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Ages of Expansions of ω-Categorical Structures

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Abstract The age of a structure M is the set of all isomorphism types of finite substructures of M. We study ages of generic expansions of ω-stable ω-categorical structures.

1 Introduction

Expansions of ω-stable structures by so-called generic relations have become one of the main sources of examples of simple theories (see [4] and the bibliography of that paper). In this paper we concentrate on generic expansions of ω-stable ω-categorical structures and show that their ages have some interesting properties.

Let M be a first-order structure of a language $\mathcal{L}$. The age of M, denoted by $\mathcal{J}(M)$, is the set of all isomorphism types of finite substructures of M. This notion appears in several places of model theory and permutation group theory (see [3] and [9]). We consider the question how the age of M determines the ages of its expansions. In particular, we study what conditions on the structure M and a relation R imply that the expansion (M, R) has the following property: for any countable locally finite $\mathcal{L}$-structure N with $\mathcal{J}(M) = \mathcal{J}(N)$, there is a relation P on N with $\mathcal{J}(M, R) = \mathcal{J}(N, P)$.

It turns out that the case when M is an ω-stable ω-categorical structure becomes very natural in questions of this kind. In Section 2 we study this case and find some conditions on M and R in terms of forking, which guarantee the property above. We also show that in our situation, generic expansions by unary predicates studied in [4] satisfy these conditions. In Section 3 we study when generic expansions in the sense of category [7] satisfy conditions introduced in Section 2.

We now describe some preliminary material and motivations. We fix a countable structure M of a language $\mathcal{L}$. We assume that M is ω-categorical. Let $\mathcal{L}'$

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be the extension of the language $\mathcal{L}$ by additional relational and functional symbols $\bar{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_t)$ (so, it is possible that some $\mathbf{r}_i$ denotes a function). Any $\mathcal{L}'$-expansion $(M, \bar{\mathbf{s}})$ is determined by the interpretation $\bar{\mathbf{s}}$ of the symbols from $\bar{\mathbf{r}}$ on M. When we write $(M, \bar{\mathbf{r}})$ we mean that such an interpretation is already defined.

We say that the expansion $(M, \bar{\mathbf{r}})$ is *$\mathcal{J}$-generic* if for any countable locally finite $\mathcal{L}$-structure N with $\mathcal{J}(M) = \mathcal{J}(N)$, there is an expansion $(N, \bar{\mathbf{s}})$ with $\mathcal{J}(M, \bar{\mathbf{r}}) = \mathcal{J}(N, \bar{\mathbf{s}})$. Any countable structure can be considered as a $\mathcal{J}$-generic expansion of the structure which is a countable set without any relations. Moreover, if M is *absolutely ubiquitous*,[1] any expansion of M is $\mathcal{J}$-generic by obvious reasons.

We expect that ω-stable ω-categorical structures are similar to absolutely ubiquitous ones in respects involving the age. For example, there is a conjecture that every absolutely ubiquitous structure in a finite language is ω-stable [7]. On the other hand, since typical ω-stable ω-categorical structures have generic automorphisms and generic unary predicates, it is natural to expect that the corresponding expansions are $\mathcal{J}$-generic.

The following example provides an expansion by constants of a simple ω-categorical structure, which is not $\mathcal{J}$-generic. It illustrates typical circumstances forbidding $\mathcal{J}$-genericity. In our main results we will make assumptions which eliminate examples of this kind.

Example 1.1 Let V be a symplectic vector space over a finite field K (a classical structure of type (b)(iii) from [8]). By $\langle -, - \rangle$ we denote the corresponding bilinear form (always satisfying $\langle v, v \rangle = 0$). In a big elementary extension of V, find a vector a orthogonal to V. Then the structure $V' = \mathrm{acl}(V \cup \{a\})$ is of the same age as V and has a nontrivial radical (containing a). It is clear that no constant expansion $(V, \langle -, - \rangle, c)$ has the same age as $(V', \langle -, - \rangle, a)$ (the same statement holds for the expansion of V' by the unary predicate "to be equal to a"). This means that $(V', \langle -, - \rangle, a)$ (and the expansion by the corresponding unary predicate) is not $\mathcal{J}$-generic. The theory of our structure is simple of SU-rank 1.

It is easy to see that in this example the structure $(V', \langle -, - \rangle)$ does not have elimination of quantifiers: there is a c satisfying the same quantifier-free formulas as a, but $\mathrm{tp}(c/\varnothing) \neq \mathrm{tp}(a/\varnothing)$. To forbid $\mathcal{J}$-nongenericity based on reasons of this kind, we will concentrate on expansions of structures M having elimination of quantifiers. In fact, the main results of the paper can be considered as an attempt heading to the statement that ω-stable ω-categorical structures with elimination of quantifiers have $\mathcal{J}$-generic expansions in natural languages. We also suspect that expansions of such structures which are generic in some traditional sense of this word usually are $\mathcal{J}$-generic. Our results below also confirm this suspicion.

The following example shows that when M is not absolutely ubiquitous there can be expansions which are not $\mathcal{J}$-generic and this can happen even when M is ω-stable ω-categorical and has elimination of quantifiers.

Example 1.2 Let E be an equivalence relation on ω consisting of infinitely many infinite classes and let $M = (\omega, E)$. It is easy to see M is ω-stable ω-categorical and has elimination of quantifiers. Let $\prec$ be an ordering of M of type ω and a_0 be the first element of the corresponding enumeration. Let $R(x, y)$ be defined by $R(a, b) \Leftrightarrow$ "b is the $\prec$-successor of a". We claim that (M, R, a_0) *is not $\mathcal{J}$-generic.* Indeed let N be a structure of an equivalence relation consisting of infinitely many finite classes whose sizes are unbounded. Then $\mathcal{J}(M) = \mathcal{J}(N)$. If $\mathcal{J}(M, R, a_0) = \mathcal{J}(N, P, c_0)$

then P is a binary relation defining an enumeration of a substructure of N (with the first element c_0) which is isomorphic to M. This is a contradiction with the fact that N does not have substructures of this kind.

When M is considered in the original language $\mathcal{L}$ we will use standard notation: $\mathrm{tp}(A/B)$, $\mathrm{stp}(A/B)$, $\mathrm{dcl}(A)$, $\mathrm{acl}(A)$, and so on. For $A \subset M$ by $\langle A \rangle$ we denote a substructure of M generated by A. When M is considered in the language extended by relations $\bar{\mathbf{r}}$ we will write $\mathrm{tp}_{\bar{\mathbf{r}}}(A/B)$, $\mathrm{stp}_{\bar{\mathbf{r}}}(A/B)$, $\mathrm{dcl}_{\bar{\mathbf{r}}}(A)$, and $\mathrm{acl}_{\bar{\mathbf{r}}}(A)$.

2 $\mathcal{J}$-Generic Relations

In this section we prove the main theorem of this paper. First we introduce the notion of $\mathcal{J}$-smooth relations on stable, ω-categorical structures.

Definition 2.1 Let M be a stable ω-categorical structure. We say that an expansion $(M, \bar{\mathbf{r}})$ is $\mathcal{J}$-smooth if for any finite substructures A and B of M there is a substructure A' so that

(1) $(A', \bar{\mathbf{r}}) \cong (A, \bar{\mathbf{r}})$,
(2) $\mathrm{tp}(A') = \mathrm{tp}(A)$,
(3) A' and B are independent over $\varnothing$.

Theorem 2.2 *Let M be an ω-stable ω-categorical structure having quantifier elimination. Let $(M, \bar{\mathbf{r}})$ be a $\mathcal{J}$-smooth ω-categorical expansion of M. Then for any countable locally finite structure N with $\mathcal{J}(M) = \mathcal{J}(N)$ there is an expansion $(N, \bar{\mathbf{s}})$ of N such that $\mathcal{J}(M, \bar{\mathbf{r}}) = \mathcal{J}(N, \bar{\mathbf{s}})$.*

Note that although the structure M from Example 1.2 is ω-stable ω-categorical and has elimination of quantifiers, the corresponding expansion (M, R, a_0) does not satisfy the assumptions of Theorem 2.2. It is neither ω-categorical (moreover, the reduct to (R, a_0) is not ω-categorical) nor $\mathcal{J}$-smooth. The latter follows from the fact that condition (1) of the definition of $\mathcal{J}$-smoothness implies that for $A \subset M$ with $a_0 \in A$, any A' as in (1) contains a_0 too, contradicting (3) for $B = \{a_0\}$.

The proof of Theorem 2.2 is based on the following lemma.

Lemma 2.3 *Under the circumstances of Theorem 2.2 let $A, B < M$, $B' < N$ be finite substructures such that $B \cong B'$. Then there are $C \subset M$, $C' \subset N$ and an isomorphism $f : \langle B, C \rangle \to \langle B', C' \rangle$ such that $f(B) = B'$ and $(A, \bar{\mathbf{r}}) \cong (C, \bar{\mathbf{r}})$.*

Proof Let p be the complete type $\mathrm{tp}(A)$ in the pure language of M. Since the $\bar{\mathbf{r}}$-expansion of M is $\mathcal{J}$-smooth, there is $C \subset M$ such that $(C, \bar{\mathbf{r}}) \cong (A, \bar{\mathbf{r}})$ and $\mathrm{tp}(C/B)$ is a nonforking extension of p. Since M is ω-categorical, the number of strong types over $\varnothing$ extending p is finite. Let $p_1, p_2, \ldots, p_n$ be an enumeration of all extensions of this form. Since the U-rank of B over $\varnothing$ is finite, we can fix a natural number $t > U(B/\varnothing)$. Find a family of subsets $C_{i,j}$ of M for $i = 1, \ldots, n$ and $j = 1, \ldots, t$ such that $C_{1,1} = C$, $C_{i,j} \models p_i$ and for $j \neq k$ sets $C_{i,j}$ and $C_{i,k}$ are independent realizations of the same strong type over $\varnothing$. Let $Q = \langle C_{i,j} \rangle_{i \leq n, j \leq t}$. Find an isomorphism $g_0 : Q \to Q'$ with $Q' \subset N$. Let $S' = \langle B', Q' \rangle$. Since M admits elimination of quantifiers, every isomorphism between finite substructures of M can be extended to an automorphism of M. Using this we can find $S < M$ and an isomorphism $g_1 : S \to S'$ such that $B \subseteq S$ and $g_1(B) = B'$. Let $D_{i,j} = g_1^{-1} g_0(C_{i,j})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, t$.

Using elimination of quantifiers we see that for all $i \neq j$ and $k \neq l$, $\mathrm{stp}(D_{i,k}) = \mathrm{stp}(D_{i,l})$, $\mathrm{stp}(D_{i,k}) \neq \mathrm{stp}(D_{j,k})$, and sets $D_{i,k}$ and $D_{i,l}$ are independent realizations of the same strong type over $\varnothing$. We claim that

for every i there is $k_i \leq t$ such that D_{i,k_i} and B are independent over $\varnothing$.

If not, consider the chain of types $q_0 \subseteq q_1 \subseteq \cdots \subseteq q_t$, where $q_0 = \mathrm{tp}(B/\varnothing)$ and $q_j = \mathrm{tp}(B/D_{i,1}D_{i,2}\ldots D_{i,j})$. It is easy to see that for every j the type q_{j+1} is a forking extension of the type q_j, so $U(B/\varnothing) \geq t$, a contradiction.

Since some permutation of the sequence $D_{1,k_1}, D_{2,k_2}, \ldots, D_{n,k_n}$ is a sequence of realizations of the corresponding types $p_1, p_2, \ldots, p_n$, then by the Finite Equivalence Relation Theorem there is j such that $\mathrm{tp}(C/B) = \mathrm{tp}(D_{j,k_j}/B)$. Thus $\langle B, C\rangle \cong \langle B', g_1(D_{j,k_j})\rangle$ and the statement of the lemma holds for $C' = g_1(D_{j,k_j})$. □

Proof of Theorem 2.2 Let $M_0 = \mathrm{acl}_{\bar{\mathbf{r}}}(\varnothing)$. Since $(M, \bar{\mathbf{r}})$ is ω-categorical, M_0 is finite. We represent $N = \bigcup_{i<\omega} N_i$ where all N_i are finite and N_0 is isomorphic to M_0. For each $i \leq |\bar{\mathbf{r}}|$ we construct the ith relation $\mathbf{s}_i$ from $\bar{\mathbf{s}}$ on N as the union of finite relations $\mathbf{s}_i^j$ defined on some N_{n_j}.

Suppose that we have already defined the relations $\bar{\mathbf{s}}^k = (\mathbf{s}_1^k, \mathbf{s}_2^k, \ldots)$ on the corresponding N_{n_k}, $n_k < \omega$. Here we assume that $(N_{n_k}, \bar{\mathbf{s}}^k)$ is isomorphic to some $(M_{n_k}, \bar{\mathbf{r}}) < (M, \bar{\mathbf{r}})$ such that $(M_{n_k}, \bar{\mathbf{r}})$ contains all isomorphism types of k-generated substructures of $(M, \bar{\mathbf{r}})$. Since $(M, \bar{\mathbf{r}})$ is ω-categorical, there is a finite substructure $(A, \bar{\mathbf{r}}) < (M, \bar{\mathbf{r}})$ containing all isomorphism types of $(k+1)$-generated substructures.

By Lemma 2.3 there are $C < M$, $C' < N$ and an isomorphism f such that f maps $\langle M_{n_k}, C\rangle$ onto $\langle N_{n_k}, C'\rangle$, $f(M_{n_k}) = N_{n_k}$ and $(A, \bar{\mathbf{r}}) \cong (C, \bar{\mathbf{r}})$. Let n_{k+1} be the minimal natural number such that $\langle N_{n_k}, C'\rangle < N_{n_{k+1}}$. Since M admits elimination of quantifiers and $\mathcal{J}(M) = \mathcal{J}(N)$, there are $M_{n_{k+1}} < M$ and an isomorphism $g : M_{n_{k+1}} \to N_{n_{k+1}}$ such that $\langle C, M_{n_k}\rangle < M_{n_{k+1}}$ and $g|_{\langle C, M_{n_k}\rangle} = f$. Let us define the relations $\bar{\mathbf{s}}^{k+1}$ on N as follows:

$$\mathbf{s}_j^{k+1}(\bar{a}) \text{ if and only if } \bar{a} \in N_{n_{k+1}} \text{ and } M \models \mathbf{r}_j(g^{-1}(\bar{a})).$$

Let $\bar{\mathbf{s}} = \bigcup_{i<\omega} \bar{\mathbf{s}}^i$. The conclusion of the theorem is obvious. □

The following notion appears in [4] as a notion of generic relations. We slightly change the terminology.

Definition 2.4 Let M be a structure such that $\mathrm{Th}(M)$ admits quantifier elimination and elimination of quantifier $\exists^\infty$. A unary relation P of some sort of M is mc-generic[2] if for every formula $\varphi(\bar{x}, \bar{z})$, $\bar{x} = (x_1, \ldots, x_n)$, for every subset I of $\{1, \ldots, n\}$, the following sentence holds in (M, P):

$$\forall \bar{z}(\exists \bar{x}(\varphi(\bar{x}, \bar{z}) \wedge (\mathrm{acl}_T(\bar{z}) \cap \bar{x} = \varnothing) \wedge \bigwedge_{i \neq j} x_i \neq x_j) \to$$

$$\exists \bar{x}(\varphi(\bar{x}, \bar{z}) \wedge \bigwedge_{i \in I} x_i \in P \wedge \bigwedge_{i \notin I} x_i \notin P)).$$

It is proved in [4] that $\mathrm{Th}(M)$ together with all sentences appearing in the definition of mc-generic relations axiomatizes the model completion of P-expansions of models of $\mathrm{Th}(M)$.

Proposition 2.5 *Let M be a stable ω-categorical structure having quantifier elimination. If P is an mc-generic unary relation on some sort of M then (M, P) is a $\mathcal{J}$-smooth expansion of M.*

Proof Let $A, B < M$ be finite and $|A \setminus \mathrm{acl}(\varnothing)| = r$. Since M is ω-categorical, we can enumerate all r-types over B as follows: $p_1, p_2, \ldots, p_k, \ldots, p_n$, where we assume that for $i = 1, \ldots, k$ the types p_i are all nonforking extensions of $\mathrm{tp}(A \setminus \mathrm{acl}(\varnothing))$. Take formulas $\varphi_i(\bar{x})$ over B isolating the corresponding types p_i. Then $\Phi(\bar{x}) = \vee_{i=1}^{k} \varphi_i(\bar{x})$ expresses that $\mathrm{tp}(\bar{x}/B)$ is a nonforking extension of $\mathrm{tp}(A \setminus \mathrm{acl}(\varnothing))$. Since P is mc-generic on M, there is $A' < M$ such that $A' \setminus \mathrm{acl}(\varnothing)$ realizes $\Phi(\bar{x})$ and $(A, P) \cong (A', P)$. This A' is composed from $A \cap \mathrm{acl}(\varnothing)$ and an appropriate realization of Φ. As a result we have that P is $\mathcal{J}$-smooth on M. $\square$

This proposition together with Theorem 2.2 implies that in the ω-stable case mc-generic relations define $\mathcal{J}$-generic expansions of M.

It is mentioned in Remark 2.12(2) of [4] that an mc-generic relation of an imaginary sort can define an n-ary relation on M for some $n > 1$. For example, the random graph can be defined in the theory of pure infinite sets as an mc-generic relation on the sort of all 2-element subsets. In this case Proposition 2.5 tells us that the random graph is a $\mathcal{J}$-smooth and $\mathcal{J}$-generic relation.

3 Generic Expansions of ω-Categorical Structures

We fix a countable structure M in a language $\mathcal{L}$. We assume that M is ω-categorical (most of the terms below make sense under the assumption that M is atomic). Let T be an extension of $\mathrm{Th}(M)$ in the language with additional relational and functional symbols $\bar{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_t)$. We assume that T is axiomatizable by sentences of the following form:

$$(\forall \bar{x})(\bigvee_i (\varphi_i(\bar{x}) \wedge \psi_i(\bar{x}))),$$

where φ_i is a quantifier-free formula in the language $\mathcal{L} \cup \bar{\mathbf{r}}$, and ψ_i is a first-order formula of the language $\mathcal{L}$. Consider the set $\mathbf{X}$ of all possible expansions of M to models of T.

Following [7] we define for a tuple $\bar{a} \subset M$ a *diagram* $\varphi(\bar{a})$ of $\bar{\mathbf{r}}$ on $\bar{a}$. To every functional symbol from $\bar{\mathbf{r}}$ we associate a partial function from $\bar{a}$ to $\bar{a}$. Choose a formula from every pair $\{\mathbf{r}_i(\bar{a}'), \neg\mathbf{r}_i(\bar{a}')\}$, where $\mathbf{r}_i$ is a relational symbol from $\bar{\mathbf{r}}$ and $\bar{a}'$ is a tuple from $\bar{a}$ of the corresponding length. Then $\varphi(\bar{a})$ consists of the conjunction of the chosen formulas and the definition of the chosen functions (so, in the functional case we look at $\varphi(\bar{a})$ as a tuple of partial maps).

Consider the class $\mathbf{B}_T$ of all theories $D(\bar{a})$, $\bar{a} \subset M$ such that each of them consists of $\mathrm{Th}(M, \bar{a})$ and a diagram of $\bar{\mathbf{r}}$ on $\bar{a}$ satisfied in some $(M, \bar{\mathbf{r}}) \models T$. We order $\mathbf{B}_T$ by extension: $D(\bar{a}) \leq D'(\bar{b})$ if $\bar{a} \subset \bar{b}$ and $D'(\bar{b})$ implies $D(\bar{a})$ under T (in particular, the partial functions defined in D' extend the corresponding partial functions defined in D). Since M is an atomic model, each element of $\mathbf{B}_T$ is determined by a formula of the form $\varphi(\bar{a}) \wedge \psi(\bar{a})$, where ψ is a complete formula for M and φ is a diagram of $\bar{\mathbf{r}}$ on $\bar{a}$. The corresponding formula $\varphi(\bar{x}) \wedge \psi(\bar{x})$ will be called *basic*.

On the set $\mathbf{X} = \{(M, \bar{\mathbf{r}}') : (M, \bar{\mathbf{r}}') \models T\}$ of all $\bar{\mathbf{r}}$-expansions of the structure M we consider the topology generated by basic open sets of the form $[D(\bar{a})] = \{(M, \bar{\mathbf{r}}') : (M, \bar{\mathbf{r}}') \models D(\bar{a})\}$, $\bar{a} \subset M$. It is easily seen that any $[D(\bar{a})]$

is clopen. The topology is metrizable: fix an enumeration $\bar{a}_0, \bar{a}_1, \ldots$ of $M^{<\omega}$ and define

$d((M, \bar{\mathbf{r}}'), (M, \bar{\mathbf{r}}'')) = \sum\{2^{-n}$: there is a symbol $\mathbf{r} \in \bar{\mathbf{r}}$ such that its interpretations on $\bar{a}_n$ in the structures $(M, \bar{\mathbf{r}}')$ and $(M, \bar{\mathbf{r}}'')$ are not the same (if $\mathbf{r}$ is a functional symbol then $\mathbf{r}'(\bar{b}) \neq \mathbf{r}''(\bar{b})$ for some $\bar{b} \subseteq \bar{a}_n$)$\}$.

It is easily seen that the metric d defines the topology determined by the sets of the form $[D(\bar{a})]$.

By the assumptions on T (T is axiomatizable by sentences which are universal with respect to symbols from $\bar{\mathbf{r}}$) the space $\mathbf{X}$ forms a closed subset of the complete metric space of all $\bar{\mathbf{r}}$-expansions of M. Thus $\mathbf{X}$ is complete and the Baire Category Theorem holds for $\mathbf{X}$. We say that $(M, \bar{\mathbf{r}}) \in \mathbf{X}$ is *generic* if the class of its images under $\mathrm{Aut}(M)$ is comeagre in $\mathbf{X}$ [7].

Notice that the space $\mathrm{Aut}(M)$ under the conjugacy action, and generic automorpisms (introduced in [10]), provide a particular example of this construction. Indeed, identify each $\alpha \in \mathrm{Aut}(M)$ with the expansion (M, α, α^{-1}). The class of structures of this form is axiomatized in the language of M with the functional symbols $\{\alpha, \beta\}$ by $\mathrm{Th}(M)$, the sentence $\alpha\beta(x) = \beta\alpha(x) = x$, and universal sentences asserting that α preserves the relations of M. Also, any partial isomorphism $\bar{a} \to \bar{a}'$ can be viewed as the diagram corresponding to the maps $\bar{a} \to \bar{a}'$ and $\bar{a}' \to \bar{a}$. It is clear that a generic automorphism α (see [10]) defines generic expansion (M, α, α^{-1}).

We now give a general statement describing when generic expansions of a stable ω-categorical structure are $\mathcal{J}$-generic.

Proposition 3.1 *Let M be a stable ω-categorical structure. Let T be an extension of* $\mathrm{Th}(M)$ *in the language with additional relational and functional symbols* $\bar{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_t)$ *satisfying all assumptions of the section. Assume that the set* $\mathbf{X}$ *of appropriate expansions of M has a generic structure and* $\mathbf{B}_T$ *has the joint embedding property of the following form:*

for any $D_1(\bar{a}), D_2(\bar{b}) \in \mathbf{B}_T$ there exist $D(\bar{c}) \in \mathbf{B}_T$ and M-elementary maps $\delta : \bar{a} \to \bar{c}$ and $\sigma : \bar{b} \to \bar{c}$ such that $D(\bar{c})$ extends $D_1(\delta(\bar{a})) \cup D_2(\sigma(\bar{b}))$ and $\delta(\bar{a})$ is independent from $\sigma(\bar{b})$ over $\varnothing$.

Then every generic expansion of M is $\mathcal{J}$-smooth.

Proof Let $(M, \bar{\mathbf{s}})$ be a generic expansion from $\mathbf{X}$. For any pair $D_1(\bar{a})$ and $D_2(\bar{b})$ from $\mathbf{B}_T$ define a set

$\mathbf{S}_{\mathrm{ind}}(D_1, D_2) = \{(M, \bar{\mathbf{r}}) \in \mathbf{X} : (M, \bar{\mathbf{r}}) \not\models D_1(\bar{a})$ or there is a partial M-elementary map σ such that $D_2(\sigma(\bar{b}))$ holds in $(M, \bar{\mathbf{r}})$ and $\bar{a}$ is independent from $\sigma(\bar{b})$ over $\varnothing\}$.

It is clear that any set of this form is open. To see that $\mathbf{S}_{\mathrm{ind}}(D_1, D_2)$ is dense take any $D^*(\bar{d}) \in \mathbf{B}_T$. If there is no $(M, \bar{\mathbf{h}})$ satisfying both $D_1(\bar{a})$ and $D^*(\bar{d})$ then any element of the basic clopen set determined by $D^*(\bar{d})$ belongs to $\mathbf{S}_{\mathrm{ind}}(D_1, D_2)$. If $D_1(\bar{a}) \cup D^*(\bar{d})$ holds in some element of $\mathbf{X}$ then by homogeneity of M and the joint embedding property from the formulation of the proposition we see that there is $D_3(\bar{c}) \in \mathbf{B}_T$ and an M-elementary map $\sigma : \bar{b} \to \bar{c}$ such that $D_3(\bar{c})$ extends $D_1(\bar{a}) \cup D^*(\bar{d}) \cup D_2(\sigma(\bar{b}))$ and $\bar{a}$ is independent from $\sigma(\bar{b})$ over $\varnothing$. Then each element of $\mathbf{X}$ satisfying $D_3(\bar{c})$ belongs to the intersection of $\mathbf{S}_{\mathrm{ind}}(D_1, D_2)$ and the basic clopen set defined by $D^*(\bar{d})$.

As a result the intersection of all sets of the form $\mathbf{S}_{\mathrm{ind}}(D_1, D_2)$ is comeager in $\mathbf{X}$. It is clear that this set is $\mathrm{Aut}(M)$-invariant. We see that the expansion $(M, \bar{\mathbf{s}})$ belongs to this intersection. This obviously implies that $(M, \bar{\mathbf{s}})$ is $\mathcal{J}$-smooth. $\square$

By this proposition and Theorem 2.2 we obtain that if M is an ω-stable ω-categorical structure with elimination of quantifiers, then the joint embedding property in the form above implies that every ω-categorical generic (with respect to an appropriate theory T) expansion is $\mathcal{J}$-generic.

The case of expansions by automorphisms provides a typical illustration of Proposition 3.1. This is based on the fact from [6] that every ω-stable ω-categorical structure has an amalgamation base. In particular, Proposition 3.4 of [6] (and its proof) implies that under additional assumptions that M has elimination of quantifiers and the property $\mathrm{dcl}^{\mathrm{eq}} = \mathrm{acl}^{\mathrm{eq}}$, all substructures of M expanded by n automorphisms form a universal class with the joint embedding property and the amalgamation property. Moreover, the corresponding amalgamation (respectively, joint embedding) has the following canonical form: if $(A, \bar{\alpha}) < (B, \bar{\beta})$ and $(A, \bar{\alpha}) < (C, \bar{\gamma})$, then $\bar{\beta}$ and $\bar{\gamma}$ can be amalgamated to a tuple of automorphisms of a substructure of M generated by copies of B and C which are independent over A.

Corollary 3.2 *Let M be an ω-stable ω-categorical $\mathcal{L}$-structure having elimination of quantifiers and the property* $\mathrm{acl}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}$. *Let T be a universal theory of expansions of $\mathcal{L}$-structures by automorphisms $(\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1})$. Assume that*

(a) *the class of finite models of T is closed with respect to the canonical amalgamation (joint embedding) above;*
(b) *there is a function $f : \omega \to \omega$ such that T implies all sentences of the following form: "any m elements generate a substructure of size $\leq f(m)$ in the language $\mathcal{L} \cup (\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1})$".*

Then M has generic T-expansions which are $\mathcal{J}$-generic.

Proof Let K be the class of all finite substructures of M. Let K_a be the class of all structures of the form $(C, \alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1})$ where $C \in K$ and $(C, \alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1}) \models T$. By conditions (a),(b) and Fraissé's Theorem (Theorem 2.4 of [5]) there is an ω-categorical structure which is universal homogeneous with respect to K_a. It is easy to see that its L-reduct is isomorphic to M. By (a) and Proposition 3.1 the structure obtained is a $\mathcal{J}$-smooth expansion of M. By Theorem 2.2 it is $\mathcal{J}$-generic. $\square$

An infinite vector space over a finite field can be taken as a structure M in the corollary. Let m be a natural number and T be the universal theory of the corresponding expansions (N, α, α^{-1}), $N < M$, defined by the axiom $\alpha^m = \mathrm{id}$. Then it is easy to see that the assumptions of Corollary 3.2 are satisfied in this case.

It is worth noting that the assumptions of Corollary 3.2 imply that the resulting expansions are ω-categorical and thus cannot be generic in the sense of [6]. Note that in fact the universal theory T corresponding to the situation of [6] is the theory of all expansions of substructures of M by automorphisms.

Similar considerations can be applied in the following general situation. Let M be an ω-categorical structure in a language $\mathcal{L}$. Let $\bar{\mathbf{r}}$ be a tuple of relations on M and T be $\mathrm{Th}(M)$ extended by all the sentences from $\mathrm{Th}(M, \bar{\mathbf{r}})$ of the form $\forall \bar{x} \neg D(\bar{x})$, where $D(\bar{x})$ is basic for $(M, \bar{\mathbf{r}})$. It is clear that T satisfies the conditions from the beginning of the section. Note that $\mathbf{B}_T$ consists of all diagrams $D(\bar{b})$ such that the corresponding formula $D(\bar{x})$ is realizable in $(M, \bar{\mathbf{r}})$. The expansion $(M, \bar{\mathbf{r}})$ is *ubiquitous in category* if $(M, \bar{\mathbf{r}})$ is generic with respect to $\mathbf{B}_T$ [7]. It is clear that generic expansions (with respect to some theory T') are always ubiquitous in category.

The easiest $\mathcal{J}$-generic expansion of this sort can be obtained when $\bar{\mathbf{r}}$ consists of one unary predicate P. This obviously follows from Proposition 2.5 and the following proposition.

Proposition 3.3 *Let M be a stable ω-categorical structure admitting elimination of quantifiers. Let (M, P) be an mc-generic expansion.*[3] *Then (M, P) is ω-categorical and ubiquitous in category.*

Proof By Corollary 2.6 of [4] for any $\bar{a} \subseteq M$ the type $\mathrm{tp}(\bar{a}/\varnothing)$ of $\mathrm{Th}(M, P)$ is determined by the isomorphism type of the substructure $\mathrm{acl}(\bar{a})$ in the extended language. Since M is ω-categorical we see that

(i) such a type is determined in (M, P) by a formula of the form $\exists \bar{y} D(\bar{x}\bar{y})$ where $D(\bar{x}\bar{y})$ is basic,
(ii) given the length of $\bar{a}$ the number of such formulas is finite.

Theorem 1.5 of [7] states that

> a structure $(M, \bar{\mathbf{r}})$ is ubiquitous in category if and only if every complete type over $\varnothing$ realizable in $(M, \bar{\mathbf{r}})$ is determined in $(M, \bar{\mathbf{r}})$ by a formula of the form $\exists \bar{y} D(\bar{x}\bar{y})$ where $D(\bar{x}\bar{y})$ is basic.

Applying this theorem we obtain that (M, P) is ubiquitous in category. Conditions (i) and (ii) immediately imply that (M, P) is ω-categorical. □

We now see that for ω-stable ω-categorical structures M, structures of the form $(M, \bar{\mathbf{r}})$ which are ubiquitous in category, may serve many natural examples of $\mathcal{J}$-generic expansions. Notice that expansions arising in Corollary 3.2 are also ubiquitous in category.

There are nice examples of $\mathcal{J}$-generic expansions of slightly different nature.

Example 3.4 This example is based on Proposition 5.8 from [9]. Let A be a vector space over $GF(2)$ with a basis $\{e_i, f_i : i < \omega\}$ and θ be the automorphism of A defined by $\theta(e_i) = f_i$ and $\theta(f_i) = e_i$ for all $i < \omega$. Consider the structure (A, θ). Since A is a $GF(2)[\theta]$-module decomposable into the direct sum of submodules generated by $\langle e_i, f_i \rangle$, the structure (A, θ) is ω-categorical and ω-stable (see [1] and Theorem 7 from Appendix of [2]). On the other hand, it is not absolutely ubiquitous. Indeed, the module $A \oplus GF(2)w$ with $\theta(w) = w$ can be embedded into A by a map taking w to $\langle e_0 + f_0 \rangle$ and all e_i, f_i to e_{i+1}, f_{i+1}. Now it is easy to see that $\mathcal{J}(A, \theta) = \mathcal{J}(A \oplus GF(2)w, \theta)$ but $(A, \theta) \not\cong (A \oplus GF(2)w, \theta)$.

To see that (A, θ) admits elimination of quantifiers it suffices to prove that a pp-formula[4] of the form

$$\varphi(x_2, \ldots, x_l) = (\exists x_1) \bigwedge_{j \le n} \left(\sum \varepsilon_{j,i} x_i + \sum \tau_{j,i} \theta(x_i) = 0 \right), \text{ where } \varepsilon_{j,i}, \tau_{j,i} \in \{0, 1\},$$

is equivalent to a quantifier-free pp-formula (we use the fact that for every module, every formula is equivalent to a Boolean combination of pp-formulas, Theorem 1.1 from [11]). The latter is clear if for some $j \le n$ exactly one of the coefficients $\varepsilon_{j,1}$, $\tau_{j,1}$ equals 0 (then we apply an appropriate substitution). If for all $j \le n$, $\varepsilon_{j,1} + \tau_{j,1} = 0$, then (assuming that $\varepsilon_{1,1} \neq 0$) we express

$x_1 + \theta(x_1) = \sum_{i\neq 1} \varepsilon_{1,i} x_i + \sum_{i\neq 1} \tau_{1,i}\theta(x_i)$ and substitute this into all remaining equations. We also add the equation

$$\sum_{i\neq 1} \varepsilon_{1,i} x_i + \sum_{i\neq 1} \tau_{1,i}\theta(x_i) = \sum_{i\neq 1} \varepsilon_{1,i}\theta(x_i) + \sum_{i\neq 1} \tau_{1,i} x_i .$$

The rest is clear.

Claim 3.5 *Let α be an automorphism of (A, θ) such that (A, θ, α) is ω-categorical. Then (A, θ, α) is a $\mathcal{J}$-generic expansion of (A, θ).*

Indeed, let (A', δ) be a countable structure of the same age as (A, θ). Consider A' as a vector space over $GF(2)$ and choose a maximal linearly independent subset B with the condition $b \in B \Rightarrow (\delta(b) \in B \wedge b \neq \delta(b))$. Let $A_0 = \langle B\rangle$ and $A' = A_0 \oplus A_1$ for appropriate A_1. It is easy to see that for any $c \in A_1$, $\delta(c) = c$. We also have that $(A_0, \delta) \cong (A, \theta)$. We will use this to obtain an appropriate expansion of (A', δ).

Since A is an ω-categorical $GF(2)[\theta, \alpha]$-module, by [1] it can be decomposed into a direct sum of the following form:

$$C_1^{(\omega)} \oplus C_2^{(\omega)} \oplus \cdots \oplus C_n^{(\omega)} \oplus C',$$

where C' is finite and each $C_i^{(\omega)}$ is the direct sum of ω copies of an indecomposable $GF(2)[\theta, \alpha]$-submodule C_i. Then A is isomorphic to

$$A \oplus C_1^{(\omega)} \oplus C_2^{(\omega)} \oplus \cdots \oplus C_n^{(\omega)}.$$

Since each C_i contains a nontrivial element fixed by α and θ (for $c \in C_i$ take the sum of the $\langle\alpha, \theta\rangle$-orbit of c), the module A has a submodule which is isomorphic to $A \oplus C$ where C is an infinite $GF(2)$-vector space fixed by α and θ pointwise. Since $(A, \theta) \cong (A_0, \delta)$ and A_1 can be embedded into C, we can define an automorphism β on $A_0 \oplus A_1$ so that A_1 is fixed pointwise and A_0 is isomorphic with A as an $GF(2)[\delta, \beta]$-module. It is obvious that $\mathcal{J}(A, \theta, \alpha) = \mathcal{J}(A', \delta, \beta)$.

Notes

1. That is, M is uniformly locally finite and any countable locally finite L-structure N with $\mathcal{J}(M) = \mathcal{J}(N)$ is isomorphic with M [9].
2. Model completion.
3. Our assumptions imply all conditions needed for its existence [4].
4. See [11].

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