Notre Dame Journal of Formal Logic
Volume 48, Number 2, 2007

Semantical Characterizations for Irreflexive and Generalized Modal Languages

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Abstract This paper deals with two main topics: One is a semantical investigation for a bimodal language with a modal operator ■ associated with the intersection of the accessibility relation R and the inequality $\neq$. The other is a generalization of some of the former results to general extended languages with modal operators. First, for our language $\mathcal{L}_{\Box\blacksquare}$, we prove that Segerberg's theorem (equivalence between finite frame property and finite model property) fails and establish both van Benthem-style and Goldblatt-Thomason-style characterizations. We extract the notion of ■-realizer (a generalization of bulldozing) as an essence from the proofs of our results. Second, we generalize the notion of ■-realizer and prove quite general versions of these semantical characterization results. The known and previously unknown characterization results for almost all of the languages extended with modal operators already proposed will be immediate corollaries.

1 Introduction

As is well known, the standard modal propositional language $\mathcal{L}_{\Box}$ with $\Box$ cannot define all the natural assumptions about the accessibility relation R on the set W of states, for example, irreflexivity and partial ordering. However, we need irreflexivity to define the notion of strict partial ordering, which often appears in mathematics as well as the notion of partial ordering. Additionally, in the context of temporal logic, we usually postulate at least that R is irreflexive and transitive (see, e.g., *the flow of time* in [8]). It has been known that we can prove the completeness with respect to the frames with undefinable properties ([7], [8]) by adding Gabbay-style nonorthodox rules without changing the language. Especially, these studies have focused on irreflexivity. In order to overcome the lack of expressive power of $\mathcal{L}_{\Box}$,

Received May 1, 2006; accepted November 28, 2006; printed May 14, 2007
2000 Mathematics Subject Classification: Primary, 03B45; Secondary, 03B44, 03C40
Keywords: extended modal model theory, (modality for) irreflexivity, Goldblatt-Thomason theorem, characterization of modal definability, finite model property, finite frame property

on the other hand, extensions with various tools have been proposed, for example, the difference operator D (see [19], $\overline{\mathrm{D}}$ is the dual of D), the nominals i (see [1]), the global modality A (see [9]), and the satisfaction operator $@_i$ (see [3]). Their precise expressive powers (i.e., modal definability) have been captured (see [9], [11], [12], [19], [4]) within semantical frameworks for them, that is, the notion of Kripke frame (set of states with an accessibility relation) and the notion of Kripke model (frame with a valuation assigning a set of states to each propositional letter).

Most languages for these extensions are strong enough to express trichotomy, $R = W^2$, and so on, as well as irreflexivity and partial ordering. However, they are too strong for us since the validity in them is not preserved under taking disjoint unions or generated subframes. In other words, if we add, for example, the global modality A or the difference operator D to the unimodal language, then we can refer to states that are not R-connected from the current state. In the context of tense logic, for example, this means that we can refer to states (i.e., instants) that do not belong to the time-series in which the current (i.e., the present) exists. Since all those extended languages already introduced that can define irreflexivity and partial ordering can also define linearity, $R = W^2$ and so on, one may think that the expressive powers needed to define these properties are the same, or that, to define irreflexivity and partial ordering, it is needed to break the closure properties mentioned above. The purpose of our investigation is to show that this is not the case, that is, to separate the expressive powers for irreflexivity and partial ordering from those for linearity, for $R = W^2$ and so on, by proposing a new modest extension with the following features: (i) its validity is closed under disjoint unions and generated subframes; (ii) its expressive power is strong enough to express irreflexivity and partial ordering at least; (iii) it has nice properties, for example, Kripke completeness and finite frame property (FFP).

For this purpose, the first author of the present paper proposed a new extension of the standard modal language ([21], [20]), which consisted in adding an operator $\blacksquare$ associated with the intersection of the accessibility relation R and the inequality $\neq$ (i.e., $(R \cap \neq)$). He has already proved ([21], [20]) that various normal modal logics (some of which are kinds of Lemmon-Scott's Axioms $\Diamond^m \Box^n A \supset \Box^j \Diamond^k A$ $(m, n, j, k \in \omega)$) in his extended language enjoy Kripke completeness and FFP. He has also shown that $\blacksquare p \supset \Box p$ defines irreflexivity and $\blacksquare p \supset \blacksquare\blacksquare p$ defines the conjunction of antisymmetry and transitivity, though antisymmetry is not independently definable. The definability of these properties of frames witnesses the strength of his extension.

We can also point out that his extension has the following two connections to other works. First, a similar operator has been proposed as an auxiliary operator in the context of proof search for unimodal logics ([18], [17]) in order to avoid loop checking. Especially, when we construct a counter model in **KT** according to [18], Ch. 2, this auxiliary operator has the same interpretation as $\blacksquare$. Second, in the context of dynamic logic, the loop operator $\boxdot$ (or, the loop constant loop), associated with the intersection of R and the equality $=$ (i.e., $(R \cap =)$), has been discussed ([10], [12], [5]). Since the loop operator and $\blacksquare$'s intersections with $\Box$ are relative complements of each other, our investigation of the bimodal language with $\blacksquare$ could be an indirect study of the loop operator.

In the spirit of the Goldblatt-Thomason theorem ([11], Theorem 8), which captures the expressive power of modal logic, that is, characterizes definable elementary classes of Kripke frames in modal-model-theoretic terms, this paper will give characterizations for the precise expressive power of our extension. By this characterization, we can see, from the modal-model-theoretic viewpoints, how much stronger the expressive power of our extension is. We also compare our language with two other languages, $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$, the unimodal language plus D associated with the inequality $\neq$ on W, and we conclude that our extension is more modest than $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ because of the closure properties under disjoint unions and under generated subframes; that is, our extension has a more internal and local perspective on Kripke frames than $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ and so matches with our original motivation. In particular, we will prove that, within the class of all finite and transitive frames, our extension is more expressive than the unimodal language and less expressive than $\mathcal{L}_{\Box\overline{\mathrm{D}}}$.

To obtain these results, we employ a modal-model-theoretic approach with our new notion of *realizer*, which has some advantages over the algebraic one and Gabbay-style one. First, in order to characterize the definability in our extension, we need a method of connecting an algebraically nice class of frames (bimodal or multimodal frames) with our interesting class of frames (unimodal frames as $\mathcal{L}_{\Box\blacksquare}$-frames or $\mathcal{L}_{\{\beta_i\}_i}$-frames (see Section 6)). The notion of $\blacksquare$-realizer gives us a simpler method for the connection in our approach, whereas, in the algebraic approach, such a method is very complicated. Second, in most works extending modal languages, the Gabbay-style nonorthodox rule [7] allows us to prove the Kripke completeness of a wide range of logics in a uniform way. Without such a rule, we have to use some complicated frame constructions, such as bulldozing, to prove Kripke completeness in each case. In this paper, we will extract the notion of $\blacksquare$-realizer from such constructions and use it essentially. If we bring both Kripke completeness and semantical characterization into view, the notion of $\blacksquare$-realizer turns out to be quite useful, whereas Gabbay-style rules could conceal the essence of the construction.

In order to emphasize the advantages of our approach and the strong utility of our notion of realizer, in Section 6, we will generalize this notion and the characterization of definability to a wide class of multimodal languages. These languages allow any operators associated with relations defined by Boolean combinations of the accessibility relation and the equality, that is, quantifier-free formulas, and they include almost all extensions by modal operators that have already been introduced. As corollaries, in a quite uniform way, we obtain several known characterizations, for example, for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ and those of the extensions whose characterizations were previously unknown.

Up to now, the characterization for each of these extensions has been given case by case (except in [12], though the base language there is fixed and too strong for us) and, whenever operators are proposed, the new problem of characterization comes up with them. Our results, however, clear up such problems once and for all. It must be emphasized that this is a fundamental result in the extended modal model theory, comparable with the Goldblatt-Thomason theorem in modal model theory and with series of works for capturing the expressive power of first-order logic, due to Kochen [15], Keisler [14], and Shelah [22] (or [6], Corollary 6.1.16), in first-order model theory.

Let us explain the contents briefly. Section 2 introduces the basic notions including $\blacksquare$-realizer. In Section 3, we prove that Segerberg's theorem fails in our extension. It has already been shown that this theorem fails in $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ ([19], p. 578) and so this can be seen as a price for the strong expressive power compared to the unimodal language.

Section 4 investigates the characterization of modal definability for classes of $\mathcal{L}_{\Box\blacksquare}$-models. First, we introduce the notion of bisimulations and prove the results corresponding to the van Benthem Characterization Theorem in our extension. From these results, we show that our extension of the unimodal language differs from $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ at the level of Kripke models. Finally, we prove that modally definable classes of pointed $\mathcal{L}_{\Box\blacksquare}$-models (i.e., Kripke models with a state) can be characterized by using ultraproducts and bisimulations between $\mathcal{L}_{\Box\blacksquare}$-models.

Section 5 investigates the modal definability of classes for $\mathcal{L}_{\Box\blacksquare}$-frames. For our extension, we prove Goldblatt-Thomason-style theorems both for classes of finite transitive $\mathcal{L}_{\Box\blacksquare}$-frames and for elementary classes of $\mathcal{L}_{\Box\blacksquare}$-frames.

Section 6 introduces a multimodal setting and generalizes the notion of the $\blacksquare$-realizer and characterization results from Sections 4 and 5. First, we mention a van Benthem-style characterization theorem on this setting, though it seems to be a kind of folklore among modal model theorists. Second, we define the notion of absoluteness, which allows us to consider multimodally generated subframes as $\mathcal{L}_{\{\beta_i\}_i}$-frames (roughly, unimodal frames), and a new frame construction called *amalgamation*, a generalization of disjoint union. Finally, we prove Goldblatt-Thomason-style theorems on this generalized setting.

2 Preliminaries

The modal languages we consider have (i) an infinite set Φ of propositional letters p_i ($i \in I$), (ii) the propositional connectives $\sim$, $\supset$, and (iii) finitely many unary modal operators, for example, $\Box$, $\blacksquare$, $\overline{\mathrm{D}}$. The standard modal language $\mathcal{L}_{\Box}$ has the operator $\Box$. (In general, $\mathcal{L}_{O_1 \cdots O_n}$ denotes the modal language with only operators $O_1, \ldots, O_n$, for example, $\mathcal{L}_{\Box\blacksquare}$, $\mathcal{L}_{\Box\overline{\mathrm{D}}}$.) The well-formed formulas of each language are defined as usual. In addition to the usual abbreviations for conjunction $\wedge$, disjunction $\vee$, logical equivalence $\equiv$, the verum $\top$, and the falsum $\bot$, we use the following: $\Diamond A := \sim \Box \sim A$, $\blacklozenge A := \sim \blacksquare \sim A$, and $\mathrm{D}A := \sim \overline{\mathrm{D}} \sim A$. We use $A, B, C, \ldots$ to denote formulas and $\Gamma, \Delta, \ldots$ to denote sets of formulas.

A *bimodal frame* is a triple $\mathfrak{F} = \langle W, R, S \rangle$ of a nonempty set W, called a *domain*, and two binary relations R, S on W. A *bimodal model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ of a bimodal frame $\mathfrak{F} = \langle W, R, S \rangle$ and a mapping $V : \Phi \to \mathcal{P}(W)$. A *unimodal frame* and a *unimodal model* are defined similarly. When there is no room for misunderstanding, as a general convention we assume that $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', V' \rangle$. $|\mathfrak{M}|$ or $|\mathfrak{F}|$ means the domain of a model $\mathfrak{M}$ or a frame $\mathfrak{F}$, respectively. For any binary relation Q on W, $Q[w]$ denotes $\{ w' \mid wQw' \}$.

For a bimodal model $\mathfrak{M} = \langle W, R, S, V \rangle$ or unimodal model $\mathfrak{M} = \langle W, R, V \rangle$, $w \in |\mathfrak{M}|$ and a formula A of a bimodal or unimodal language, the satisfaction relation $\mathfrak{M}, w \Vdash A$ is defined as usual. In modal languages containing $\Box$, we assume that $\Box$ is associated with R; that is, $\mathfrak{M}, w \Vdash \Box A$ if and only if $(\forall w' \in W)$ $[wRw'$ implies $\mathfrak{M}, w' \Vdash A]$ and that the other modal operator (e.g., $\blacksquare$ or $\overline{\mathrm{D}}$), if it exists, is associated with S.

For a model $\mathfrak{M}$ and w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \Sigma$ means that $\mathfrak{M}, w \Vdash A$ for any $A \in \Sigma$. We say that w and w' are *modally equivalent* (written $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$ or, simply, $w \leftrightsquigarrow w'$ if the models are clear from the context) if $\mathfrak{M}, w \Vdash A \iff \mathfrak{M}', w' \Vdash A$, for every modal formula A. Note that we define this notion in each language mentioned above.

A formula A is *valid in a model* $\mathfrak{M}$ (written $\mathfrak{M} \Vdash A$) if $\mathfrak{M}, w \Vdash A$ for any w in $\mathfrak{M}$. A is *valid in a frame* $\mathfrak{F}$ (written $\mathfrak{F} \Vdash A$) if $\langle \mathfrak{F}, V \rangle \Vdash A$ for any valuation $V : \Phi \to \mathcal{P}(|\mathfrak{F}|)$. A is *satisfiable in a model* $\mathfrak{M}$ *or a frame* $\mathfrak{F}$ if $\mathfrak{M} \nVdash \sim A$ or $\mathfrak{F} \nVdash \sim A$, respectively. A is *valid in a class* M *of models* or F *of frames* (written $\mathsf{M} \Vdash A$ or $\mathsf{F} \Vdash A$) if it is valid in every $\mathfrak{M} \in \mathsf{M}$ or $\mathfrak{F} \in \mathsf{F}$, respectively. For a set of formulas, these notions are defined similarly.

A bimodal frame $\langle W, R, S \rangle$ satisfying $S = (R \cap \neq)$ is called an $\mathcal{L}_{\Box\blacksquare}$*-frame*, where $w(R\cap\neq)w'$ means that wRw' and $w \neq w'$. A bimodal frame satisfying $S = \{\langle w, w' \rangle \mid w \neq w'\}$ is called an $\mathcal{L}_{\Box\overline{\mathrm{D}}}$*-frame*. The notions of $\mathcal{L}_{\Box\blacksquare}$*-model* and of $\mathcal{L}_{\Box\overline{\mathrm{D}}}$*-model* are defined similarly. Thus, for any $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \blacksquare A$ if and only if $(\forall w' \in W)$ $[w(R\cap\neq)w'$ implies $\mathfrak{M}, w' \Vdash A]$ and, for any $\mathcal{L}_{\Box\overline{\mathrm{D}}}$-model $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \overline{\mathrm{D}}A$ if and only if $(\forall w' \in W)$ $[w \neq w'$ implies $\mathfrak{M}, w' \Vdash A]$.

Note that an $\mathcal{L}_{\Box\blacksquare}$-frame $\langle W, R, (R\cap\neq) \rangle$ or -model is determined by the unimodal frame $\langle W, R \rangle$ or model, respectively. Therefore, we often confuse $\langle W, R \rangle$ and $\langle W, R, (R\cap\neq) \rangle$ for an $\mathcal{L}_{\Box\blacksquare}$-frame. We are mainly interested in unimodal frames and models regarded as $\mathcal{L}_{\Box\blacksquare}$- or $\mathcal{L}_{\Box\overline{\mathrm{D}}}$-frames and models. Bimodal frames and models, in general, are only for technical purposes, not of our original interest.

The *first-order frame language* $\mathcal{L}^f$ is the first-order language that has the identity symbol $\approx$ together with the binary predicate symbol $\mathbf{R}$. We denote $\mathcal{L}^m(\Phi)$ (if Φ is clear from the context, we simply write $\mathcal{L}^m$) as the *first-order model language* which is the expanded language of $\mathcal{L}^f$ with unary predicates $\mathbf{P}_i$ associated with the propositional letters $p_i \in \Phi$. We write $\alpha(x)$ or $\beta(v_1, v_2)$ to denote a formula α with at most one free variable x or two distinct free variables v_1, v_2, respectively.

Note that an $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{M} = \langle W, R, V \rangle$ can be seen as the $\mathcal{L}^m$-structure defined as follows: $|\mathfrak{M}| = W$, $\mathbf{R}^{\mathfrak{M}} = R$, $\mathbf{P}_i{}^{\mathfrak{M}} = V(p_i)$, for any $p_i \in \Phi$. An $\mathcal{L}_{\Box\blacksquare}$-frame $\mathfrak{F}$ can be seen as the $\mathcal{L}^f$-structure defined similarly. $\mathfrak{M} \models \alpha[\overrightarrow{a}]$, where $\overrightarrow{a} = \langle a_1, \ldots, a_n \rangle$ is an n-tuple from $|\mathfrak{M}|$, means the usual satisfaction relation (for details, see, e.g., [6], Ch. 1).

Definition 2.1 (Bimodal p-morphism) Let $\mathfrak{F} = \langle W, R, S \rangle$ and $\mathfrak{F}' = \langle W', R', S' \rangle$ be bimodal frames. A mapping $f : W \to W'$ is a *bimodal p-morphism from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ (written $f : \mathfrak{F} \to \mathfrak{F}'$) if it satisfies the following:

(R-forth) If wRv, then $f(w)R'f(v)$,

(R-back) If $f(w)R'v'$, then wRv and $f(v) = v'$ for some $v \in W$,

and (S-forth) and (S-back) defined similarly. For bimodal models $\mathfrak{M}$ and $\mathfrak{M}'$, $f : |\mathfrak{M}| \to |\mathfrak{M}'|$ is a bimodal p-morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ (written $f : \mathfrak{M} \to \mathfrak{M}'$) if $f : \mathfrak{F} \to \mathfrak{F}'$ with $w \in V(p)$ if and only if $f(w) \in V'(p)$ for each $p \in \Phi$ and each $w \in |\mathfrak{M}|$. If there is a bimodal p-morphism f from $\mathfrak{M}$ to $\mathfrak{M}'$ such that f is surjective as a mapping between domains, $\mathfrak{M}'$ is called a p-morphic image of $\mathfrak{M}$ (written $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$). For any bimodal frames $\mathfrak{F}, \mathfrak{F}'$, $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ is defined similarly.

Given two unimodal frames or models, (unimodal) p-morphism between them is defined by using the clauses (R-forth) and (R-back).

It is known that the following hold (see, e.g., [2], Proposition 2.14, and [2], Theorem 3.14).

Fact 2.2

1. For any bimodal models $\mathfrak{M}$ and $\mathfrak{M}'$ with $f : \mathfrak{M} \to \mathfrak{M}'$, and for any w in $\mathfrak{M}$, $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', f(w)$.
2. For any bimodal frames $\mathfrak{F}$ and $\mathfrak{F}'$ with $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, and for any formula A, $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$.

Observe that these two facts hold even if we restrict our attention to $\mathcal{L}_{\Box\blacksquare}$-frames and -models. For bimodal p-morphisms between $\mathcal{L}_{\Box\blacksquare}$-frames, observe that ($R$-back) implies (($R\cap\neq$)-back). This is proved as follows: if $f(w)(R'\cap\neq)v'$, then by (R-back), we have $v \in W$ with $v' = f(v)$, wRv and $w \neq v$ by $f(w) \neq f(v)$.

Next, we introduce the notion of realizer and realizations. These are generalizations of bulldozing ([2], ch. 4.5, pp. 222–4).

Definition 2.3 ($\blacksquare$-realizer, $\blacksquare$-realization) Let $\mathfrak{F}$ be a bimodal frame, $\mathfrak{F}'$ an $\mathcal{L}_{\Box\blacksquare}$-frame. If a bimodal p-morphism $f : \mathfrak{F}' \to \mathfrak{F}$ is surjective as a mapping between domains, f is called a $\blacksquare$-*realizer* and $\mathfrak{F}'$ is a $\blacksquare$-*realization* of $\mathfrak{F}$. For a bimodal model $\mathfrak{M}$ and an $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{M}'$, these notions are defined in the same way as for frames.

Proposition 2.4 *Any bimodal frame and bimodal model with* $(R\cap\neq) \subset S \subset R$ *have* $\blacksquare$*-realizations.*

Proof Suppose $(R\cap\neq) \subset S \subset R$. Let $C = \{ w \in |\mathfrak{F}| \mid wSw \}$, $W^- = W \setminus C$, $2 = \{0, 1\}$, and $W' = W^- \cup (C \times 2)$, where we may assume $W^- \cap (C \times 2) = \varnothing$. $f : W' \to W$ is defined as follows: $f(a) := a$ if $a \in W^-$; w if $a = \langle w, i \rangle$ for some $i \in 2$ and some $w \in C$. Write an $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{M}' = \langle W', R', S', V' \rangle$, where $R' = \{ \langle a, b \rangle \mid f(a)Rf(b) \}$, $S' = (R'\cap\neq)$, and $V'(p) = f^{-1}[V(p)] = \{ a \mid f(a) \in V(p) \}$ for any $p \in \Phi$. It suffices to prove that f is a bimodal p-morphism from $\langle W', R', S' \rangle$ to $\langle W, R, S \rangle$. Since f is surjective by definition and $aR'b$ is equivalent to $f(a)Rf(b)$ for any $a, b \in W'$, we have (R-forth) and (R-back). Finally, we show that (S-forth) and (S-back) hold.

(S-forth) Assume $a(R'\cap\neq)b$.

Case 1 If $f(a) = f(b)$, then, by $a \neq b$, $f(a) \in C$ and so $f(a)Sf(b)$.

Case 2 Suppose $f(a) \neq f(b)$. Since $aR'b$, we have $f(a)Rf(b)$ and, by $(R\cap\neq) \subset S$, we have $f(a)Sf(b)$.

(S-back) Assume $f(a)Sy$. We prove that $a(R'\cap\neq)b$ and $f(b) = y$ for some $b \in W'$.

Case 1 Suppose $f(a) \in C$. Write $c = f(a)$. We can assume $a = \langle c, 1 \rangle$.

Case 1A Suppose $c = y$. Since cSc and $S \subset R$, we have cRc and $a(R'\cap\neq)\langle c, 0 \rangle$ with $f(\langle c, 0 \rangle) = c = y$.

Case 1B Suppose $c \neq y$. Since f is surjective, $f(b) = y$ for some $b \in W'$. Since $f(a)Sf(b)$ and $S \subset R$, we have $f(a)Rf(b)$ and so $aR'b$. By $f(a) \neq f(b)$, $a \neq b$.

Case 2 Suppose $f(a) \in W^-$. Since f is surjective, $f(b) = y$ for some $b \in W'$. Since $f(a)Sf(b)$ and $S \subset R$, $f(a)Rf(b)$. Then we have $aR'b$. By $f(a) \in W^-$, $f(a) \neq f(b)$, and so $a \neq b$. □

For bimodal frames and models, the notions of $\overline{\mathrm{D}}$-*realizer* and of $\overline{\mathrm{D}}$-*realization* are defined similarly. The same construction as $\overline{\mathrm{D}}$-realization was used in the completeness proof of the basic system of $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ ([16], Theorem 4.3.3); namely, any bimodal frame and bimodal model with $(S\cup =) = W^2$ have $\overline{\mathrm{D}}$-realizations.

3 FMP ≠ FFP

Segerberg (1971) proved that FMP and FFP (defined below) are equivalent in $\mathcal{L}_\Box$; that is, a normal modal logic Λ of $\mathcal{L}_\Box$ enjoys FFP if and only if it enjoys FMP. One can find the proof, for example, in [2], Theorem 3.28. We give a counterexample **KsT**$_\blacksquare$ (defined below) in $\mathcal{L}_{\Box\blacksquare}$ to Segerberg's Theorem. We prove that **KsT**$_\blacksquare$ is $\mathcal{L}_{\Box\blacksquare}$-frame incomplete (Lemma 3.5) and that **KsT**$_\blacksquare$ enjoys FMP with respect to some class M_1 (see Definition 3.6) of "nice" models (Lemma 3.12).

It is known that the classical modal logic **K** is Kripke complete with respect to the class of unimodal frames ([2], Theorem 4.23). Therefore, we define the notion of normal modal logic of $\mathcal{L}_\Box$ as follows: A *normal modal logic* Λ (of $\mathcal{L}_\Box$) is a set of formulas that contains all classical tautologies and $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and that is closed under modus ponens (MP), uniform substitution and $\Box$-rule, that is, the rule that from A we may infer $\Box A$.

Definition 3.1 Hilbert Calculus **Ks** of $\mathcal{L}_{\Box\blacksquare}$ consists of *classical tautologies* and the following schemata and rules:

(■1)	$\blacksquare(A \supset B) \supset (\blacksquare A \supset \blacksquare B)$.	(□1)	$\Box(A \supset B) \supset (\Box A \supset \Box B)$.
(M1)	$A \wedge \blacksquare A \supset \Box A$.	(M2)	$\Box A \supset \blacksquare A$.
(□-rule)	From A, we may infer $\Box A$.		
(MP)	From $A \supset B$ and A, we may infer B.		

The deduction relation $\vdash$ is defined as usual.

Observe that (■-rule), that is, the rule that from A we may infer $\blacksquare A$, can be derived from (□-rule) and (M2).

A formula A is *refuted in a model* $\mathfrak{M}$ or *in a frame* $\mathfrak{F}$ if $\mathfrak{M} \nVdash A$ or $\mathfrak{F} \nVdash A$, respectively. A bimodal logic Λ enjoys *finite model property* (FMP) or *finite frame property* (FFP) with respect to a class M of $\mathcal{L}_{\Box\blacksquare}$-models or F of $\mathcal{L}_{\Box\blacksquare}$-frames if $\mathsf{M} \Vdash \Lambda$ or $\mathsf{F} \Vdash \Lambda$ and every formula not in Λ is refuted in a finite model in M or a finite frame in F, respectively. Λ enjoys FMP or FFP if it enjoys that property with respect to some class of models or frames.

The set of all those formulas that are valid in a class F of frames is called the *logic* of F (written Λ_{F}).

Fact 3.2 **Ks** enjoys FFP with respect to $\mathsf{K}_{\mathrm{all}}$, the class of all $\mathcal{L}_{\Box\blacksquare}$-frames. Therefore, **Ks** $= \Lambda_{\mathsf{K}_{\mathrm{all}}}$.

To prove FFP of **Ks**, which implies the Kripke completeness, we apply bimodal filtration (Definition 3.9) and Lemma 3.11 to the canonical model of **Ks** and apply Lemma 2.4. For an alternative proof, see [21].

According to this fact, we define normal modal logic in $\mathcal{L}_{\Box\blacksquare}$ as follows: A *normal modal logic* Λ of $\mathcal{L}_{\Box\blacksquare}$ is a set of formulas that contains all classical tautologies, (□1), (■1), (M1), and (M2) and that is closed under MP, uniform substitution, and □-rule. Note that **Ks** is the smallest normal modal logic of $\mathcal{L}_{\Box\blacksquare}$.

Definition 3.3 A normal modal logic Λ is *($\mathcal{L}_{\Box\blacksquare}$-frame) complete* if $\Lambda = \Lambda_{\mathsf{F}}$ for some class F of $\mathcal{L}_{\Box\blacksquare}$-frames. Λ is *($\mathcal{L}_{\Box\blacksquare}$-frame) incomplete* if Λ is not complete.

By this definition, **Ks** is $\mathcal{L}_{\Box\blacksquare}$-frame complete.

Definition 3.4 Hilbert Calculus $\mathbf{KsT}_{\blacksquare}$ of $\mathcal{L}_{\Box\blacksquare}$ consists of all schemata and rules of **Ks** and ($\mathbf{T}_{\blacksquare}$) $\blacksquare A \supset A$.

Lemma 3.5 *$\mathbf{KsT}_{\blacksquare}$ is $\mathcal{L}_{\Box\blacksquare}$-frame incomplete. Therefore, $\mathbf{KsT}_{\blacksquare}$ does not enjoy* FFP.

Proof Assume for contradiction $\mathbf{KsT}_{\blacksquare} = \Lambda_{\mathsf{K}}$ for some class K of $\mathcal{L}_{\Box\blacksquare}$-frames. Since no $\mathcal{L}_{\Box\blacksquare}$-frame $\mathfrak{F}$ satisfies $\mathfrak{F} \Vdash \mathbf{T}_{\blacksquare}$, $\mathsf{K} = \varnothing$, and $\mathsf{K} \Vdash \bot$. By our assumption, $\vdash_{\mathbf{KsT}_{\blacksquare}} \bot$. However, $\nvdash_{\mathbf{KsT}_{\blacksquare}} \bot$; that is, $\mathbf{KsT}_{\blacksquare}$ is consistent, since $\langle W, R, R \rangle \Vdash \mathbf{T}_{\blacksquare}$ where R is reflexive and where $\langle W, R, R \rangle$ is not an $\mathcal{L}_{\Box\blacksquare}$-frame but a bimodal frame. □

Definition 3.6 (M_1) Let $\mathfrak{M} = \langle W, R, V \rangle$ be an $\mathcal{L}_{\Box\blacksquare}$-model and $\pi : W \times 2 \to W$ the projection, where 2 is $\{0, 1\}$. Write $\mathfrak{M} \times 2 = \langle W \times 2, R', V' \rangle$, where $R' = \{\langle a, b \rangle \mid \pi(a) R \pi(b)\}$ and $V'(p) = \pi^{-1}[V(p)] = \{a \in W \times 2 \mid \pi(a) \in V(p)\}$ for any $p \in \Phi$. Define $\mathsf{M}_1 = \{\mathfrak{M} \times 2 \mid \mathfrak{M} : R\text{-reflexive } \mathcal{L}_{\Box\blacksquare}\text{-model}\}$.

Lemma 3.7 *For any $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{M}$, any w in $\mathfrak{M}$, $\mathfrak{M}\times 2, \langle w, 0 \rangle \leftrightsquigarrow \mathfrak{M}\times 2, \langle w, 1 \rangle$.*

Lemma 3.8 $\mathsf{M}_1 \Vdash \mathbf{KsT}_{\blacksquare}$.

Proof Let $\mathfrak{M}$ be an R-reflexive $\mathcal{L}_{\Box\blacksquare}$-frame. Suppose $\mathfrak{M} \times 2, \langle w, i \rangle \Vdash \blacksquare A$. Since R is reflexive, $\langle w, 1-i \rangle R' \langle w, i \rangle$ and $\langle w, i \rangle R' \langle w, 1-i \rangle$. By (1), $\mathfrak{M} \times 2, \langle w, 1-i \rangle \Vdash \blacksquare A$ whence $\mathfrak{M} \times 2, \langle w, i \rangle \Vdash A$. □

To prove FMP of $\mathbf{KsT}_{\blacksquare}$ with respect to M_1, we have to prove that A is refuted in some finite model in M_1 for every A with $\nvdash_{\mathbf{KsT}_{\blacksquare}} A$. Let us introduce the notion of filtration to construct the desired finite model in M_1.

Definition 3.9 (Bimodal finest filtration) Let $\mathfrak{M} = \langle W, R, S, V \rangle$ be a bimodal model and Σ a subformula-closed set of formulas. $w \sim_\Sigma v$ if and only if ($\mathfrak{M}, w \Vdash A$ if and only if $\mathfrak{M}, v \Vdash A$) for every $A \in \Sigma$. The quotient $W/\sim_\Sigma$ is defined as usual and $[w] \in W/\sim_\Sigma$ denotes the equivalence class of w. A bimodal model $\mathfrak{M}^f_\Sigma = \langle W^f, R^f, S^f, V^f \rangle$ is called *a bimodal finest filtration of $\mathfrak{M}$ through Σ* if $W^f = W/\sim_\Sigma$, $[w] R^f [v]$ if and only if $(\exists w' \in [w])(\exists v' \in [v])\, w' R v'$, $[w] S^f [v]$ if and only if $(\exists w' \in [w])(\exists v' \in [v])\, w' S v'$ and $V^f(p) = \{[w] \mid \mathfrak{M}, w \Vdash p\}$ for all $p \in \Sigma \cap \Phi$.

Fact 3.10 (Filtration lemma) For any $A \in \Sigma$ and any state w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash A$ if and only if $\mathfrak{M}^f_\Sigma, [w] \Vdash A$.

You can find the proof, for example, [2], Theorem 2.39, since finest filtration is a special case of “filtration” in [2].

Lemma 3.11

1. *$(R\cap\neq) \subset S$ implies $(R^f\cap\neq) \subset S^f$,*
2. *$S \subset R$ implies $S^f \subset R^f$, and*
3. *if S is reflexive, so is S^f.*

Proof Clearly, (2) and (3) hold. We prove (1). Suppose $[w](R^f \cap \neq)[w']$; that is, vRv' for some $v \in [w]$ and some $v' \in [w']$. Since $[w] \neq [w']$, $v \neq v'$ whence $v(R\cap\neq)v'$. By $(R\cap\neq) \subset S$, vSv' whence $[w]S^f[w']$. □

Lemma 3.12 $\mathbf{KsT}_{\blacksquare}$ *enjoys* FMP *with respect to* M_1. *Therefore,* $\mathbf{KsT}_{\blacksquare}$ *is decidable.*

Proof Let $\mathfrak{M} = \langle W, R, S, V \rangle$ be the canonical (*bimodal*) model of $\mathbf{KsT}_{\blacksquare}$ (for the definition of the canonical model of the unimodal case, see [2], Definition 4.18; the generalization to the bimodal case is obvious). $S \subset R$ (due to (M2)) and S is reflexive (due to $\blacksquare A \supset A$). Therefore, $\mathfrak{M}$ is an R-reflexive model. In addition, we prove $(R\cap\neq) \subset S$ as follows: Assume that xRx' and $x \neq x'$. $B \in x$ and $\sim B \in x'$ for some B of $\mathcal{L}_{\Box\blacksquare}$ since x and x' are maximal consistent. Suppose for contradiction that not xSx'. Then, $\blacksquare C \in x$ and $\sim C \in x'$ for some C of $\mathcal{L}_{\Box\blacksquare}$. We have $\sim B \wedge \sim C \in x'$ whence $B \vee C \notin x'$. Since xRx', $\Box(B \vee C) \notin x$. By $B, \blacksquare C \in x$, we have $B \vee C, \blacksquare(B \vee C) \in x$ and so $(B \vee C) \wedge \blacksquare(B \vee C) \in x$, which implies $\Box(B \vee C)$, a contradiction.

Suppose $\nvdash_{\mathbf{KsT}_{\blacksquare}} A$. Then $\mathfrak{M}, w \nVdash A$ for some w in $\mathfrak{M}$. Take the finest bimodal filtration $\mathfrak{M}^f_\Sigma$ of $\mathfrak{M}$ through the set Σ of all subformulas of A. By the Filtration Lemma and $\mathfrak{M}, w \nVdash A$, we deduce that $\mathfrak{M}^f_\Sigma, [w] \nVdash A$. Note that, by Lemma 3.11, $\mathfrak{N} \times 2$ is a $\blacksquare$-realization of $\mathfrak{M}^f_\Sigma$ in the way of the proof of Lemma 2.4 where $\mathfrak{N} = \langle W/{\sim_\Sigma}, R^f, V^f \rangle$, and so $\mathfrak{N} \times 2 \nVdash A$. Finally, $\mathfrak{N} \times 2$ is finite and $\mathfrak{N} \times 2$ is in M_1 by Lemma 3.11(2) and (3). □

From Lemma 3.5 and Lemma 3.12, we have the following main theorem in this section. Even if we confine logics to decidable ones, the equivalence of FFP and FMP fails in our extension. (It is true that with respect to *bimodal* frames or models, $\mathbf{KsT}_{\blacksquare}$ enjoys FFP or FMP. Here, however, this possibility is excluded as being nonintended. Our restriction to $\mathcal{L}_{\Box\blacksquare}$-frames or $\mathcal{L}_{\Box\blacksquare}$-models is essential).

Theorem 3.13 $\mathbf{KsT}_{\blacksquare}$ *does not enjoy* FFP*, but* FMP.

4 Characterizing Modal Definability in Terms of Kripke Models

The van Benthem Characterization Theorem (see, e.g., [2], Theorem 2.68) states that the set of unimodal formulas, interpreted on Kripke models, is equivalent to the set of bisimulation-invariant $\mathcal{L}^m$-formulas. In this section, we show that adding the $\blacksquare$-operator does not change this situation (Theorem 4.9) with an appropriate notion of bisimulation. We also characterize modally definable classes of pointed models (i.e., Kripke models with states).

Let us introduce two tools we need to capture modal expressivity over models.

Definition 4.1 The *standard translation* ST_x assigning the first-order formulas of $\mathcal{L}^m(\Phi)$ to $\mathcal{L}_{\Box\blacksquare}$-formulas is defined as usual (e.g., [2], Section 2.4) except that, $ST_x(\blacksquare A) := \forall y((x\boldsymbol{R}y \wedge \sim x \approx y) \supset ST_y(A))$ where y is a fresh variable. We also define the standard translations from $\mathcal{L}_\Box$ and from $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ to $\mathcal{L}^m(\Phi)$ by $ST_x(\overline{\mathrm{D}}B) := \forall y(\sim x \approx y \supset ST_y(B))$. For a set Σ of formulas, we define $ST_x[\Sigma] = \{\, \alpha(x) \mid (\exists A \in \Sigma) \models ST_x(A) \equiv \alpha(x) \,\}$ in each language.

We can easily prove the following.

Fact 4.2 $\mathfrak{M}, w \Vdash A$ if and only if $\mathfrak{M} \models ST_x(A)[w]$ for all $\mathfrak{M}$, w in $\mathfrak{M}$, and A of $\mathcal{L}_{\Box\blacksquare}$ (or $\mathcal{L}_\Box$, $\mathcal{L}_{\Box\overline{\mathrm{D}}}$).

Definition 4.3 Let $\mathfrak{M} = \langle W, R, S, V \rangle$ and $\mathfrak{M}' = \langle W', R', S', V' \rangle$ be two bimodal models. A nonempty binary relation $Z \subset W \times W'$ is called a *bimodal bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following are satisfied:

(Fact)	If wZw', then $w \in V(p)$ if and only if $w' \in V(p)$ for all $p \in \Phi$;
(R-forth)	If wZw' and wRv, then vZv' and $w'R'v'$ for some v' (in $\mathfrak{M}'$);
(R-back)	If wZw' and $w'R'v'$, then vZv' and wRv for some v (in $\mathfrak{M}$);
(S-forth) and (S-back)	Defined similarly.

When Z is a bimodal bisimulation with wZw', we say these states are *bisimilar by* Z (written $Z : \mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$). When $Z : \mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$ for some Z, we write $\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$, or $w \underline{\leftrightarrow} w'$ if the models are clear from the context. Given two unimodal models, the notion of *unimodal bisimulation* is defined similarly.

We have the following fact (see, e.g., [2], Theorem 2.20): For any bimodal model $\mathfrak{M}$ and $\mathfrak{M}'$, any w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$, $w \underline{\leftrightarrow} w'$ implies $w \leftrightsquigarrow w'$. To understand the connection between bisimulations and the notions from first-order model theory, we need the following property for classes of models.

Definition 4.4 A class M of bimodal models enjoys *the bimodal Hennessy-Milner property* (bimodal HMP) if every $\mathfrak{M}, \mathfrak{M}' \in \mathsf{M}$ and any $w \in |\mathfrak{M}|, w' \in |\mathfrak{M}'|$, $w \leftrightsquigarrow w' \iff w \underline{\leftrightarrow} w'$.

Definition 4.5 Let $\mathfrak{M} = \langle W, R, S, V \rangle$ be a bimodal model, $X \subset W$, Σ a set of formulas. Σ is *satisfiable* in X, if $\mathfrak{M}, x \Vdash \Sigma$ for some $x \in X$. Σ is *finitely satisfiable* in X, if every finite subset of Σ is satisfiable in X. $\mathfrak{M}$ is called *bimodally saturated* if for every $w \in W$ and every set Σ, (i) if Σ is finitely satisfiable in $R[w]$, then Σ is satisfiable in $R[w]$, and (ii) it satisfies the same condition about S.

The following fact is known (e.g., [2], Proposition 2.54).

Fact 4.6 Any class of bimodally saturated $\mathcal{L}_{\Box\blacksquare}$-models enjoys bimodal HMP.

To construct a modally saturated model, we have two ways, by ultrafilter extensions (introduced in Section 5.2) and via ω-saturated first-order structures. In this section, we use the second way.

We use some notions from first-order model theory, for example, submodel, elementary embedding $\prec$, realizing a type, ω-saturatedness. For a first-order language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathfrak{M}$, and a set $D \subset |\mathfrak{M}|$, $\mathcal{L}(D)$ denotes the expanded language of $\mathcal{L}$ with new constants d for $d \in D$. $\mathfrak{M}(D)$ denotes the expansion of $\mathfrak{M}$ for $\mathcal{L}(D)$ where the interpretation of $\underline{d}$ is $d \in D$. The reader unfamiliar with them or the following fact can refer to, for example, [6].

Fact 4.7 An $\mathcal{L}^m$-structure $\mathfrak{M}$ has an ω-saturated elementary extension.

Lemma 4.8 *An $\mathcal{L}_{\Box\blacksquare}$-model $\langle W, R, V \rangle$, which is ω-saturated as an $\mathcal{L}^m$-structure, is bimodally saturated, that is, in terms of R and $(R\cap\neq)$. It follows that a class of ω-saturated $\mathcal{L}_{\Box\blacksquare}$-models enjoys bimodal* HMP.

Proof We prove this lemma in a way similar to the unimodal case ([2], Theorem 2.65). Suppose that $\langle W, R, V \rangle$ is ω-saturated. Let w be a state in W. Since the clause about $R[w]$ is proved in [2], we show one about $(R\cap\neq)[w]$. Assume that Σ is finitely satisfiable in $(R\cap\neq)[w]$. Define $\Sigma' = \{ \underline{w}\mathbf{R}x \} \cup \{ \sim \underline{w} \approx x \} \cup ST_x[\Sigma]$.

It is clear that $\mathfrak{M}(\{ w \})$ realizes every finite subset of Σ', namely, in some $(R\cap\neq)$-successor of w. Thus, by the ω-saturation of $\mathfrak{M}$, Σ' itself is realized in some state v. By $\mathfrak{M}(\{ w \}) \models ST_x(A)[v]$ for all $A \in \Sigma$, it follows that $\mathfrak{M}, v \Vdash \Sigma$ and $w(R\cap\neq)v$. □

An $\mathcal{L}^m$-formula $\alpha(x)$ is *invariant for bisimulations* if $\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{N}, v$ implies ($\mathfrak{M} \models \alpha(x)[\, w \,]$ if and only if $\mathfrak{N} \models \alpha(x)[\, v \,]$).

Theorem 4.9 (van Benthem characterization theorem in $\mathcal{L}_{\Box\blacksquare}$) *For an $\mathcal{L}^m$-formula $\alpha(x)$, the following are equivalent:*

(A) $\alpha(x) \in ST_x[\mathcal{L}_{\Box\blacksquare}]$,
(B) $\alpha(x)$ *is invariant for bimodal bisimulations between $\mathcal{L}_{\Box\blacksquare}$-models.*

The direction from (A) to (B) is immediate from Fact 4.2 and Definition 4.3. We can prove the converse in the same way as in the cases in $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ by Lemma 4.8 and Fact 4.7 (for details, see, for example, [2], pp. 103–6, or [19], Theorem 4.7), though it seems to be folklore among modal model theorists according to [13].

Let us investigate how the three languages, $\mathcal{L}_{\Box}$, $\mathcal{L}_{\Box\blacksquare}$, and $\mathcal{L}_{\Box\mathrm{D}}$, differ.

Proposition 4.10

1. $ST_x(\blacksquare p) \notin ST_x[\mathcal{L}_{\Box\overline{\mathrm{D}}}]$.
2. $ST_x(\overline{\mathrm{D}}p) \notin ST_x[\mathcal{L}_{\Box\blacksquare}]$.
3. *Thus,* $ST_x(\blacksquare p), ST_x(\overline{\mathrm{D}}p) \notin ST_x[\mathcal{L}_{\Box}]$.

Proof

1 Suppose for contradiction, $\models ST_x(A) \equiv ST_x(\blacksquare p)$ for some A of $\mathcal{L}_{\Box\overline{\mathrm{D}}}$. Consider two models $\mathfrak{M}$, $\mathfrak{M}'$ and a bimodal bisimulation Z between $\mathcal{L}_{\Box\overline{\mathrm{D}}}$-models: $\mathfrak{M} = \langle \{ a, b \}, \{ \langle a, a \rangle, \langle b, b \rangle \}, V \rangle$, $\mathfrak{M}' = \langle \{ 0, 1 \}, \{ 0, 1 \}^2, V' \rangle$, where $V(p) = V'(p) = \varnothing$ for any $p \in \Phi$ and $Z = \{ a, b \} \times \{ 0, 1 \}$. Then, $\mathfrak{M}, a \Vdash A$, but $\mathfrak{M}', 0 \nVdash A$.

2 Suppose for contradiction, $\models ST_x(A) \equiv ST_x(\overline{\mathrm{D}}p)$ for some A of $\mathcal{L}_{\Box\blacksquare}$. Consider two models $\mathfrak{M}$, $\mathfrak{M}'$ and a bimodal bisimulation Z between $\mathcal{L}_{\Box\blacksquare}$-models: $\mathfrak{M} = \langle \{ a \}, \varnothing, V \rangle$, $\mathfrak{M}' = \langle \{ b, c \}, \varnothing, V' \rangle$, where $V(p) = V'(p) = \varnothing$ for any $p \in \Phi$, and $Z = \{ a \} \times \{ b, c \}$. Then, $\mathfrak{M}, a \Vdash A$, but $\mathfrak{M}', c \nVdash A$. □

Therefore, we conclude that $\mathcal{L}_{\Box\blacksquare}$ is a different extension of $\mathcal{L}_{\Box}$ from $\mathcal{L}_{\Box\overline{\mathrm{D}}}$. However, we have not proved that $ST_x[\mathcal{L}_{\Box\blacksquare}] \cap ST_x[\mathcal{L}_{\Box\overline{\mathrm{D}}}] = ST_x[\mathcal{L}_{\Box}]$, nor have we found a counterexample to this.

A *pointed $\mathcal{L}_{\Box\blacksquare}$-model* is a pair $\langle \mathfrak{M}, w \rangle$ where w is a state of $\mathfrak{M}$. P denotes a class of pointed $\mathcal{L}_{\Box\blacksquare}$-models. P is *closed under bisimulation* if $\langle \mathfrak{M}, w \rangle \in \mathsf{P}$ and $\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{N}, v$ imply $\langle \mathfrak{N}, v \rangle \in \mathsf{P}$. P is *closed under ultraproduct* if any ultraproduct $\prod_U(\mathfrak{M}_i, w_i)$ of a family of pointed $\mathcal{L}_{\Box\blacksquare}$-models $\langle \mathfrak{M}_i, w_i \rangle$ in P $(i \in I)$ belongs to P. P is *definable by a set of formulas* or *definable by a single formula* if there is a set Γ or singleton set Γ, respectively, of modal formulas such that, for any pointed $\mathcal{L}_{\Box\blacksquare}$-model $\langle \mathfrak{M}, w \rangle$, $\langle \mathfrak{M}, w \rangle \in \mathsf{P} \iff \mathfrak{M}, w \Vdash \Gamma$. These notions are also defined for pointed unimodal models and pointed $\mathcal{L}_{\Box\overline{\mathrm{D}}}$-models in similar ways.

Theorem 4.11 *Let P be a class of pointed $\mathcal{L}_{\Box\blacksquare}$-models. Then (1) P is definable by a set of $\mathcal{L}_{\Box\blacksquare}$-formulas if and only if P is closed under bimodal bisimulations between $\mathcal{L}_{\Box\blacksquare}$-models and under ultraproducts, and $\overline{\mathsf{P}}$, the complement of P, is closed* *under ultrapowers. (2)* P *is definable by a single* $\mathcal{L}_{\Box\blacksquare}$*-formula if and only if both* P *and* $\overline{\mathsf{P}}$ *are closed under bimodal bisimulations between* $\mathcal{L}_{\Box\blacksquare}$*-models and under ultraproducts.*

Theorem 4.11 is proved in the same way as in the case of $\mathcal{L}_\Box$ and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ (see, e.g., [2], Theorems 2.75 and 2.76, for $\mathcal{L}_\Box$ and [19], Theorem 4.8, for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$). We have characterized the modal definability of a class of pointed $\mathcal{L}_{\Box\blacksquare}$-models.

Results shown in this section are summarized in the following Table 1, where 'reflect' means that the complement of a given class is closed under the intended operation. The proofs of Propositions 4.10(3), (2), and (1) tell us 'No's in the second (for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ and $\mathcal{L}_\Box$), third, and fourth columns, respectively.

	closed under?				reflect?
	unimodal bisimulations	bimodal bisimulations for $\mathcal{L}_{\Box\blacksquare}$	bimodal bisimulations for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$	ultraproducts	ultrapowers
$\mathcal{L}_\Box$	**Yes**	Yes	Yes	**Yes**	**Yes**
$\mathcal{L}_{\Box\blacksquare}$	No	**Yes**	No	**Yes**	**Yes**
$\mathcal{L}_{\Box\overline{\mathrm{D}}}$	No	No	**Yes**	**Yes**	**Yes**

Yes (boldface) can characterize the modal definability.

Table 1 Comparison of modal definability of a class of pointed models

5 Characterizing Modal Definability in Terms of Kripke Frames

Definition 5.1 (Frame definability) A set Γ of formulas *defines a class* K *of frames* if, for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \Gamma \iff \mathfrak{F} \in \mathsf{K}$. A class K of frames is *modally definable* if there is some set of formulas that defines K.

Fact 5.2 The following define the corresponding properties.

(1) $\blacksquare p \supset \Box p$; irreflexivity,
(2) $\blacksquare p \supset \Box\Box p$; strict partial ordering,
(3) $\blacksquare p \supset \blacksquare\blacksquare p$; antisymmetry and transitivity,
(4) $(\Box p \supset p) \wedge (\blacksquare p \supset \blacksquare\blacksquare p)$; partial ordering.

For details, see [20]. (1) holds since $\blacksquare p \supset \Box p$ defines $R \subset (R\cap\neq)$. It is known that irreflexivity and antisymmetry are undefinable in $\mathcal{L}_\Box$ ([2], Example 3.15 and Exercise 3.3.2(a)). For example, with regard to irreflexivity, a counterexample is as follows: The mapping which collapses the set of natural numbers in their usual strict ordering to a single reflexive state is a unimodal p-morphism and the p-morphic images preserve validity on frames ([2], Theorem 3.14). We can, however, define irreflexivity in $\mathcal{L}_{\Box\blacksquare}$ [20]. Therefore, unimodal p-morphic images do not preserve validity on $\mathcal{L}_{\Box\blacksquare}$-frames. On the other hand, we cannot define antisymmetry in $\mathcal{L}_{\Box\blacksquare}$ as well as in $\mathcal{L}_\Box$ (see Fact 5.7). However, by Fact 5.2(3), it is definable "within" transitivity where "within" means as follows.

Definition 5.3 (Relative frame definability) Let C be a class of frames. A set Γ of formulas *defines a class* K *of frames within* C if, for all frames $\mathfrak{F}$ in C, $\mathfrak{F} \Vdash \Gamma \iff \mathfrak{F} \in \mathsf{K}$. A class K of frames is *modally definable within* C if there is some set of formulas that defines it within C.

In this section, we discuss the precise expressivity of $\mathcal{L}_{\Box\blacksquare}$. First, within finite and transitive frames, we characterize the definability of frame classes by a modified Jankov-Fine formula (Theorem 5.12). Second, we characterize that of elementary frame classes (Theorem 5.17). To do so, we introduce new frame constructions.

Definition 5.4 The *(unimodal) disjoint union* of pairwise disjoint frames $\mathfrak{F}_j = \langle W_j, R_j \rangle$ ($j \in J$) is the structure $\biguplus_{j\in J} \mathfrak{F}_j = \left\langle \bigcup_{j\in J} W_j, \bigcup_{j\in J} R_j \right\rangle$.

Note that we may assume that, up to isomorphism, any family of frames is pairwise disjoint.

Definition 5.5 Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{F}' = \langle W', R' \rangle$ be two frames. $\mathfrak{F}'$ is a *(unimodally) generated subframe* of $\mathfrak{F}$ (written $\mathfrak{F}' \rightarrowtail \mathfrak{F}$) if (i) $W' \subset W$, (ii) $R' = R \cap (W')^2$, and (iii) $w \in W'$ implies $R[w] \subset W'$. For a subset $X \subset W$, the *subframe generated by* X (notation: $\mathfrak{F}_X$) is the smallest generated subframe of $\mathfrak{F}$ whose domain contains X. The *point-generated frame by* w (notation: $\mathfrak{F}_w$) is $\mathfrak{F}_{\{w\}}$ where w is called a *root* of the frame.

Lemma 5.6 *Let A be a formula of $\mathcal{L}_{\Box\blacksquare}$. Let $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}_j$ ($j \in J$) be $\mathcal{L}_{\Box\blacksquare}$-frames.*

(1) If $\mathfrak{F}_j \Vdash A$ for every $j \in J$, then $\biguplus_{j\in J} \mathfrak{F}_j \Vdash A$.
(2) If $\mathfrak{F}' \rightarrowtail \mathfrak{F}$, then $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$.
(3) If $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, then $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$. (Recall that $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ means that $\mathfrak{F}'$ is a "bimodal" p-morphic image of $\mathfrak{F}$.)

We can prove (1) and (2) of this lemma as in the unimodal case ([2], Theorem 3.14). (3) is a special case of Fact 2.2(2). By these closures, we can prove that some properties are undefinable in $\mathcal{L}_{\Box\blacksquare}$ as follows [20].

Fact 5.7 The following are undefinable in $\mathcal{L}_{\Box\blacksquare}$: (1) antisymmetry, (2) trichotomy, (3) connectedness (wRw' or $w'Rw$ for any w, w'), (4) total ordering, (5) strict total ordering.

Proof We only prove (1). Consider $\mathfrak{F} = \langle \{a, b\}, \{a, b\}^2 \rangle$ and $\mathfrak{F}' = \langle \omega, \{ \langle m, m \rangle, \langle m, m+1 \rangle \mid m \in \omega \} \rangle$. Note that $\mathfrak{F}'$ is antisymmetric and $\mathfrak{F}$ is not. Define f by $f(2m) = a$, $f(2m+1) = b$ for any $m \in \omega$. Then we can prove that f is a surjective bimodal p-morphism $f : \mathfrak{F}' \to \mathfrak{F}$, which violates antisymmetry. □

Lemma 5.8 *An $\mathcal{L}_{\Box\blacksquare}$-frame $\mathfrak{F}$ is a bimodal p-morphic image of the disjoint union of its point-generated subframes.*

We can easily generalize the proof in the case of $\mathcal{L}_{\Box}$ ([2], Exercises 3.3.4) to this case (note that $\biguplus_{w\in|\mathfrak{F}|} \mathfrak{F}_w \twoheadrightarrow \mathfrak{F}$).

5.1 Goldblatt-Thomason theorem for finite and transitive $\mathcal{L}_{\Box\blacksquare}$-frames

Definition 5.9 (Jankov-Fine formula of $\mathcal{L}_{\Box\blacksquare}$) Let $\mathfrak{F} = \langle W, R \rangle$ be a finite frame which is generated by w. Enumerate W as $w = w_0, \ldots, w_n$. Associate a distinct p_i in Φ with each w_i. Let $A_{\mathfrak{F},w}$ be the conjunction of the following:

(1) p_0,
(2) $\Box(p_0 \vee \cdots \vee p_n)$,
(3) $(p_i \supset\sim p_j) \wedge \Box(p_i \supset\sim p_j)$ for each i, j with $i \neq j$,
(4) $(p_i \supset \Diamond p_j) \wedge \Box(p_i \supset \Diamond p_j)$ for each i, j with $w_i R w_j$,

(5) $(p_i \supset\sim \Diamond p_j) \wedge \Box(p_i \supset\sim \Diamond p_j)$ for each i, j with not $w_i R w_j$,
(6) $(p_i \supset\sim \blacklozenge p_i) \wedge \Box(p_i \supset\sim \blacklozenge p_i)$ for each i, where $0 \leq i, j \leq n$.

$A_{\mathfrak{F},w}$ is called the *Jankov-Fine formula of* $\mathcal{L}_{\Box\blacksquare}$ *for* $\mathfrak{F}$ *and* w.

Lemma 5.10 *Let* $\mathfrak{F}$ *be a transitive and finite frame generated by* w*. Then, for any transitive frame* $\mathfrak{G}$*, (A)* $A_{\mathfrak{F},w}$ *is satisfiable in* $\mathfrak{G}$ *if and only if (B) there exists a surjective* bimodal p-morphism *from* $\mathfrak{G}_v$ *onto* $\mathfrak{F}$ *for some state* v *in* $\mathfrak{G}$*.*

Proof It is easy to prove that (B) implies (A). Conversely, assume (A); that is, $\langle \mathfrak{G}, V \rangle, v \Vdash A_{\mathfrak{F},w}$ for some valuation V and some state v in $\mathfrak{G}$. Therefore, $\langle \mathfrak{G}_v, V' \rangle, v \Vdash A_{\mathfrak{F},w}$, where V' is the restriction of V. Note that $\mathfrak{G}_v$ is R-transitive. We can prove the following: $\bigcup_{0 \leq i \leq n} V(p_i) = |\, \mathfrak{G}_v\,|$; for any $0 \leq i, j \leq n$, $i \neq j$ implies $V(p_i) \cap V(p_j) = \varnothing$. Thus, we can *define* the mapping f from $\mathfrak{G}_v$ to $\mathfrak{F}$ so that $f^{-1}[\{w_i\}] = V(p_i)$ for any $0 \leq i \leq n$.

Since the Jankov-Fine formula of $\mathcal{L}_{\Box\blacksquare}$ implies that of $\mathcal{L}_{\Box}$, we can prove that f is surjective and satisfies (R-forth), (R-back), as a consequence of the $\mathcal{L}_{\Box}$-case (see [2], Lemma 3.20). Recall that (R-back) implies (($R\cap\neq$)-back). We prove the contraposition of (($R\cap\neq$)-forth); that is, [(not $f(x) R f(y)$) or $f(x) = f(y)$] implies [(not xRy) or $x = y$]. Since (R-forth) holds, it suffices to show that $f(x) = f(y)$ implies [(not xRy) or $x = y$]. We can see this by the conjunct (6) and the transitivity of $\mathfrak{G}_v$. □

Definition 5.11 Let F be a class of frames. F is *closed under bimodal p-morphic images* if $\mathfrak{F} \in$ F and $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ imply $\mathfrak{F}' \in$ F. F is *closed under generated subframes* if $\mathfrak{F} \in$ F and $\mathfrak{F}' \rightarrowtail \mathfrak{F}$ imply $\mathfrak{F}' \in$ F. F is *closed under disjoint unions* if $\mathfrak{F}_j \in$ F $(j \in J)$ implies $\biguplus_{j \in J} \mathfrak{F}_j \in$ F.

Theorem 5.12 *Let* $\mathsf{C}_{\text{fintra}}$ *be the class of all finite transitive* $\mathcal{L}_{\Box\blacksquare}$*-frames. Then (A)* K *is modally definable within* $\mathsf{C}_{\text{fintra}}$ *if and only if (B)* K *is closed under (finite) disjoint unions, generated subframes, and bimodal p-morphic images.*

By Lemmas 5.10 and 5.8, we can prove this theorem as in the case of $\mathcal{L}_{\Box}$. See, for example, [2], Theorem 3.21. It is known that a class K of finite frames is definable in $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ if and only if it is closed under isomorphisms ([19], Proposition 4.3). In particular, in $\mathcal{L}_{\Box\overline{\mathrm{D}}}$, this equivalence holds within the class of finite transitive $\mathcal{L}_{\Box\overline{\mathrm{D}}}$-frames.

Results proved in this subsection are summarized in Table 2 where $\mathsf{C}_{\text{fintra}}$ is the class of finite transitive frames. We can write 'No' for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ in the second column because linearity is defined by $p \supset \Diamond q \vee \overline{\mathrm{D}}(q \supset \Diamond p)$ [19] but linearity is not always preserved under disjoint unions; 'No' for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ in the third column because $R \neq \varnothing$ is defined by $\mathrm{D}\Diamond\top \vee \Diamond\top$ [19], but $R \neq \varnothing$ is not always preserved under generated subframes; 'No' for $\mathcal{L}_{\Box\blacksquare}$ and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ in the fourth column because irreflexivity is definable in $\mathcal{L}_{\Box\blacksquare}$ (Fact 5.2(1)) and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ [19], respectively, but irreflexivity is not always preserved under unimodal p-morphic images ([2], Example 3.15); 'No' for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ in the fifth column since antisymmetry is defined by $(p \wedge \overline{\mathrm{D}} \sim p) \supset \Box(\Diamond p \supset p)$ in $\mathcal{L}_{\Box\overline{\mathrm{D}}}$ [19], but antisymmetry is not always preserved under bimodal p-morphic images between $\mathcal{L}_{\Box\blacksquare}$-frames (see the proof of Fact 5.7(1)). Therefore, within the class of finite transitive frames, we can conclude that $\mathcal{L}_{\Box\blacksquare}$ is strictly stronger than $\mathcal{L}_{\Box}$ and strictly weaker than $\mathcal{L}_{\Box\overline{\mathrm{D}}}$.

	closed under?				
	disjoint unions	generated subframes	unimodal p-morphic images	bimodal p-morphic images in $\mathcal{L}_{\Box\blacksquare}$-frames	isomorphisms
$\mathcal{L}_{\Box}$	**Yes**	**Yes**	**Yes**	Yes	Yes
$\mathcal{L}_{\Box\blacksquare}$	**Yes**	**Yes**	No	**Yes**	Yes
$\mathcal{L}_{\Box\overline{\mathrm{D}}}$	No	No	No	No	**Yes**

Yes (boldface) can characterize the modal definability within $\mathsf{C}_{\text{fintra}}$.

Table 2 Comparison of relative modal definability within $\mathsf{C}_{\text{fintra}}$

5.2 Characterizing modal definability of elementary classes For the characterization, we need another construction, bimodal ultrafilter extensions and combine it with $\blacksquare$-realizations ($\blacksquare$-realizations of bimodal ultrafilter extensions).

Definition 5.13 (Bimodal ultrafilter extension)

(1) Given a binary relation Q on a set W, we define a unary operation l_Q on $\mathcal{P}(W)$: $l_Q(X) := \{\, w \in W \mid Q[w] \subset X \,\}$.

(2) The *bimodal ultrafilter extension* (bimodal UE) $\mathfrak{ue}\,\mathfrak{F}$ of $\mathfrak{F} = \langle\, W, R, S \,\rangle$ is the frame $\langle\, W^{ue}, R^{ue}, S^{ue} \,\rangle$, where W^{ue} is the set of (principal and nonprincipal) ultrafilters over W, $u R^{ue} u'$ if and only if for any $X \subset W$, $l_R(X) \in u$ implies $X \in u'$, and S^{ue} defined similarly by l_S. *The unimodal ultrafilter extension* (unimodal UE) of $\mathfrak{F} = \langle\, W, R \,\rangle$ is $\langle\, W^{ue}, R^{ue} \,\rangle$.

(3) The *bimodal* UE $\mathfrak{ue}\,\mathfrak{M}$ of a bimodal model $\mathfrak{M} = \langle\, \mathfrak{F}, V \,\rangle$ is the model $\langle\, \mathfrak{ue}\,\mathfrak{F}, V^{ue} \,\rangle$, where $V^{ue}(p) = \{\, u \in W^{ue} \mid p \in u \,\}$ for any $p \in \Phi$.

Let $\mathfrak{F}$ be a unimodal frame and A a formula of $\mathcal{L}_{\Box}$ or $\mathcal{L}_{\Box\overline{\mathrm{D}}}$. It is known that $\langle\, W^{ue}, R^{ue} \,\rangle \Vdash A$ implies $\langle\, W, R \,\rangle \Vdash A$ (see, e.g., [2], Corollary 3.16, for $\mathcal{L}_{\Box}$ and [19], Proposition 1.3, for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$). However, we have not proved this implication in $\mathcal{L}_{\Box\blacksquare}$ nor have we found any counterexample.

Fact 5.14

(1) $\{\mathfrak{ue}\,\mathfrak{M} \mid \mathfrak{M}$ is a bimodal model$\}$ enjoys bimodal HMP.

(2) For a bimodal frame $\mathfrak{F}$ and a formula A of $\mathcal{L}_{\Box\blacksquare}$, $\mathfrak{ue}\,\mathfrak{F} \Vdash A$ implies $\mathfrak{F} \Vdash A$.

Note that the bimodal $\mathfrak{ue}\,\mathfrak{F}$ of $\mathfrak{F}$ is not necessarily an $\mathcal{L}_{\Box\blacksquare}$-frame even if $\mathfrak{F}$ is. One can find the proof of (1), for example, in [2], Proposition 2.61. For the proof of (2), we can easily generalize the proof in the case of $\mathcal{L}_{\Box}$ ([2], Corollary 3.16) to the bimodal case. From Fact 2.2 and Fact 5.14, we deduce the following corollary.

Corollary 5.15 *For any $\mathcal{L}_{\Box\blacksquare}$-frame $\mathfrak{F}$, any A of $\mathcal{L}_{\Box\blacksquare}$, and any $\blacksquare$-realization $\mathfrak{G}$ of $\mathfrak{ue}\,\mathfrak{F}$, $\mathfrak{G} \Vdash A$ implies $\mathfrak{F} \Vdash A$.*

Definition 5.16 (Reflect) Let K be a class of unimodal frames.

(1) K *reflects* $\blacksquare$-realizations or $\overline{\mathrm{D}}$-realizations of bimodal UEs if $\mathfrak{G} \in \mathsf{K}$ implies $\mathfrak{F} \in \mathsf{K}$ for any frames $\mathfrak{F}$ and any $\blacksquare$-realization or $\overline{\mathrm{D}}$-realization $\mathfrak{G}$ of $\mathfrak{ue}\,\mathfrak{F}$, respectively.

(2) K *reflects* unimodal UEs if $\langle\, W^{ue}, R^{ue} \,\rangle \in \mathsf{K}$ implies $\langle\, W, R \,\rangle \in \mathsf{K}$.

Although we have not shown that a modally definable class F in $\mathcal{L}_{\Box\blacksquare}$ reflects unimodal UEs, we have proved that F reflects $\blacksquare$-realizations of bimodal UEs.

A class K of frames is *elementary* if there exists a set Θ of $\mathcal{L}^f$-sentences such that, for any $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{K} \iff \mathfrak{F} \models \Theta$. Then we can answer the question which elementary classes of frames are definable by formulas of $\mathcal{L}_{\Box\blacksquare}$.

Theorem 5.17 *Let K be an elementary class of $\mathcal{L}_{\Box\blacksquare}$-frames. Then (A) K is modally definable in $\mathcal{L}_{\Box\blacksquare}$ if and only if (B) it is closed under (i) disjoint unions, (ii) generated subframes, and (iii) bimodal p-morphic images and reflects (iv) $\blacksquare$-realizations of bimodal* UE*s.*

Proof Trivially, (A) implies (B) by Lemma 5.6 and Corollary 5.15. Conversely, assume (B). Let Λ_{K} be the logic of K, that is, $\{\, A \text{ of } \mathcal{L}_{\Box\blacksquare} \mid \mathsf{K} \Vdash A \,\}$. We prove that, for any $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{K} \iff \mathfrak{F} \Vdash \Lambda_{\mathsf{K}}$. The direction from left to right is obvious by definition. Conversely, suppose $\mathfrak{F} \Vdash \Lambda_{\mathsf{K}}$.

Step 1 By Lemma 5.8, we can assume that $\mathfrak{F}$ is point-generated by a root w.

Step 2 We can suppose that Φ contains a proposition letter p_X for each subset X of $|\,\mathfrak{F}\,|$. Let $\mathfrak{M} = \langle\, \mathfrak{F}, V \,\rangle$, where V is a natural valuation with $V(p_X) = X$. Let Δ be $\{\, A \text{ of } \mathcal{L}_{\Box\blacksquare}(\Phi) \mid \mathfrak{M}, w \Vdash A \,\}$.

Step 3 Δ is satisfiable in K (for details, see [2], pp. 180–1).

Step 4 By Step 3, we have $\langle\, \mathfrak{G}, V' \,\rangle, v \Vdash \Delta$ for some valuation V' and some v in $\mathfrak{G}$ and for some $\mathfrak{G} \in \mathsf{K}$. Then we have $\langle\, \mathfrak{G}_v, V'' \,\rangle, v \Vdash \Delta$ easily, where V'' is the restriction of V'. By the closure condition and $\mathfrak{G} \in \mathsf{K}$, $\mathfrak{G}_v \in \mathsf{K}$. Take an ω-saturated elementary extension $\mathfrak{N} = \langle\, \mathfrak{H}, V''' \,\rangle$ of $\langle\, \mathfrak{G}_v, V'' \,\rangle$ by Fact 4.7.

Step 5 We show that $\mathfrak{H} \twoheadrightarrow \mathfrak{ue}\,\mathfrak{F}$ in this and the next steps. Since $\langle\, \mathfrak{G}_v, V'' \,\rangle \prec \mathfrak{N}$, by $\langle\, \mathfrak{G}_v, V'' \,\rangle, v \Vdash \Delta$ and elementariness of $\Vdash$, Δ is satisfiable in $\mathfrak{N}$. By the ω-saturatedness of $\mathfrak{N}$, the $\mathcal{L}_{\Box\blacksquare}$-model $\mathfrak{N}$ is bimodally saturated.

Step 6 For any s in $\mathfrak{N}$, $\{\, X \subset |\,\mathfrak{F}\,| \mid \mathfrak{N}, s \Vdash p_X \,\}$ is an ultrafilter. In order to prove this, we use the following equivalence: $\mathfrak{M} \Vdash A$ if and only if $\mathfrak{N} \Vdash A$ for any A of $\mathcal{L}_{\Box\blacksquare}$ (see [2], p. 180). We can prove it as follows ($\Box^n A$ stands for A preceded by n $\Box$s): $\mathfrak{M} \Vdash A \iff \mathfrak{M} \Vdash \Box^n A$ for any $n \in \omega \iff \Box^n A \in \Delta$ for any $n \in \omega \iff \langle\, \mathfrak{G}_v, V'' \,\rangle, v \Vdash \Box^n A$ for any $n \in \omega \iff \langle\, \mathfrak{G}_v, V'' \,\rangle \Vdash A \iff \mathfrak{N} \Vdash A$.

This defines the mapping $f : |\,\mathfrak{N}\,| \to |\,\mathfrak{ue}\,\mathfrak{F}\,|$. We can prove that f is surjective by ω-saturatedness. Since $\mathfrak{N}$ and $\langle\, \mathfrak{ue}\,\mathfrak{F}, V^{ue} \,\rangle$ are bimodally saturated, we can deduce that $f : \mathfrak{N} \to \langle\, \mathfrak{ue}\,\mathfrak{F}, V^{ue} \,\rangle$ (for details, see [2], pp. 180–1). Then we have proved $\mathfrak{H} \twoheadrightarrow \mathfrak{ue}\,\mathfrak{F}$.

Step 7 Since K is elementary and $\mathfrak{G}_v \in \mathsf{K}$, $\mathfrak{H} \in \mathsf{K}$. Since $\mathfrak{H}$ is a $\blacksquare$-realization of $\mathfrak{ue}\,\mathfrak{F}$, $\mathfrak{F} \in \mathsf{K}$. $\Box$

For $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Box\overline{\mathrm{D}}}$, the following are known (see, e.g., [2], Theorem 3.19, for (1), and [9], Theorem 4.15, for (2)).

Fact 5.18 Let K be an elementary class of frames.

(1) K is modally definable in $\mathcal{L}_{\Box}$ if and only if it is closed under taking disjoint unions, generated subframes, and unimodal p-morphic images and reflects unimodal UEs.

(2) K is modally definable in $\mathcal{L}_{\Box\overline{D}}$ if and only if it reflects $\overline{D}$-realizations of bimodal UEs.

Proposition 5.19 *$\blacksquare$-realizations of bimodal* UE*s do not antipreserve the frame validity in $\mathcal{L}_{\Box\overline{D}}$.*

Proof Let $\mathfrak{F} = \langle \{a\}, \varnothing \rangle$. Then the unique mapping $f : \mathfrak{F} \uplus \mathfrak{F} \to \mathfrak{F}$ is a $\blacksquare$-realizer of $\mathfrak{ue}\,\mathfrak{F}$. Note that $\mathfrak{ue}\,\mathfrak{F} \cong \mathfrak{F}$ since $|\mathfrak{F}|$ is finite. However, $\mathfrak{F} \nVdash \mathrm{D}\top$ but $\mathfrak{F} \uplus \mathfrak{F} \Vdash \mathrm{D}\top$. $\Box$

Results proved in this subsection are summarized in Table 3. From Table 2 in Section 5.1, we easily fill out the answers from the second column to the fifth column. We can put 'Yes' for $\mathcal{L}_{\Box\overline{D}}$ in the sixth column because of [19], Proposition 1.3. Proposition 5.19 tells us 'No' for $\mathcal{L}_{\Box\overline{D}}$ in the seventh column of this table.

In the table, for each language, we also characterize modal definability of an elementary class of frames. The following elementary properties demonstrate 'No's for the corresponding columns: linearity for the second column, $R \neq \varnothing$ for the third, irreflexivity for the fourth, antisymmetry for the fifth, $(\forall x)(\exists y)\, x \neq y$ for the seventh (see the proof of Proposition 5.19). Note that each of these properties is definable in the languages having 'No' for the corresponding column (e.g., $\mathrm{D}\top$ defines $(\forall x)(\exists y)\, x \neq y$ and, for the others, see the remark at the end of Section 5.1).

Bimodal p-morphisms have stronger clauses than unimodal ones. Thus, condition (iii) in Theorem 5.17 is weaker than the corresponding condition for $\mathcal{L}_{\Box}$. Condition (iv) in Theorem 5.17 is weaker than the corresponding condition for $\mathcal{L}_{\Box}$, because it is not necessary to satisfy the following: Closure under unimodal p-morphic images and reflection of UEs, but it suffices to reflect the composition of them, that is, $\blacksquare$-realizations of bimodal UEs. Therefore, the condition in $\mathcal{L}_{\Box\blacksquare}$ becomes weaker than that in $\mathcal{L}_{\Box}$. More classes of frames become definable in $\mathcal{L}_{\Box\blacksquare}$. Thus, the difference of the expressivity between $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Box\blacksquare}$ is captured in terms of frame constructions.

	closed under?				reflect?		
	disjoint unions	generated subframes	unimodal p-morhic images	bimodal p-morhic images in $\mathcal{L}_{\Box\blacksquare}$-frames	unimodal UEs	$\blacksquare$-realizations of bimodal UEs	$\overline{D}$-realizations of bimodal UEs
$\mathcal{L}_{\Box}$	**Yes**	**Yes**	**Yes**	Yes	**Yes**	Yes	Yes
$\mathcal{L}_{\Box\blacksquare}$	**Yes**	**Yes**	No	**Yes**	Unknown	**Yes**	Unknown
$\mathcal{L}_{\Box\overline{D}}$	No	No	No	No	Yes	No	**Yes**

Yes (boldface) can characterize the MD of an elementary class.

Table 3 Comparison of MD of an elementary class of frames

6 Generalizing Characterizations

In this section, we generalize the notion of realizer to get characterizations in more general languages that have finite modal operators. To avoid repetition, we give only the outlines. The theorems and corollaries in this section, except Theorem 6.13 and Corollary 6.14, can be obtained even if the language has infinitely many operators.

For a formula $\beta(v_1, v_2)$ of $\mathcal{L}^f$, we define the modal operator $[\beta]$ as follows: For any unimodal model $\mathfrak{M}$ and any $w \in |\mathfrak{M}|$, $\mathfrak{M}, w \Vdash [\beta]A$ if and only if $(\forall w' \in W)[wR_\beta w'$ implies $\mathfrak{M}, w' \Vdash A]$, where $R_\beta = \{\langle a, b\rangle \mid \mathfrak{F} \models \beta[a, b]\}$. Write $\beta_\Box(v_1, v_2) := v_1\mathbf{R}v_2$, $\beta_\blacksquare(v_1, v_2) := v_1\mathbf{R}v_2 \wedge \sim v_1 \approx v_2$, $\beta_{\overline{\mathrm{D}}}(v_1, v_2) := \sim v_1 \approx v_2$. Then $[\beta_\Box]A$ is $\Box A$, $[\beta_\blacksquare]A$ is $\blacksquare A$, and $[\beta_{\overline{\mathrm{D}}}]A$ is $\overline{\mathrm{D}}A$. By set theoretical notations, we often write simply $[R]$, $[(R\cap\neq)]$, and $[\neq]$ for them. As a matter of convention, $\{\beta_i\}_i$ denotes $\{\beta_i \mid 1 \leq i \leq n\}$ where $\beta_i(v_1, v_2)$ is a formula of $\mathcal{L}^f$.

We consider the modal language $\mathcal{L}_{\{\beta_i\}_i}$, which denotes $\mathcal{L}_{[\beta_1]\ldots[\beta_n]}$. A *multimodal frame* is a pair $\langle W, \{R_i\}_i\rangle$ where R_i is a binary relation on $W \neq \varnothing$. A *multimodal model* and *multimodal pointed model* are defined similarly to the bimodal case. The satisfaction relation $\Vdash$ is defined as usual. We define *multimodal bisimulation*, *multimodal Hennessey-Milner property* (multimodal HMP), and *multimodal saturation* as in the bimodal case.

A multimodal frame where $\beta_i(v_1, v_2)$ of $\mathcal{L}^f$ determines $R_i \subset W^2$ $(1 \leq i \leq n)$ for some R is called an $\mathcal{L}_{\{\beta_i\}_i}$*-frame*. An $\mathcal{L}_{\{\beta_i\}_i}$*-model* is defined similarly. Observe that an $\mathcal{L}_{\{\beta_i\}_i}$-frame (or -model) is determined by the unimodal frame $\langle W, R\rangle$ (or model, respectively). Therefore, we confuse $\langle W, R\rangle$ and $\langle W, \{R_i\}_i\rangle$ for an $\mathcal{L}_{\{\beta_i\}_i}$-frame.

The *standard translation* ST_x taking $\mathcal{L}_{\{\beta_i\}_i}$-formulas to the first-order formulas of $\mathcal{L}^m(\Phi)$ is defined as before except that $ST_x([\beta_i]A) := \forall y(\beta_i(x, y) \supset ST_y(A))$ $(1 \leq i \leq n)$, where y is a fresh variable. Define $ST_x[\Sigma] = \{\alpha(x) \mid (\exists A \in \Sigma) \models ST_x(A) \equiv \alpha(x)\}$ for any subsets Σ of $\mathcal{L}_{\{\beta_i\}_i}$. For a formula A of $\mathcal{L}_{\{\beta_i\}_i}$, we can prove that $\mathfrak{M}, w \Vdash A$ if and only if $\mathfrak{M} \models ST_x(A)[w]$ for all $\mathcal{L}_{\{\beta_i\}_i}$-models $\mathfrak{M}$ and all w in $\mathfrak{M}$ in the same way. Then we have the following as before.

Lemma 6.1

(1) *Any class of all multimodally saturated $\mathcal{L}_{\{\beta_i\}_i}$-models enjoys multimodal* HMP.

(2) *An ω-saturated $\mathcal{L}_{\{\beta_i\}_i}$-model $\langle W, R, V\rangle$ is multimodally saturated.*

For (2), the only difference from the proof of Lemma 4.8 is that we define $\Sigma' = \{\beta_i(\underline{w}, x)\} \cup ST_x[\Sigma]$.

The next theorem (Theorem 6.2) seems to be a kind of folklore among the modal model theorists as mentioned before.

Theorem 6.2 (van Benthem characterization theorem in $\mathcal{L}_{\{\beta_i\}_i}$) *For an $\mathcal{L}^m$-formula $\alpha(x)$, $\alpha(x) \in ST_x[\mathcal{L}_{\{\beta_i\}_i}]$ if and only if $\alpha(x)$ is invariant for multimodal bisimulations between $\mathcal{L}_{\{\beta_i\}_i}$-models.*

Theorem 6.3 *Let* P *be a class of pointed $\mathcal{L}_{\{\beta_i\}_i}$-models. Then*

(1) P *is definable by a set of $\mathcal{L}_{\{\beta_i\}_i}$-formulas if and only if* P *is closed under multimodal bisimulations between $\mathcal{L}_{\{\beta_i\}_i}$-models and under ultraproducts, and $\overline{\mathsf{P}}$ is closed under ultrapowers.*

(2) P *is definable by a single $\mathcal{L}_{\{\beta_i\}_i}$-formula if and only if both* P *and $\overline{\mathsf{P}}$ are closed under multimodal bisimulations between $\mathcal{L}_{\{\beta_i\}_i}$-models and under ultraproducts.*

These are proved in the same way as in the case of $\mathcal{L}_\Box$ (for details, see [2], Theorems 2.75 and 2.76). As corollaries of Theorem 6.2 and Theorem 6.3, we derive the characterizations for $\mathcal{L}_{\square\blacksquare}$ (Theorem 4.9 and Theorem 4.11, respectively) and for $\mathcal{L}_{\square\overline{\mathrm{D}}}$ ([19], Theorem 4.7 and Theorem 4.8, respectively).

Multimodal p-morphisms and the related notations are defined as before. Then we can prove that if $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, then $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$ for any formula A of a multimodal language.

We can define *multimodally generated subframes* as in the unimodal case. Observe that $\mathfrak{F} \Vdash A$ implies $\mathfrak{F}' \Vdash A$ for any formula A, for any multimodally generated subframe $\mathfrak{F}'$ of $\mathfrak{F}$ (written $\mathfrak{F}' \rightarrowtail \mathfrak{F}$). For generated subframes of $\mathcal{L}_{\{\beta_i\}_i}$-frames, we use the following different notations from unimodally generated subframes: *The multimodal subframe* $\mathfrak{F}_X^{\{R_i\}_i}$ *generated by* X, the *point-generated multimodal frame* $\mathfrak{F}_w^{\{R_i\}_i}$ *by* w.

For an $\mathcal{L}_{\{\beta_i\}_i}$-frame, multimodally generated subframes might differ from unimodally generated subframes since in the multimodal case the closure under all associated relations is required. For example, in the case of $\mathcal{L}_{\square\overline{\mathrm{D}}}$, a multimodal subframe must be the original frame itself. Another example is an $\mathcal{L}_{[R\circ R]}$-frame $\langle \omega, R \circ R \rangle$ where $R = \{\langle m, m+1\rangle \mid m \in \omega\}$ and $\circ$ denotes the composition. $|\mathfrak{F}_0^{R\circ R}| = \{2m \mid m \in \omega\}$ and if we put $R' = R \cap |\mathfrak{F}_0^{R\circ R}|^2$, then $R' = \varnothing$ whence $R' \circ R' \neq (R \circ R) \cap |\mathfrak{F}_0^{R\circ R}|^2$. These motivate us to define the following notion, *absolute*. If $\{\beta_i\}_i$ is absolute, any multimodally generated subframe of an $\mathcal{L}_{\{\beta_i\}_i}$-frame can be seen as a unimodally generated $\mathcal{L}_{\{\beta_i\}_i}$-frame.

Definition 6.4 (Absolute) Let $\{\beta_i(v_1, v_2)\}_i$ be a set of formulas of $\mathcal{L}^f$. $\{\beta_i\}_i$ is *absolute* if for any $\mathcal{L}_{\{\beta_i\}_i}$-frame $\langle W, R\rangle$, any multimodally generated subframe $\mathfrak{F}' = \langle W', \{R_i\}_i\rangle$ of $\mathfrak{F}$, and any $\overrightarrow{w}$ from W', the following holds: $\langle W, R\rangle \models \beta_i[\overrightarrow{w}]$ if and only if $\langle W', R'\rangle \models \beta_i[\overrightarrow{w}]$ where $R' = R \cap (W')^2$.

Proposition 6.5 *If $\{\beta_i\}_i$ is a set of quantifier-free formulas, then $\{\beta_i\}_i$ is absolute.*

For example, any nonempty subsets of $\{R, R^{-1}, -R, (R\cap =), (R\cap\neq), W^2, \neq\}$, are absolute, where R^{-1} is the converse of R and $-R$ is the complement of R, that is, $W^2 \setminus R$. $\{(R\cap\neq), R \circ R\}$ is also absolute, while $\{R \circ R\}$ is not absolute as seen before. Thus, we need to define the notion of absoluteness for each combination of formulas, not for each formula.

For a relation Q on a nonempty set, Q^* denotes the reflexive transitive closure of Q. Note that the domain $|\mathfrak{F}_w^{\{R_i\}_i}|$ of multimodal point-generated subframe $\mathfrak{F}_w^{\{R_i\}_i}$ of $\mathfrak{F}$ is $\{w' \in W \mid w \left(\bigcup_i R_i\right)^* w'\}$.

Definition 6.6 (γ-relativized formulas of $\mathcal{L}^f$) Fix a quantifier-free formula $\gamma(v_1, v_2)$ of $\mathcal{L}^f$.

(i) $x\mathbf{R}y$ and $x \approx y$ are γ-relativized formulas.

(ii) If α and β are γ-relativized formulas, then so are $\sim \alpha$ and $\alpha \supset \beta$.

(iii) If x, y are distinct variables and α is a γ-relativized formula, then so is $\forall y\, (\gamma(x, y) \supset \alpha)$.

Proposition 6.7 *Fix a quantifier-free formula $\gamma(v_1, v_2)$ of $\mathcal{L}^f$. Let $\{\beta_i(v_1, v_2)\}_i$ be γ-relativized formulas. Suppose that $R_\gamma \subset \left(\bigcup_i R_{\beta_i}\right)^*$ for any $\mathcal{L}_{\{\beta_i\}_i}$-frame $\mathfrak{F} = \langle W, R\rangle$. Then $\{\beta_i\}_i$ is absolute.*

Proof Suppose that $\langle W', \{R'_{\beta_i}\}_i\rangle \rightarrowtail \mathfrak{F}$ and $\overrightarrow{w} = \langle w_1, \ldots, w_m\rangle \in W'^m$. We prove that $\langle W, R\rangle \models \alpha[\overrightarrow{w}]$ if and only if $\langle W', R \cap (W')^2\rangle \models \alpha[\overrightarrow{w}]$

by induction on a γ-relativized formula α. Let us check the case where α is $\forall y\,(\gamma(x, y) \supset \alpha')$ only. Since the direction from left to right is obvious, we show the converse. Suppose that $w_k R_\gamma v$ and $v \in W$. $w_k R_\gamma v$ implies $w_k (\bigcup_i R_\beta)^* v$, whence $v \in W'$ since W' is multimodally generated and $w_k \in W'$. By assumption, $\langle W', R \cap (W')^2 \rangle \models \alpha'[\overrightarrow{w}, v]$, whence $\langle W, R \rangle \models \alpha'[\overrightarrow{w}, v]$ by induction hypothesis. □

Observe that $\bot$ and $\top$ are quantifier-free formulas. Proposition 6.5 is a corollary of this Proposition 6.7, since $\bot$ defines $R_\bot = \varnothing$ and since all quantifier-free formulas are $\bot$-relativized. We have the following corollary since the quantifier-free formula $\top$ defines $R_\top = W^2$ and since all formulas are $\top$-relativized.

Corollary 6.8 *Suppose that* $\left(\bigcup_i R_{\beta_i}\right)^* = W^2$ *for any* $\mathcal{L}_{\{\beta_i\}_i}$*-frame* $\mathfrak{F} = \langle W, R \rangle$*. Then* $\{\beta_i\}_i$ *is absolute.*

Next we introduce a new construction, amalgamation, which is a generalization of disjoint union.

Definition 6.9 A multimodal frame $\mathfrak{F}$ is an *amalgamation* of $\{\mathfrak{G}_j \mid j \in J\}$ if $\mathfrak{G}_j$ is, up to isomorphism, a multimodally generated subframe of $\mathfrak{F}$ for any $j \in J$ and $\bigcup_{j \in J} |\mathfrak{G}_j| = |\mathfrak{F}|$.

Note that the closure properties under unimodal disjoint unions and bimodal p-morphic images imply the closure property under amalgamations in $\mathcal{L}_{[R][(R\cap\neq)]}$, that is, $\mathcal{L}_{\Box\blacksquare}$.

Proposition 6.10 *For an amalgamation* $\mathfrak{F}$ *of* $\{\mathfrak{G}_j \mid j \in J\}$ *and a formula* A*, if* $\mathfrak{G}_j \Vdash A$ *for all* $j \in J$*, then* $\mathfrak{F} \Vdash A$*.*

Proof We prove the contraposition. Suppose that $\mathfrak{F} \nVdash A$; that is, $\langle \mathfrak{F}, V \rangle, v \nVdash A$ for some V and some v. Then $v \in |\mathfrak{G}_j|$ for some $j \in J$ since $\bigcup_{j \in J} |\mathfrak{G}_j| = |\mathfrak{F}|$. Since $\langle \mathfrak{G}_j, V' \rangle, v \nVdash A$ where V' is the restriction of V, $\mathfrak{G}_j \nVdash A$. □

As a matter of convention, write $R_{\beta=} = \left(\left(\bigcup_{1\le i\le n} R_{\beta_i}\right) \cup =\right)$ and $[\beta_=]A := A \wedge [\beta_1]A \wedge \cdots \wedge [\beta_n]A$.

Definition 6.11 (Jankov-Fine formula of $\mathcal{L}_{\{\beta_i\}_i}$) Let $\mathfrak{F} = \langle W, R \rangle$ be a finite $\mathcal{L}_{\{\beta_i\}_i}$-frame, which is multimodally generated by w. Enumerate W as $w = w_0, \ldots, w_m$. Associate a distinct p_i in Φ with each w_i. Let $A_{\mathfrak{F},w}$ be the conjunction of the following:

(1) p_0,
(2) $[\beta_=](p_1 \vee \cdots \vee p_n)$,
(3) $[\beta_=](p_i \supset\sim p_j)$ for each i, j with $i \neq j$,
(4) $[\beta_=](p_i \supset \langle \beta_l \rangle p_j)$ for each i, j, l with $w_i R_{\beta_l} w_j$,
(5) $[\beta_=](p_i \supset\sim \langle \beta_l \rangle p_j)$ for each i, j, l with not $w_i R_{\beta_l} w_j$.

Lemma 6.12 *Let* $\{\beta_i\}_i$ *be absolute. Let* $\mathfrak{F}$ *be a finite* $R_{\beta=}$*-transitive* $\mathcal{L}_{\{\beta_i\}_i}$*-frame, which is multimodally generated by* w*. Then for any* $R_{\beta=}$*-transitive* $\mathcal{L}_{\{\beta_i\}_i}$*-frame* $\mathfrak{G}$*, (A)* $A_{\mathfrak{F},w}$ *is satisfiable in* $\mathfrak{G}$ *if and only if (B) there exists a surjective multimodal* p*-morphism from* $\mathfrak{G}_v$ *onto* $\mathfrak{F}$ *for some state* v *in* $\mathfrak{G}$*.*

We can prove this as a consequence of the multimodal case of [2], Lemma 3.20.

Theorem 6.13 *Let $\{\beta_i\}_i$ be absolute. Let $\mathsf{D}_{\text{fintra}}$ be the class of all finite $R_{\beta=}$-transitive $\mathcal{L}_{\{\beta_i\}_i}$-frames. Then (A) K is modally definable within $\mathsf{D}_{\text{fintra}}$ if and only if (B) K is closed under amalgamations, multimodally generated subframes, and multimodal p-morphic images.*

By Lemma 6.12, we can prove the rest similarly to the unimodal case [2], Theorem 3.21. Since $\mathfrak{F}$ is an amalgamation of $\{\mathfrak{F}_w \mid w \in |\mathfrak{F}|\}$ by absoluteness, we may assume that $\mathfrak{F}$ is multimodally point-generated.

We can generalize this theorem for a finite absolute set $\{\beta_i\}_i$ of higher order formulas by defining absoluteness in the same way. In particular, its language may have, for example, an absolute and finite set of operators whose accessibility relations are obtained by Boolean combinations and the $*$-operation from R and the equality. In general, if we add the operator $[(R_{\beta=})^*]$ to $\mathcal{L}_{\{\beta_i\}_i}$, because the clauses (in the three closure operations) and the absoluteness for $\{R_{\beta_i}\}_i$ imply those for $\{R_{\beta_i}\}_i \cup \{(R_{\beta=})^*\}$, we can extend $\mathsf{D}_{\text{fintra}}$ in Theorem 6.13 to the class of all finite frames without changing the closure conditions. Thus, for example, for $\mathcal{L}_{[R][R^*]}$, K is modally definable within the class of finite frames if and only if K is closed under disjoint unions and unimodally generated subframes and unimodal p-morphic images.

Combining this theorem and Proposition 6.5, we get the following corollary.

Corollary 6.14 *Let $\{\beta_i\}_i$ be a set of quantifier-free formulas. Let $\mathsf{D}_{\text{fintra}}$ be the class of all finite $R_{\beta=}$-transitive $\mathcal{L}_{\{\beta_i\}_i}$-frames. Then (A) K is modally definable within $\mathsf{D}_{\text{fintra}}$ if and only if (B) K is closed under amalgamations, multimodally generated subframes, and multimodal p-morphic images.*

Multimodal UE is defined as in the bimodal case. We can prove the following as before.

Definition 6.15 (Realizer, realization) Let $\mathfrak{F}$ be a multimodal frame and $\mathfrak{F}'$ be an $\mathcal{L}_{\{\beta_i\}_i}$-frame. If $f : \mathfrak{F}' \to \mathfrak{F}$ is surjective as a mapping between domains, f is a $\{\beta_i\}_i$-*realizer* and $\mathfrak{F}'$ is a $\{\beta_i\}_i$-*realization* of $\mathfrak{F}$.

Proposition 6.16 *For any $\mathcal{L}_{\{\beta_i\}_i}$-frame $\mathfrak{F}$, any A of $\mathcal{L}_{\{\beta_i\}_i}$ and any $\{\beta_i\}_i$-realization $\mathfrak{G}$ of $\mathfrak{ue}\,\mathfrak{F}$, $\mathfrak{G} \Vdash A$ implies $\mathfrak{F} \Vdash A$.*

A class K of $\mathcal{L}_{\{\beta_i\}_i}$-frames *reflects* $\{\beta_i\}_i$-realizations of multimodal UEs if $\mathfrak{G} \in \mathsf{K}$ implies $\mathfrak{F} \in \mathsf{K}$ for any $\mathcal{L}_{\{\beta_i\}_i}$-frame $\mathfrak{F}$ and any $\{\beta_i\}_i$-realization $\mathfrak{G}$ of $\mathfrak{ue}\,\mathfrak{F}$.

Theorem 6.17 *Let $\{\beta_i\}_i$ be absolute. Let K be an elementary class of $\mathcal{L}_{\{\beta_i\}_i}$-frames. Then (A) K is modally definable in $\mathcal{L}_{\{\beta_i\}_i}$ if and only if (B) it is (i) closed under amalgamations, (ii) closed under multimodally generated subframes, and (iii) reflects $\{\beta_i\}_i$-realizations of multimodal* UE*s.*

Proof We have established (A) $\Longrightarrow$ (B). Conversely, assume (B). Let Λ_{K} be the logic of K, that is, $\{A \text{ of } \mathcal{L}_{\{\beta_i\}_i} \mid \mathsf{K} \Vdash A\}$. It suffices to prove that, for any $\mathfrak{F}$, $\mathfrak{F} \Vdash \Lambda_{\mathsf{K}}$ implies $\mathfrak{F} \in \mathsf{K}$. Suppose $\mathfrak{F} \Vdash \Lambda_{\mathsf{K}}$. We only mention differences from the proof of Theorem 5.17 for $\mathcal{L}_{\Box\blacksquare}$. The arguments except Step 1 and Step 6 are the same.

Step 1 Since $\mathfrak{F}$ is an amalgamation of $\{\mathfrak{F}_w \mid w \in |\mathfrak{F}|\}$, we can assume that $\mathfrak{F}$ is multimodally point-generated.

Step 6 The proof is similar except the following point. In order to show the equivalence, $\mathfrak{M} \Vdash A$ if and only if $\mathfrak{N} \Vdash A$ for any A of $\mathcal{L}_{\{\beta_i\}_i}$ (see [2], p. 180), we need to show the following: For any $\mathcal{L}_{\{\beta_i\}_i}$-model $\mathfrak{M}'$, if $\mathfrak{M}'$ is point-generated by x, then $\mathfrak{M}' \Vdash A \iff [\mathfrak{M}', x \Vdash [\beta_{j_1}] \dots [\beta_{j_m}]A$ for any finite sequence $\langle j_1, \dots, j_m \rangle$ from $\{ i \mid 1 \leq i \leq n \}]$. □

	Relative GT (Cor. 6.14)	GT (Cor. 6.18)	Examples
$R^* = (R_{\beta=})^*$	**(closed under)** -disjoint unions -unimodal generated subframes -multimodal p-morphic images	**(closed under)** -ditto -ditto -ditto **(reflect)** -$\{\beta_i\}_i$-realizations of multimodal UEs	$\mathcal{L}_{\Box}$ $\mathcal{L}_{\Box\blacksquare}$ $\mathcal{L}_{\Box\boxdot}$ $\mathcal{L}_{\blacksquare\boxdot}$
$(R \cup R^{-1})^* = (R_{\beta=})^*$	**(closed under)** -disjoint unions -$(R \cup R^{-1})$-generated subframes -multimodal p-morphic images	**(closed under)** -ditto -ditto -ditto **(reflect)** -$\{\beta_i\}_i$-realizations of multimodal UEs	$\mathcal{L}_{[R][R^{-1}]}$
$W^2 = (R_{\beta=})^*$	**(closed under)** -multimodal p-morphic images	**(reflect)** -$\{\beta_i\}_i$-realizations of multimodal UEs	$\mathcal{L}_{\Box\overline{\mathrm{D}}}$ $\mathcal{L}_{[R][W^2]}$ $\mathcal{L}_{[R][-R]}$

Table 4 Our results cover almost all extensions by modal operators

Corollary 6.18 *Let $\{\beta_i\}_i$ be a set of quantifier-free formulas and* K *an elementary class of $\mathcal{L}_{\{\beta_i\}_i}$-frames. Then (A)* K *is modally definable in $\mathcal{L}_{\{\beta_i\}_i}$ if and only if (B) it is (i) closed under amalgamations, (ii) closed under multimodally generated subframes, and (iii) reflects $\{\beta_i\}_i$-realizations of multimodal* UE*s.*

Corollaries 6.14 and 6.18 cover almost all extensions by modal operators that have already been introduced, for example, any nonempty subsets of $\{[R], [R^{-1}], [-R], [-R^{-1}], [(R\cap =)], [(R\cap\neq)], [W^2], [\neq]\}$. These examples include modal logic ($[R]$) and one with our ■ or the loop operator ⊡, tense logic ($[R]$ and $[R^{-1}]$) [8], modal and tense logics with the global modality $[W^2]$ [9], with the window operators $[-R]$ and $[-R^{-1}]$ [12], and with the difference operator $\overline{D}$ [19].

If, in Corollaries 6.14 and 6.18, '$R^* = (R_{\beta=})^*$' (or $(R \cup R^{-1})^* = (R_{\beta=})^*$) holds and we add the closure property of K under multimodal p-morphic images, we can replace '(i) amalgamations' and '(ii) multimodally generated subframes' with '(i′) unimodal disjoint unions' and '(ii′) unimodally generated subframes' (or $(R \cup R^{-1})$-generated subframes, respectively), because (i′) & (ii′) imply (i) & (ii) by the closure under multimodal p-morphic images. Therefore, with respect to $\mathcal{L}_{\Box\blacksquare}$, we get Theorems 5.12 and 5.17 as corollaries. To get Theorem 5.12, note that $R_{\beta=}$-transitivity is equivalent to R-transitivity in the case where $R = \bigcup_i R_{\beta_i}$.

Where $(R_{\beta=})^* = W^2$ for any $\mathcal{L}_{\{\beta_i\}_i}$-frame $\langle W, R \rangle$, they are absolute by Corollary 6.8 and we can delete the clauses about amalgamations and multimodally generated subframes in Corollaries 6.14 and 6.18 since such notions are trivialized. To get relative Goldblatt-Thomason-style characterization, observe that, if $W^2 = R_{\beta=}$ holds, then $R_{\beta=}$-transitivity holds trivially. Thus, for $\mathcal{L}_{\Box\overline{\mathrm{D}}}$, we get [19], Proposition 4.3, and Fact 5.18(2) as corollaries.

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Acknowledgments

Both of the authors would like to thank KABA (Kyoto Artificial Brain Associates) for their kind support for our research seminars. The first author would like to thank Yasushi Hashimoto, Izumi Takeuti, Ryo Kashima, and Soshichi Uchii for their invaluable comments and encouragement.

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