \ x_n^{-s} \leq_L x_m^k \leq_L x_n^s \right\}.$$

To show the existence of F in RCA$_0$, we need to prove that the Σ_1^0 formula appearing in the above definition is provably Δ_1^0.

Claim 5.2.3   If $t = \max(|k|, n)$, then $\exists s \ x_n^{-s} \leq_L x_m^k \leq_L x_n^s$ is equivalent to $x_n^{-t} \leq_L x_m^k \leq_L x_n^t$.

Proof   One direction of the equivalence is obvious, so assume that $x_n^{-s} \leq_L x_m^k \leq_L x_n^s$ for some $s \neq t$. If $s < t$ the conclusion follows immediately from $x_n^{-t} <_L x_n^{-s}$ and $x_n^s <_L x_n^t$. If $s > t$ then $x_m^k \in L_{s-1}$ (because $m \leq |k| \leq t < s$) and $n < s$: hence by (iii) we have $x_n^{-s+1} \leq_L x_m^k \leq_L x_n^{s-1}$. Repeating this argument we obtain $x_n^{-t} \leq_L x_m^k \leq_L x_n^t$. $\square$

Claim 5.2.3 shows that F exists. It is immediate that (i1) is satisfied, so we need only to check (i2). If $p_n <_P p_m$ then by (i) we have $x_n^s <_L x_m^{-s}$ for every $s \geq \max(n, m)$ and this easily implies $\forall x \in F(p_n) \forall y \in F(p_m) x <_L y$. If $p_n \not<_P p_m$ then $x_m^{-s} <_L x_n^s$ where $s = \max(n, m)$: since $x_n^s \in F(p_n)$ and $x_m^{-s} \in F(p_m)$, $\forall x \in F(p_n) \forall y \in F(p_m) x <_L y$ fails. $\square$

6   Equivalences with WKL$_0$

We first show that WKL$_0$ suffices to prove that the six characterizations of interval orders we introduced are equivalent.

Lemma 6.1 (WKL$_0$)   *Every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is a distinguishing interval order.*

Proof   Let **P** be a partial order not containing $\mathbf{2} \oplus \mathbf{2}$. By Lemma 4.7 we can assume P is infinite and let $\{ p_n \mid n \in \mathbb{N} \}$ be a one-to-one enumeration of P. If $s \in \mathbb{N}$ let $\mathbf{P}_s = (\{ p_n \mid n \leq s \}, \leq_P)$ and $\mathbf{P}_s^*$ be the conjoint linear quasi order associated to the finite partial order $\mathbf{P}_s$. $\mathbf{P}_s^*$ is a linear quasi order by Lemma 4.2 because **P**, and hence the finite partial order $\mathbf{P}_s$, does not contain $\mathbf{2} \oplus \mathbf{2}$.

Let T be the set defined by setting $\sigma \in T$ if and only if σ is a finite sequence of length $\mathrm{lh}(\sigma)$ such that, for all $s, t < \mathrm{lh}(\sigma)$,

1. $\sigma(s)$ is (the code for) a linear order (denoted by $\leq_{\sigma(s)}$) compatible with $\mathbf{P}_s^*$ (see Definition 4.4);
2. if $s < t < \mathrm{lh}(\sigma)$ then $\sigma(t)$ extends $\sigma(s)$; that is, $x \leq_{\sigma(s)} y \iff x \leq_{\sigma(t)} y$ for all $x, y \in P_s^*$.

T exists by Δ_1^0-comprehension. It is immediate that T is a tree. Since $\sigma(s)$ can assume only finitely many values—corresponding to the (codes of the) finitely many linear orders on the finite set P_s^*—T is bounded in the sense of [21], Definition IV.1.3. By Lemma 4.6, for every s, there exists a linear order compatible with $\mathbf{P}_s^*$. By taking its restrictions to P_t^* for $t < s$ we construct a sequence in T of length s. Thus T is infinite.

By Bounded König's Lemma, which is provable in WKL$_0$ ([21], Lemma IV.1.4), T has an infinite path. This path is a sequence $\{ \alpha(s) \mid s \in \mathbb{N} \}$ of (codes for) finite linear orders, each one extending the previous ones and such that $\alpha(s)$ is compatible

with $\mathbf{P}_s^*$. If $x, y \in P^*$, let $x \leq_L y$ if and only if $x \leq_{\alpha(s)} y$ for any (or, equivalently, each) s with $x, y \in P_s^*$. (Notice that here we are considering P^* just as a set, without the ordering $\leq_{\mathbf{P}}^*$ which is not definable in WKL_0.) $\leq_L$ exists by $\mathbf{\Delta}_1^0$-comprehension.

It is straightforward to check that $(P^*, \leq_L)$ is a linear order and that conditions (c1), (c2), and (c4) are satisfied by the functions $p \mapsto p^-$, $p \mapsto p^+$ (because they are satisfied by each $\leq_{\alpha(s)}$, by the proof of Lemma 4.7). Hence $\mathbf{P}$ is a distinguishing interval order.   □

Corollary 6.2 (WKL_0)   *The five notions of interval order of Definition 3.7 and the property of not containing $\mathbf{2} \oplus \mathbf{2}$ are all equivalent.*

Proof   This follows from Theorem 3.13 and Lemma 6.1.   □

We now show that the implications that cannot be obtained by composing arrows appearing in Figure 1 are equivalent to WKL_0. In particular, these implications are not provable in RCA_0.

The following well-known characterization of WKL_0 ([21], Lemma IV.4.4) is useful.

Lemma 6.3 (RCA_0)   *The following are equivalent:*

   (i) WKL_0;
   (ii) *if $f, g : \mathbb{N} \to \mathbb{N}$ are one-to-one functions such that $\forall n, m\ f(n) \neq g(m)$ then there exists a set X such that $\forall n(f(n) \in X \wedge g(n) \notin X)$.*

Lemma 6.4 (RCA_0)   *If every interval order is a 1-1 interval order then WKL_0 holds.*

Proof   We will show that under our hypothesis (ii) of Lemma 6.3 holds. Fix one-to-one functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $\forall n, m\ f(n) \neq g(m)$. We want to find a set X such that $\forall n(f(n) \in X \wedge g(n) \notin X)$.

We define a partial order $\leq_P$ on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where, for each k, $P_k = \{a_k, b_k\} \cup \{c_k^n \mid n \in \mathbb{N}\}$. If $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$ if and only if $k <_{\mathbb{N}} h$. The elements of each P_k are pairwise $\leq_P$-incomparable with the following exceptions:

   1. if n is such that $f(n) = k$, then $c_k^n <_P a_k <_P c_k^{n+1}$;
   2. if n is such that $g(n) = k$, then $c_k^n <_P b_k <_P c_k^{n+1}$.

Notice that our hypothesis on f and g imply that for each k at most one of the two possibilities occurs, and for at most one n. $\leq_P$ can be defined within RCA_0.

Let $\mathbf{P} = (P, \leq_P)$: it is immediate that $\mathbf{P}$ does not contain $\mathbf{2} \oplus \mathbf{2}$. By Theorem 5.2 $\mathbf{P}$ is an interval order and by our hypothesis $\mathbf{P}$ is a 1-1 interval order. Hence there exist a linear order $\mathbf{L} = (L, \leq_L)$ and $F \subseteq P \times L$ satisfying conditions (i1)–(i3) of Definition 3.7. Let $\varphi(k)$ and $\psi(k)$ be the $\mathbf{\Pi}_1^0$ formulas

$$F(a_k) \subseteq F(b_k) \qquad \text{and} \qquad F(b_k) \subseteq F(a_k),$$

respectively. Since (i3) holds (i.e., F is one-to-one) we have $\forall k\ \neg(\varphi(k) \wedge \psi(k))$ and we are in the hypothesis of $\mathbf{\Pi}_1^0$-separation ([21], Exercise IV.4.8), which is provable in RCA_0: hence there exists a set X satisfying

$$\forall k((\varphi(k) \implies k \in X) \wedge (\psi(k) \implies k \notin X)).$$

We claim that X satisfies also $\forall n(f(n) \in X \wedge g(n) \notin X)$, thus completing the proof. To this end it suffices to show that $\exists n\ f(n) = k$ implies $\varphi(k)$ and $\exists n\ g(n) = k$ implies $\psi(k)$.

We prove only the first of these implications, the second being similar. Suppose n is such that $f(n) = k$: then $c_k^n <_P a_k <_P c_k^{n+1}$, $c_k^n \not\leq_P b_k$, and $b_k \not\leq_P c_k^{n+1}$. The last two conditions and (i2) imply the existence of $x \in F(c_k^n)$, $x' \in F(b_k)$, $y \in F(c_k^{n+1})$, and $y' \in F(b_k)$ such that $x' \leq_L x$ and $y \leq_L y'$. By the first condition and (i2), for all $z \in F(a_k)$, we have $x <_L z <_L y$, and hence $x' <_L z <_L y'$. Now we use (i1), obtaining $F(a_k) \subseteq F(b_k)$, that is, $\varphi(k)$. □

Lemma 6.5 (RCA$_0$)   *If every 1-1 interval order is a closed interval order then* WKL$_0$ *holds.*

Proof   Again we will show that under our hypothesis (ii) of Lemma 6.3 holds and we fix one-to-one functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $\forall n, m\ f(n) \neq g(m)$. We want to find X such that $\forall n(f(n) \in X \wedge g(n) \notin X)$.

We define a partial order $\leq_P$ on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where $P_k = \{a_k, b_k, c_k\} \cup \{ d_k^n \mid n \in \mathbb{N} \}$ for each k. As in the previous proof, if $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$ if and only if $k <_{\mathbb{N}} h$. Within each P_k we have

1. $a_k \perp_P b_k$, $a_k \perp_P c_k$, and $c_k <_P b_k$;
2. if $f(n) \neq k \neq f(m)$ and $g(n) \neq k \neq g(m)$, then $d_k^n <_P d_k^m$ if and only if $n <_{\mathbb{N}} m$;
3. if $f(n) \neq k$ and $g(n) \neq k$, then $a_k, b_k, c_k <_P d_k^n$;
4. if $f(n) = k$ and $m \neq n$, then $a_k, c_k <_P d_k^n <_P d_k^m$ and $b_k \perp_P d_k^n$;
5. if $g(n) = k$ and $m \neq n$, then $b_k, c_k <_P d_k^n <_P d_k^m$ and $a_k \perp_P d_k^n$.

Figure 3 contains the Hasse diagram of the most significant part of the restriction of $\leq_P$ to P_k in the three possible cases.

$\leq_P$ can be defined in RCA$_0$. Let $\mathbf{P} = (P, \leq_P)$.

Claim 6.5.1 $\mathbf{P}$ is a 1-1 interval order.
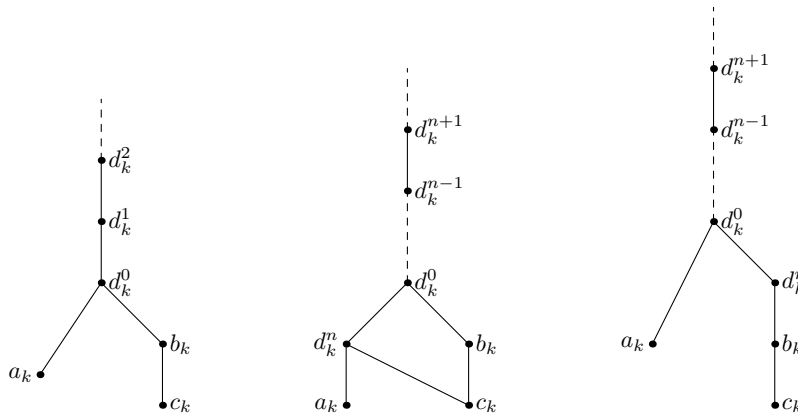
Figure 3 The three cases of $\leq_P$ restricted to P_k in the proof of Lemma 6.5: from left to right $\forall n\ f(n) \neq k \neq g(n)$, $f(n) = k$, and $g(n) = k$.

Proof   It is easy to check that **P** does not contain $\mathbf{2} \oplus \mathbf{2}$ and hence it is an interval order by Theorem 5.2. To prove the claim, by Lemma 3.12 it suffices to show that

$$\forall p, q \in P \; (p \neq q \implies p{\uparrow} \neq q{\uparrow} \vee p{\downarrow} \neq q{\downarrow}).$$

Fix $p, q \in P$ with $p \neq q$. If $p <_P q$ or $q <_P p$ then both $p{\uparrow} \neq q{\uparrow}$ and $p{\downarrow} \neq q{\downarrow}$ hold. If $p \perp_P q$ then $p, q \in P_k$ for some k, and we consider the different possibilities. In each case we exhibit an element of P witnessing either $p{\uparrow} \neq q{\uparrow}$ or $p{\downarrow} \neq q{\downarrow}$: $c_k \in b_k{\downarrow} \setminus a_k{\downarrow}$, $b_k \in c_k{\uparrow} \setminus a_k{\uparrow}$, if $f(n) = k$ then $a_k \in d_k^n{\downarrow} \setminus b_k{\downarrow}$, and if $g(n) = k$ then $b_k \in d_k^n{\downarrow} \setminus a_k{\downarrow}$.   □

By our hypothesis **P** is a closed interval order and there exist a linear order **L** and $f_0, f_1 : P \to L$ satisfying (c1) and (c2). Let $X = \{ k \mid f_1(a_k) \leq_L f_1(b_k) \}$. To complete the proof we need to check that $f(n) \in X$ and $g(n) \notin X$ for every n. If $k = f(n)$ then $f_1(a_k) <_L f_0(d_k^n) \leq_L f_1(b_k)$ and $k \in X$. If $k = g(n)$ then $f_1(b_k) <_L f_0(d_k^n) \leq_L f_1(a_k)$ and $k \notin X$.   □

We summarize our results in the following theorem (a few more implications equivalent to WKL_0 can be stated using the information contained in Figure 1, Corollary 6.2, and Lemmas 6.4 and 6.5).

Theorem 6.6 (RCA_0)   *The following are equivalent:*

(i) WKL_0;
(ii) *every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is a 1-1 interval order;*
(iii) *every interval order is a 1-1 interval order;*
(iv) *every 1-1 interval order is a distinguishing interval order;*
(v) *every 1-1 interval order is a closed interval order.*

Proof   The forward direction, that is, the fact that (i) implies each of (ii)–(v), is a consequence of Corollary 6.2. The implication (ii) $\implies$ (iii) follows from Theorem 3.13(iv). Lemma 6.4 shows that (iii) implies (i). The implication (iv) $\implies$ (v) is immediate by Theorem 3.13. Lemma 6.5 shows that (v) implies (i).   □

7   Proper Interval Orders

In this section we deal with proper interval orders. Throughout most of the section we point out the changes needed in the definitions and proofs of Sections 3 – 6. However, Theorem 7.16 is new, because its statement without "proper" is false by Lemma 6.5. The proof of Lemma 7.21 is also new, because the interval order used in the proof of Lemma 6.4 is not proper.

We start with the definitions and elementary facts corresponding to Section 3.

Definition 7.1 (RCA_0)   A partial order **P** is a *proper interval order* if there exist a linear order **L** and a set $F \subseteq P \times L$ such that (i1), (i2) of Definition 3.7 hold and, moreover,

(i4) $F(p) \subseteq F(q)$ implies $F(p) = F(q)$ for all $p, q \in P$.

P is a *proper 1-1 interval order* if (i3) of Definition 3.7 holds as well.

P is a *proper closed interval order* if there exist a linear order **L** and functions $f_0, f_1 : P \to L$ such that (c1), (c2) of Definition 3.7 hold and, moreover,

(c5) $f_0(p) <_L f_0(q)$ if and only if $f_1(p) <_L f_1(q)$ for all $p, q \in P$.

P is a *proper 1-1 closed interval order* if (c3) of Definition 3.7 holds as well. **P** is a *proper distinguishing interval order* if besides (c1), (c2), and (c5) we have also (c4).

Definition 7.2 (RCA$_0$)   A partial order **P** *does not contain* $3 \oplus 1$ if

$$\forall p_0, p_1, p_2, q \in P(p_0 <_P p_1 <_P p_2 \implies p_0 \leq_P q \vee q \leq_P p_2).$$

Lemma 7.3 (RCA$_0$)   *If* **P** *does not contain* $3 \oplus 1$ *then for every* $p, q \in P$ *we have either* $p \downarrow \subseteq q \downarrow$ *or* $p \uparrow \subseteq q \uparrow$.

Proof   Toward a contradiction assume that $p \downarrow \nsubseteq q \downarrow$ and $p \uparrow \nsubseteq q \uparrow$. If $p_0 \in p \downarrow \setminus q \downarrow$ and $p_2 \in p \uparrow \setminus q \uparrow$, then p_0, p, p_2, q witness that **P** contains $3 \oplus 1$.   □

Theorem 7.4 (RCA$_0$)

   (i) *Every proper (distinguishing) (1-1) (closed) interval order is a (distinguishing) (1-1) (closed) interval order.*
  (ii) *Every proper distinguishing interval order is a proper 1-1 closed interval order.*
 (iii) *Every proper 1-1 (closed) interval order is a proper (closed) interval order.*
 (iv) *Every proper (1-1) closed interval order is a proper (1-1) interval order.*
  (v) *Every proper interval order contains neither* $2 \oplus 2$ *nor* $3 \oplus 1$.

Proof   Statement (i) is immediate from the definitions. The statements in (ii) – (iv) are proved exactly as the corresponding statements in Theorem 3.13. To prove (v) let **P** be a proper interval order: by (i) above **P** is an interval order and by Theorem 3.13(iv) **P** does not contain $2 \oplus 2$.

To show that **P** does not contain $3 \oplus 1$ let L and F witness that **P** is a proper interval order, and suppose toward a contradiction that $p_0, p_1, p_2, q \in P$ are such that $p_0 <_P p_1 <_P p_2$, $p_0 \nleq_P q$ and $q \nleq_P p_2$. The second condition implies the existence of $x, y \in L$ such that $x \in F(p_0)$, $y \in F(q)$, and $y \leq_L x$. Similarly, by the third condition, there exist y', x' such that $y' \in F(q)$, $x' \in F(p_2)$, and $x' \leq_L y'$. For every $z \in F(p_1)$ the first condition implies $x <_L z <_L x'$: this implies, on one hand, $y, y' \notin F(p)$ and, on the other hand, $y <_L z <_L y'$ and hence $z \in F(q)$ by (i1), for all $z \in F(p_1)$. Therefore, $F(p_1) \subsetneqq F(q)$, contradicting condition (i4).   □

We now analyze finite partial orders containing neither $2 \oplus 2$ nor $3 \oplus 1$, imitating what we did in Section 4.

Definition 7.5 (RCA$_0$)   Given a finite partial order **P**, let $P^{\#} = P^*$ be defined as in Definition 4.1. Define a binary relation $\leq_{\mathbf{P}}^{\#}$ on $P^{\#}$ as follows:

$$p^+ \leq_{\mathbf{P}}^{\#} q^+ \iff p \uparrow^{\mathbf{P}} \nsupseteq q \uparrow^{\mathbf{P}} \vee (p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}} \wedge p \downarrow^{\mathbf{P}} \subseteq q \downarrow^{\mathbf{P}});$$

$$p^- \leq_{\mathbf{P}}^{\#} q^- \iff p \downarrow^{\mathbf{P}} \nsubseteq q \downarrow^{\mathbf{P}} \vee (p \downarrow^{\mathbf{P}} = q \downarrow^{\mathbf{P}} \wedge p \uparrow^{\mathbf{P}} \supseteq q \uparrow^{\mathbf{P}});$$

$$p^+ \leq_{\mathbf{P}}^{\#} q^- \iff p <_P q;$$

$$p^- \leq_{\mathbf{P}}^{\#} q^+ \iff q \nleq_P p.$$

$\mathbf{P}^{\#} = (P^{\#}, \leq_{\mathbf{P}}^{\#})$ is the *proper conjoint linear quasi order associated to* **P**. When **P** is clear from the context, we write $\leq^{\#}$ in place of $\leq_{\mathbf{P}}^{\#}$.

Remark 7.6   Notice that $\leq_{\mathbf{P}}^{\#}$ and $\leq_{\mathbf{P}}^{*}$ are defined on the same set. It is immediate that $\leq_{\mathbf{P}}^{\#} \subseteq \leq_{\mathbf{P}}^{*}$, and, in general, equality does not hold: in fact, if $p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}}$ it is always the case that $p^{+} \leq_{\mathbf{P}}^{*} q^{+}$, whereas $p^{+} \leq_{\mathbf{P}}^{\#} q^{+}$ fails when $p \downarrow^{\mathbf{P}} \not\subseteq q \downarrow^{\mathbf{P}}$.

The following lemma justifies the use of the words "linear quasi order" in Definition 7.5.

Lemma 7.7 (RCA$_0$)   *If* $\mathbf{P}$ *is a finite partial order which does not contain* $\mathbf{2} \oplus \mathbf{2}$ *then* $\leq^{\#}$ *is a linear quasi order. Moreover, if* $\mathbf{P}$ *does not contain* $\mathbf{3} \oplus \mathbf{1}$ *then* $\mathbf{P}^{\#}$ *and the functions* $p \mapsto p^{-}$, $p \mapsto p^{+}$ *show that* $\mathbf{P}$ *is a proper closed interval order.*

Proof   The proofs that $\leq^{\#}$ is a linear quasi order and that the functions $p \mapsto p^{-}$, $p \mapsto p^{+}$ witness that $\mathbf{P}$ is a closed interval order are identical to the same proofs for $\leq^{*}$ in Lemma 4.2. Hence we need only to show that condition (c5) of Definition 7.1 is met, that is, that $p^{-} <^{\#} q^{-}$ if and only if $p^{+} <^{\#} q^{+}$ for all $p, q \in P$.

Suppose $p, q \in P$ are such that $p^{-} <^{\#} q^{-}$ holds. Then either $p \downarrow \subsetneq q \downarrow$ or $p \downarrow = q \downarrow$ and $p \uparrow \not\supseteq q \uparrow$. In the first case, Lemma 7.3 implies that $q \uparrow \subseteq p \uparrow$; even if $q \uparrow = p \uparrow$ we have $q^{+} \not\leq^{\#} p^{+}$ (because $q \downarrow \not\subseteq p \downarrow$) and hence $p^{+} <^{\#} q^{+}$. In the second case, $p^{+} <^{\#} q^{+}$ is immediate. The reverse implication is proved similarly. $\square$

Remark 7.8   Remark 4.3 applies also to $\leq^{\#}$; that is, each $\equiv^{\#}$-equivalence class is contained in either P^{+} or P^{-}. Moreover, $p^{+} \equiv^{\#} q^{+}$ if and only if $p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}}$ and $p \downarrow^{\mathbf{P}} = q \downarrow^{\mathbf{P}}$ if and only if $p^{-} \equiv^{\#} q^{-}$. Therefore, the $\equiv^{\#}$-equivalence classes contained in P^{+} are paired in a straightforward way with those contained in P^{-}.

Definition 7.9 (RCA$_0$)   Given a finite partial order $\mathbf{P}$ which contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$, let $\mathbf{P}^{\#}$ be the proper conjoint linear quasi order associated to $\mathbf{P}$. A linear order $(P^{\#}, \leq_{L})$ is *compatible with* $\mathbf{P}^{\#}$ if

$$\forall x, y \in P^{\#} (x <^{\#} y \implies x <_{L} y),$$

$$\forall p, q \in P(p \neq q \wedge p^{+} \equiv^{\#} q^{+} \wedge p^{+} <_{L} q^{+} \implies p^{-} <_{L} q^{-}), \text{ and}$$

$$\forall p, q \in P(p \neq q \wedge p^{-} \equiv^{\#} q^{-} \wedge p^{-} <_{L} q^{-} \implies p^{+} <_{L} q^{+}).$$

(Actually, the second and third conditions imply each other.)

Remark 7.10   Defining $(P^{\#}, \leq_{L})$ compatible with $\mathbf{P}^{\#}$ means defining a linear order on each $\equiv^{\#}$-equivalence class and keeping the order between $\equiv^{\#}$-inequivalent elements unchanged. Moreover, we require that the linear orders on the $\equiv^{\#}$-equivalence classes containing p^{+} and p^{-} are the same.

Lemma 7.11 (RCA$_0$)   *If* $\mathbf{P}$ *is a finite partial order which contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$ *then there exists a linear order compatible with* $\mathbf{P}^{\#}$.

Proof   For example, let

$$x \leq_{L} y \iff x <^{\#} y \vee (x \equiv^{\#} y \wedge x \leq_{\mathbb{N}} y).$$

$\leq_{L}$ is a linear order compatible with $\mathbf{P}^{\#}$. $\square$

Lemma 7.12 (RCA$_0$)   *Any finite partial order which contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$ *is a proper distinguishing interval order.*

Proof   Let **P** be a finite partial order which contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$, and, by Lemma 7.11, $\leq_L$ a linear order compatible with $\mathbf{P}^{\#}$. Then $(P^{\#}, \leq_L)$ and the functions $p \mapsto p^-$, $p \mapsto p^+$ show that **P** is a proper distinguishing interval order. Indeed, if $p \neq q$ and, say, $p^+ \equiv^{\#} q^+$, then we have also $p^- \equiv^{\#} q^-$: if $p^+ <_L q^+$ then the second condition of Definition 7.9 implies $p^- <_L q^-$. $\square$

Combining Lemma 7.12 with Theorem 7.4 we obtain that RCA_0 proves the equivalence of the six characterizations of proper interval orders in the finite case.

Remark 7.13   Remark 4.8 applies also to what we have done with $\leq^{\#}$ in the previous lemmas, and we can conclude that ACA_0 suffices to prove the equivalence of the six characterizations of proper interval orders for countable partial orders. As with interval orders, we will obtain sharper results also for proper interval orders, in particular showing that all equivalences can be proved in WKL_0.

Remark 7.14   Notice that Lemma 4.9 does not hold with $\mathbf{P}^{\#}$ in place of $\mathbf{P}^*$. If $P = \{p, q, r\}$ is ordered by $\leq_P$ as $\mathbf{2} \oplus \mathbf{1}$ (i.e., the only nonreflexive relation is $p <_P q$) then $p^- <^{\#} r^- <^{\#} p^+ <^{\#} q^- <^{\#} r^+ <^{\#} q^+$.

Now we show that the upward pointing implications of Figure 2 are provable in RCA_0 much as we did with Figure 1 in Section 5.

Theorem 7.15 (RCA_0)   *Every proper closed interval order is a proper distinguishing interval order.*

Proof   We can repeat the proof of Theorem 5.1. One needs only to check that the construction preserves properness. We leave this to the reader. $\square$

As already noticed, the next theorem has no counterpart for arbitrary interval orders.

Theorem 7.16 (RCA_0)   *Every proper 1-1 interval order is a proper closed interval order.*

Proof   Let $\mathbf{L} = (L, F)$ witness that the partial order **P** is a proper 1-1 interval order.

Claim 7.16.1   For all $p, q \in P$ the following are equivalent:

(1) $p = q \vee \exists x, y \in L(x \in F(p) \setminus F(q) \wedge y \in F(q) \wedge x <_L y)$;
(2) $\forall x, y \in L(x \in F(p) \setminus F(q) \wedge y \in F(q) \implies x <_L y)$.

Proof   First assume that (1) holds and (2) fails. Since $p = q$ implies (2), there exist $x, y, x', y' \in L$ with $x, x' \in F(p) \setminus F(q)$, $y, y' \in F(q)$, $x <_L y$ and $y' <_L x'$. Let $z \in F(q)$: we have neither $z \leq_L x$ (because $x \notin F(q)$) nor $x' \leq_L z$ (because $x' \notin F(q)$). Hence $x <_L z <_L x'$ and $F(q) \subseteq F(p)$. Since it is immediate that $F(q) \neq F(p)$, we are contradicting condition (i4) in Definition 7.1.

Now assume (2) holds and (1) fails, so that in particular $p \neq q$ and hence $F(p) \neq F(q)$ because condition (i3) holds. If $F(p) \setminus F(q) = \varnothing$ then $F(q) \supseteq F(p)$ and we are again contradicting (i4). Therefore, we can choose $x \in F(p) \setminus F(q)$ and $y \in F(q)$: (2) implies $x <_L y$ and then we have (1), against our assumption. $\square$

Obviously, (1) is $\mathbf{\Sigma}_1^0$ and (2) is $\mathbf{\Pi}_1^0$. We denote either of them by $\varphi(p, q)$: φ is a provably $\mathbf{\Delta}_1^0$ formula and we can use it in the comprehension scheme. The following two claims about φ are useful.

Claim 7.16.2 $\varphi(p, q)$ implies $q \uparrow \subseteq p \uparrow$ and $p \downarrow \subseteq q \downarrow$.

Proof Let $r \in q\uparrow$: to show $r \in p\uparrow$, that is, $p <_P r$, by (i2) it suffices to show that $x <_L z$ for all $x \in F(p)$ and $z \in F(r)$. If $x \in F(q)$ this follows from $q <_P r$. If $x \in F(p) \setminus F(q)$ let $y \in F(q)$: we have $x <_L y <_L z$ and we are done. The proof that $p\downarrow \subseteq q\downarrow$ is even simpler. □

Claim 7.16.3 For every $p, q \in P$ either $\varphi(p, q)$ or $\varphi(q, p)$ holds.

Proof When $p = q$ the claim is obvious, so we assume $p \neq q$. Then $F(p) \neq F(q)$ by (i3), and by (i4) $F(p) \setminus F(q)$ and $F(q) \setminus F(p)$ are both nonempty. Let $x \in F(p) \setminus F(q)$ and $y \in F(q) \setminus F(p)$: if $x <_L y$ then $\varphi(p, q)$ holds; if $y <_L x$ then we have $\varphi(q, p)$. □

Let $P^{\#} = P^{+} \cup P^{-}$ and define $\leq_{L'}$ by

$$p^{+} \leq_{L'} q^{+} \iff \varphi(p, q);$$
$$p^{-} \leq_{L'} q^{-} \iff \varphi(p, q);$$
$$p^{+} \leq_{L'} q^{-} \iff p <_P q;$$
$$p^{-} \leq_{L'} q^{+} \iff q \not<_P p.$$

Reflexivity of $\leq_{L'}$ is immediate from the fact that $\varphi(p, p)$ holds for every p. To check transitivity start by noticing that using (2) it is immediate that $\varphi(p, q)$ and $\varphi(q, r)$ imply $\varphi(p, r)$. This gives two of the eight cases. The other four cases where some hypothesis is of the form $\varphi(p, q)$ are easily handled using Claim 7.16.2. Only two cases are left.

1. If $p^{+} \leq_{L'} q^{-} \leq_{L'} r^{+}$ then $p <_P q$ and $r \not<_P q$. Thus there exist $z \in F(r)$ and $y \in F(q)$ with $y \leq_L z$. Since $p \neq r$ we can pick $x \in F(p) \setminus F(r)$: we have $x <_L y$ and hence $x <_L z$. Therefore, $\varphi(p, r)$ and $p^{+} \leq_{L'} r^{+}$.
2. If $p^{-} \leq_{L'} q^{+} \leq_{L'} r^{-}$ then $q \not<_P p$ and $q <_P r$. Let $x \in F(p)$ and $y \in F(q)$ be such that $x \leq_L y$. Since $p \neq r$ we can choose $z \in F(r) \setminus F(p)$: $x <_L z$ follows immediately and hence we have that $\varphi(r, p)$ does not hold. By Claim 7.16.3 we have $\varphi(p, r)$ and $p^{-} \leq_{L'} r^{-}$.

The fact that $(P^{\#}, \leq_{L'})$ is linear follows immediately from the definition and Claim 7.16.3.

Define $f_0, f_1 : P \to P^{\#}$ as usual by $f_0(p) = p^{-}$ and $f_1(p) = p^{+}$. Conditions (c1), (c2), and (c5) follow immediately from the definition of $\leq_{L'}$. Therefore, **P** is a proper closed interval order. □

Remark 7.17 The reader may have noticed the construction of the proof of Theorem 7.16 satisfies also condition (c4). Therefore, the proof actually shows that RCA$_0$ suffices to prove that every proper 1-1 interval order is a proper distinguishing interval order. This result is also obtained combining the statements of Theorems 7.16 and 7.15.

Theorem 7.18 (RCA$_0$) *Every partial order which contains neither* $\mathbf{2} \oplus \mathbf{2}$ *nor* $\mathbf{3} \oplus \mathbf{1}$ *is a proper interval order.*

Proof The proof follows the pattern of the proof of Theorem 5.2: throughout the proof we replace $\mathbf{P}_s^{*}$ with $\mathbf{P}_s^{\#}$, the proper conjoint linear quasi order associated to $\mathbf{P}_s$. We point out only the spots where differences occur.

To prove the analogous of Claim 5.2.1, we need to consider the case of $n, m < s$ such that $p_n^+ <_{s-1}^{\#} p_m^+$ because $p_n \uparrow^{\mathbf{P}_{s-1}} = p_m \uparrow^{\mathbf{P}_{s-1}}$ and $p_n \downarrow^{\mathbf{P}_{s-1}} \subsetneqq p_m \downarrow^{\mathbf{P}_{s-1}}$. Besides Lemma 3.11, Lemma 7.3 (which uses the hypothesis that $\mathbf{P}$ does not contain $\mathbf{3} \oplus \mathbf{1}$) also is needed here: since $p_m \downarrow^{\mathbf{P}_s} \not\subseteq p_n \downarrow^{\mathbf{P}_s}$, we have $p_m \uparrow^{\mathbf{P}_s} \subseteq p_n \uparrow^{\mathbf{P}_s}$ and, therefore, $p_m \uparrow^{\mathbf{P}_s} \supsetneqq p_n \uparrow^{\mathbf{P}_s}$ cannot occur. Hence $p_n^+ <_s^{\#} p_m^+$. The analogous of Claim 5.2.2 states that at most two $\equiv_{s-1}^{\#}$-equivalence class contained in P_{s-1}^+ contain elements separated at s, and the same for $\equiv_{s-1}^{\#}$-equivalence classes contained in P_{s-1}^-.

The definition of $\leq_L$ on L_s requires considering a few more possible situations. When $n < s$ and p_n^+ is separated above at s, fix p_m^+ separated below at s with $p_m^+ \equiv_{s-1}^{\#} p_n^+$ and hence $x_m^{s-1} \equiv_L x_n^{s-1}$. If $p_n \uparrow^{\mathbf{P}_s} \subsetneqq p_m \uparrow^{\mathbf{P}_s}$ then no changes are needed, but now it might happen that $p_n \uparrow^{\mathbf{P}_s} = p_m \uparrow^{\mathbf{P}_s}$ (because $p_n \downarrow^{\mathbf{P}_s} \supsetneqq p_m \downarrow^{\mathbf{P}_s}$ forces $p_m^+ <_s^{\#} p_n^+$). In the latter case, x_n^s is an immediate successor of x_m^s, which by the other clauses in the definition is an immediate successor of $x_m^{s-1} \equiv_L x_n^{s-1}$. If p_n^- is separated below at s, act similarly.

If p_s^+ is neither the maximum of $\mathbf{P}^{\#}$ nor $\equiv_s^{\#} p_n^+$ for some $n < s$, let $z \in P_s^{\#}$ be an immediate successor of p_n^+ (now we cannot be sure that $z \in P_s^-$) and let x_s^s be an immediate predecessor of the element of $L_s \setminus L_{s-1}$ which corresponds to z. Proceed analogously for x_s^{-s}.

The definition of F (including Claim 5.2.3) and the proof that $\mathbf{L}$ witnesses that $\mathbf{P}$ is an interval order needs no changes. Thus we need only to show that condition (i4) is met. Assume $F(p_n) \subseteq F(p_m)$ and fix $s \geq \max(n, m)$. By condition (iii) we have $x_n^{-s} \leq_L x_m^{-s} <_L x_m^s \leq_L x_n^s$, and hence $p_n^- \leq_s^{\#} p_m^- <_s^{\#} p_m^+ \leq_s^{\#} p_n^+$. By Lemma 7.7 this implies that $p_n^- \equiv_s^{\#} p_m^-$ and $p_m^+ \equiv_s^{\#} p_n^+$, and hence $x_n^{-s} \equiv_L x_m^{-s}$ and $x_m^s \equiv_L x_n^s$. From the definition of F we get $F(p_n) = F(p_m)$, and the proof is complete. $\square$

We now conclude with results similar to the ones obtained in Section 6, showing that the implications missing from Figure 2 are equivalent to WKL_0.

Lemma 7.19 (WKL_0)   *Every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper distinguishing interval order.*

Proof   The proof of Lemma 6.1 works without major changes, replacing $\mathbf{P}_s^*$ with $\mathbf{P}_s^{\#}$. Obviously, we use Lemmas 7.7, 7.11, and 7.12 in place of Lemmas 4.2, 4.6, and 4.7. Notice that since (c5) is satisfied by each $\leq_{\alpha(s)}$ it is satisfied also by $(P^{\#}, \leq_L)$. $\square$

Corollary 7.20 (WKL_0)   *The five notions of proper interval order of Definition 7.1 and the property of containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ are all equivalent.*

Proof   This follows from Theorem 7.4 and Lemma 7.19. $\square$

Lemma 7.21 (RCA_0)   *If every closed interval order which is also a proper interval order is a proper closed interval order then WKL_0 holds.*

Proof   We will show that under our hypothesis (ii) of Lemma 6.3 holds. Fix one-to-one functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $\forall n, m\ f(n) \neq g(m)$. We want to find a set X such that $\forall n(f(n) \in X \wedge g(n) \notin X)$.

We define a partial order $\leq_P$ on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where $P_k = \{a_k, b_k\} \cup \{c_k^n \mid n \in \mathbb{N}\}$ for each k. If $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$

if and only if $k <_{\mathbb{N}} h$. The elements of each P_k are pairwise $\leq_P$-incomparable with the following exceptions:

1. if n is such that $f(n) = k$, then $a_k <_P c_k^n$;
2. if n is such that $g(n) = k$, then $c_k^n <_P a_k$.

$\leq_P$ can be defined within RCA_0. Let $\mathbf{P} = (P, \leq_P)$.

Claim 7.21.1 **P** is a closed interval order.

Proof Let $\mathbf{N} = (\mathbb{N}, \leq_{\mathbb{N}})$ and define $f_0, f_1 : \mathbb{N} \to P$ by setting

$$
\begin{aligned}
f_0(a_k) = f_1(a_k) &= 3k + 1; \\
f_0(b_k) &= 3k; \\
f_1(b_k) &= 3k + 2; \\
f_0(c_k^n) &= 3k && \text{if } f(n) \neq k; \\
f_1(c_k^n) &= 3k + 2 && \text{if } g(n) \neq k; \\
f_0(c_k^n) &= 3k + 2 && \text{if } f(n) = k; \\
f_1(c_k^n) &= 3k && \text{if } g(n) = k.
\end{aligned}
$$

It is straightforward to check that conditions (c1) and (c2) of Definition 3.7 are met. □

Claim 7.21.2 **P** is a proper interval order.

Proof Claim 7.21.1 and Theorem 3.13 imply that **P** does not contain $\mathbf{2} \oplus \mathbf{2}$. Our hypothesis on f and g imply that $c_k^n <_P a_k <_P c_k^m$ cannot occur: hence **P** does not contain $\mathbf{3} \oplus \mathbf{1}$. By Theorem 7.18, **P** is a proper interval order. □

Claims 7.21.1 and 7.21.2 and our hypothesis imply that **P** is a proper closed interval order. Hence there exist a linear order $\mathbf{L} = (L, \leq_L)$ and $f_0, f_1 : P \to L$ satisfying conditions (c1), (c2) of Definition 3.7 and condition (c4) of Definition 7.1. Let $X = \{ k \in \mathbb{N} \mid f_1(a_k) <_L f_1(b_k) \}$.

We now show that X satisfies $\forall n (f(n) \in X \wedge g(n) \notin X)$, thus completing the proof. If $f(n) = k$ then $a_k <_P c_k^n$ and $b_k \not<_P c_k^n$: hence $f_1(a_k) <_L f_0(c_k^n) \leq_L f_1(b_k)$ and $k \in X$. If $g(n) = k$ then $c_k^n <_P a_k$ and $c_k^n \not<_P b_k$: hence $f_0(b_k) \leq_L f_1(c_k^n) <_L f_0(a_k)$. From $f_0(b_k) <_L f_0(a_k)$, (c4) yields $f_1(b_k) <_L f_1(a_k)$ and hence $k \notin X$. □

Theorem 7.22 (RCA_0) *The following are equivalent:*

(i) WKL_0;
(ii) *every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper 1-1 interval order;*
(iii) *every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper closed interval order;*
(iv) *every proper interval order is a proper 1-1 interval order;*
(v) *every closed interval order which is also a proper interval order is a proper closed interval order.*

Proof The forward direction, that is, the fact that (i) implies each of (ii)–(v), is a consequence of Corollary 7.20. The implications (ii) $\implies$ (iii) and (iv) $\implies$ (v)

follow from Theorem 7.16. Theorem 7.4(v) shows (ii) $\implies$ (iv). The implication (iii) $\implies$(v) is immediate by Theorem 7.4. Lemma 7.21 shows that (v) implies (i). $\qquad\square$

References

[1] Balof, B., and K. Bogart, "Simple inductive proofs of the Fishburn and Mirkin theorem and the Scott-Suppes theorem," *Order*, vol. 20 (2003), pp. 49–51. Zbl 1031.06001. MR 1993409. 426

[2] Bogart, K. P., and D. B. West, "A short proof that 'proper = unit'," *Discrete Mathematics*, vol. 201 (1999), pp. 21–23. Zbl 0932.05065. MR 1687858. 426

[3] Cenzer, D., and J. B. Remmel, "Proof-theoretic strength of the stable marriage theorem and other problems," pp. 67–103 in *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1087.03040. MR 2185428. 427

[4] Cholak, P., A. Marcone, and R. Solomon, "Reverse mathematics and the equivalence of definitions for well and better quasi-orders," *The Journal of Symbolic Logic*, vol. 69 (2004), pp. 683–712. Zbl 1075.03030. MR 2078917. 427

[5] Downey, R. G., D. R. Hirschfeldt, S. Lempp, and R. Solomon, "Computability-theoretic and proof-theoretic aspects of partial and linear orderings," *Israel Journal of Mathematics*, vol. 138 (2003), pp. 271–89. Zbl 1044.03043. MR 2031960. 427

[6] Downey, R. G., and S. Lempp, "The proof-theoretic strength of the Dushnik-Miller theorem for countable linear orders," pp. 55–57 in *Recursion Theory and Complexity (Kazan, 1997)*, vol. 2 of *de Gruyter Series in Logic Applications*, de Gruyter, Berlin, 1999. Zbl 0951.03053. MR 1724931. 427

[7] Ershov, Yu. L., S. S. Goncharov, A. Nerode, J. B. Remmel, and V. W. Marek, editors, *Handbook of Recursive Mathematics. Vol. 2. Recursive Algebra, Analysis and Combinatorics*, vol. 139 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1998. Zbl 0905.03002. MR 1673582. 430

[8] Fishburn, P. C., "Intransitive indifference with unequal indifference intervals," *Journal of Mathematical Psychology*, vol. 7 (1970), pp. 144–49. Zbl 0191.31501. MR 0253942. 425, 426

[9] Fishburn, P. C., *Interval Orders and Interval Graphs. A Study of Partially Ordered Sets*, Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons Ltd., Chichester, 1985. Zbl 0551.06001. MR 776781. 425, 426, 427

[10] Friedman, H. M., "Metamathematics of comparability," pp. 201–18 in *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1083.03051. MR 2185435. 427

[11] Hirst, J. L., "A survey of the reverse mathematics of ordinal arithmetic," pp. 222–34 in *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1088.03047. MR 2185437. 427

[12] Marcone, A., "Wqo and bqo theory in subsystems of second order arithmetic," pp. 303–30 in *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1083.03052. MR 2185443. 427

[13] Mirkin, B. G., "Description of some relations on the set of real-line intervals," *Journal of Mathematical Psychology*, vol. 9 (1972), pp. 243–52. Zbl 0236.06002. MR 0316345. 426

[14] Montalbán, A., "Equivalence between Fraïssé's conjecture and Jullien's theorem," *Annals of Pure and Applied Logic*, vol. 139 (2006), pp. 1–42. Zbl 1094.03045. MR 2206250. 427

[15] Pouzet, M., and N. Sauer, "From well-quasi-ordered sets to better-quasi-ordered sets," 31 pages, available at http://arxiv.org/abs/math.CO/0601119, 2006. 426

[16] Roberts, F. S., "Indifference graphs," pp. 139–46 in *Proof Techniques in Graph Theory (Proceedings of the Second Ann Arbor Graph Theory Conference, Ann Arbor, Michigan, 1968)*, Academic Press, New York, 1969. Zbl 0193.24205. MR 0252267. 426

[17] Schmerl, J. H., "Reverse mathematics and graph coloring: Eliminating diagonalization," pp. 331–48 in *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1083.03053. MR 2185444. 427

[18] Schröder, B. S. W., *Ordered Sets. An Introduction*, Birkhäuser Boston Inc., Boston, 2003. Zbl 1010.06001. MR 1944415. 425

[19] Scott, D., and P. Suppes, "Foundational aspects of theories of measurement," *The Journal of Symbolic Logic*, vol. 23 (1958), pp. 113–28. Zbl 0084.24603. MR 0115919. 426

[20] Simpson, S. G., editor, *Reverse Mathematics 2001*, vol. 21 of *Lecture Notes in Logic*, Association for Symbolic Logic, La Jolla, 2005. Zbl 1075.03002. MR 2186912. 426

[21] Simpson, S. G., *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1999. Zbl 0909.03048. MR 1723993. 426, 430, 437, 438

[22] Trotter, W. T., "New perspectives on interval orders and interval graphs," pp. 237–86 in *Surveys in Combinatorics, 1997 (London)*, vol. 241 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1997. Zbl 0877.06001. MR 1477749. 425, 427

[23] Wiener, N., "A contribution to the theory of relative position," *Proceedings of the Cambridge Philosophy Society*, vol. 17 (1914), pp. 441–49. 425

Dipartimento di Matematica e Informatica
Università di Udine
viale delle Scienze 206
33100 Udine
ITALY
marcone@dimi.uniud.it