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Relative Vaught's Conjecture for Some Meager Groups

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Abstract Assume G is a superstable locally modular group. We describe for any countable model M of $\mathrm{Th}(G)$ the quotient group $G(M)/Gm(M)$. Here Gm is the modular part of G. Also, under some additional assumptions we describe $G(M)/Gm(M)$ relative to $G^-(M)$. We prove Vaught's Conjecture for $\mathrm{Th}(G)$ relative to Gm and a finite set provided that $\mathcal{M}(G) = 1$ and the ring of pseudoendomorphisms of G is finite.

1 Introduction

Throughout, we assume T is a superstable theory with *few* (that is, $< 2^{\aleph_0}$) countable models. We work inside a fixed monster model $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$ of T. The paper continues the work of [14] on Vaught's Conjecture for groups with meager forking. We assume that the reader is familiar with geometric stability theory ([3], [16]). Also some knowledge of meager forking ([8], [9], [10]) and pseudotypes ([6], [7]) would be helpful, although we shall present the main notions again in Section 2.

Assume $G \subseteq \mathfrak{C}$ is a 0-definable regular group with meager forking on the set of generic types (we call such groups *meager*, for short). A regular group is a group where each generic type is regular (that is, forking dependence is a pregeometry on this type). Such groups occur quite naturally in superstable structures (see [14], preliminaries). The eventual goal is a proof of Vaught's Conjecture for $\mathrm{Th}(G)$ if, for example, $U(G) = \omega$. Thus far the proofs in [14] and here follow the scheme from [1] (and [15]). In [14] we proved Vaught's Conjecture for $\mathrm{Th}(G)$ when G is meager, $U(G) = \omega$, $\mathcal{M}(G) = 1$, and the ring of pseudoendomorphisms $\mathcal{F}_G$ is a prime field.

Actually, we use the assumption that $U(G) = \omega$ there only to conclude by [2] that Vaught's Conjecture for $\mathrm{Th}(G^-)$ is true and that the structure of $G^-(M)$ is clear for countable $M \models T$. So it is reasonable to try to describe $G(M)$ provided that we know what $G^-(M)$ is, and then the assumption that $U(G) = \omega$ may be omitted.

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This leads to Vaught's Conjecture for G relative to G^-, which says that once we know what $G^-(M)$ is for a countable model M of T, then up to isomorphism there are countably many possibilities for M.

Actually, in [14] we prove Vaught's Conjecture for G relative to G^- and a finite set provided that G is meager, $\mathcal{M}(G) = 1$, and the ring $\mathcal{F}_G$ is a prime field. In this paper we try to generalize this result to the case when $\mathcal{F}_G$ is finite. In Theorem 3.12 below we prove Vaught's Conjecture for G relative to Gm (the modular part of G) and a finite set, provided that G is meager, $\mathcal{M}(G) = 1$, and $\mathcal{F}_G$ is finite. The main obstacle in the proof is the fact that the values of pseudoendomorphisms in G are "fuzzy," that is, defined up to G^-. We overcome this trouble by introducing a resolving group G'.

Also we try to weaken the assumption that $\mathcal{M}(G) = 1$. In Section 4 we deal with a general setup, where G is a locally modular regular group definable in a superstable theory T. We describe up to isomorphism the set of cosets of Gm realized in a countable model M of T. In fact, the nontrivial case in this classification is the meager one.

The proofs mostly generalize those in [14]. The important point is an analysis of $\mathrm{Aut}(Q)$-orbits of generic types of G when $Q = G^-(M)$ or $Q = Gm(M)$.

2 Preliminaries

In this section we recall some notions and results from [5]–[10] used in this paper. Throughout, Φ is a countable disjunction of formulas over $\varnothing$ and $Q = \Phi(M)$ for some countable model M of T. K_Q is the class of models N such that $\Phi(N) = Q$. For a formula $\varphi(x)$ (possibly with parameters), $[\varphi(x)]$ denotes the class of all (partial) types containing $\varphi(x)$.

Vaught's Conjecture relative to Q is an assertion that if T has few countable models then in K_Q there are countably many of them (up to isomorphism). Vaught's Conjecture relative to Φ is the conjunction of Vaught's Conjectures relative to Q, where Q varies over countable sets of the form $\Phi(N)$.

Suppose A is a countable subset of some $N \in K_Q$. We say that $p \in S(QA)$ is *good* if p is realized in some model in K_Q containing A; otherwise we say that p is *bad*. Let $\mathrm{Aut}(Q/A)$ be the group of automorphisms of $\mathfrak{C}$ fixing Q setwise and A pointwise. We say that $p \in S(QA)$ is Q-isolated over A if the $\mathrm{Aut}(Q/A)$-orbit of p is not meager as a subset of the topological space $S(QA)$. We will say often "$\mathrm{Aut}(Q)$-orbit over A" instead of "$\mathrm{Aut}(Q/A)$-orbit." A model $N \in K_Q$ containing A is called Q-atomic over A if for every finite tuple $a \subseteq N$, the type $\mathrm{tp}(a/QA)$ is Q-isolated over A. These notions are elaborated upon in [6] and [7]. For instance, we have the following lemma, whose first part relies on the few models assumption.

Lemma 2.1 *If $A \subseteq M$ is finite then there is a countable model $N \in K_Q$, which is Q-atomic over A. Moreover, such an N is unique up to isomorphism.*

To investigate isolation properties of types we use the notion of trace of a type. If A is any set of parameters and $s(x)$ is any type (possibly incomplete, or even a single formula) over $\mathfrak{C}$, then the *trace* of s over A is the set

$$\mathrm{Tr}_A(s) = \{\mathrm{tp}(a/\mathrm{acl}(A)) : a \text{ realizes } s(x)\}.$$

$\mathrm{Tr}_A(a/B)$ abbreviates $\mathrm{Tr}_A(\mathrm{tp}(a/b))$, and we omit A if $A = \varnothing$.

Suppose $\mathcal{P} \subseteq S(\mathrm{acl}(A))$ is closed. We say that forking is *meager* on $\mathcal{P}$ if for every formula $\varphi(x)$ forking over A, the set $\mathrm{Tr}_A(\varphi) \cap \mathcal{P}$ is nowhere dense in $\mathcal{P}$. Using this notion we can define meager types in an arbitrary stable theory. In the case of a small superstable theory we can give an easier definition. We say that a regular type p is *meager* if there is an isolated regular type q over some finite set A such that q is nonorthogonal to p and forking is meager on $\mathrm{Tr}_A(q)$.

We say that $X \subseteq S(\mathrm{acl}(A))$ is *small* if there are finitely many types $r_1, \ldots, r_k \in X$ such that any $r \in X$ is not almost orthogonal to some r_i. If T has few countable models, then meager types have the following property.

Theorem 2.2 ([8], [10]) *Assume $A \subseteq B$ are finite, $p \in S(A)$ is meager, and $q \in S(B)$ is a nonforking extension of p. Then exactly one of the following conditions holds.*

(1) *q is isolated (equivalently, $\mathrm{Tr}_A(q)$ is open in $S(\mathrm{acl}(A))$).*
(2) *$\mathrm{Tr}_A(q)$ is small.*

More information on meager types may be found in [8], [9], and [10]. From now on usually we assume that $p \in S(\mathrm{acl}(\emptyset))$ is a meager stationary type. In this paper we deal with meager types in $S(\mathrm{acl}(\emptyset))$ nonorthogonal to p. So "regular type" will mean a meager regular type which is nonorthogonal to p.

For an arbitrary set A we define $CL(A)$ as the set $\{r \in S(\mathrm{acl}(\emptyset)) : r|A$ is regular and modular$\}$. For $X \subseteq S(\mathrm{acl}(\emptyset))$ we set $CL(X)$ as $CL(A)$, where A is any independent set of realizations of types in X such that each type in X is realized in A.

Suppose $q \in S(\emptyset)$ is regular and A is finite. Then $CL(A) \cap \mathrm{Tr}(q)$ is closed and small.

Occasionally we will also use the notions of a p-formula and $\mathcal{M}$-rank; these may be found in [6], [8], and [9]. $\mathcal{M}$-rank measures the topological size of the sets of stationarizations of complete types over finite sets. So it tells us how much such types differ from stationary ones. If T has few countable models then $\mathcal{M}$-rank of any type is finite and $\leq U$-rank.

Throughout, we assume that G is a 0-definable regular Abelian locally modular group; the group law is written additively. $\mathcal{G} \subseteq S(\mathrm{acl}(\emptyset))$ is the set of generic types of G and $G^- = \mathrm{cl}_p(\emptyset) \cap G$ (we assume that p is the generic type of G^0). We define $\mathcal{G}m$ as the set of modular types in G; in particular, the generic type of G^0 is modular. By smallness, $\mathcal{G}m$ is closed and $\mathcal{G} \setminus \mathcal{G}m$ is open in $S(\mathrm{acl}(\emptyset))$ [8]. We say that G is *meager* if its generic types are meager. This is equivalent to saying that forking is meager on the set of generic types $\mathcal{G}$. In fact, we have the following characterization of meager groups.

Remark 2.3 ([8]) Assume G is locally modular. Then G is meager if and only if $\mathcal{G}m$ is nowhere dense in $\mathcal{G}$. Moreover, if G is meager then every pseudoendomorphism of G^0 is definable over $\mathrm{acl}(\emptyset)$. In this case the ring of pseudoendomorphisms of G (denoted by $\mathcal{F}_G$) is a locally finite field.

We can define a group law $+$ on $\mathcal{G}$ induced by the group law on G: $r+r' = \mathrm{stp}(a+a')$ for any independent realizations a, a' of r, r', respectively [5]. Then the generic type of G^0 is the neutral element of $(\mathcal{G}, +)$ and $\mathcal{G}m$ is a subgroup of $\mathcal{G}$. Given any set of types X and $M \models T$, we denote by X^M the set of types in X realized in M. Similarly, $(\mathcal{G}/\mathcal{G}m)^M$ denotes the set of $\mathcal{G}m$-cosets containing a type realized in M.

We define $\mathcal{M}(G)$ and $\mathcal{M}(\mathcal{G})$ as the $\mathcal{M}$-rank of any isolated generic type of G in $S(\varnothing)$. Then $\mathcal{M}(G) = 0$ if and only if G is connected-by-finite (i.e., $\mathcal{G}$ is finite), and if G is meager then $\mathcal{M}(G) > 0$ and $\mathcal{M}(G) = 1$ if and only if $\mathcal{G}m$ is finite (more generally, for meager G, $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{G}m) + 1$).

We denote by Gm the type-definable over $\varnothing$ subgroup of G generated by $\mathcal{G}m$, in the sense of [5] (so Gm is the modular part of G). Clearly, $\mathcal{G}m$ is the set of generic types of Gm [5].

From now on in this paper we assume that G is meager. Then we usually assume that p is the generic type of G^0. In this case, for any finite A, $CL(A) \cap \mathcal{G}$ is a closed subgroup of $\mathcal{G}$ containing $\mathcal{G}m$ as a subgroup of finite index.

3 A Relative Vaught's Conjecture

In this section we prove Vaught's Conjecture for G relative to Gm and a finite set, provided that G is meager, of $\mathcal{M}$-rank 1 (that is, $\mathcal{G}m$ is finite), and the ring of pseudoendomorphisms $\mathcal{F}_G$ is finite. p is the generic type of G^0. Notice that, more generally, when G is locally modular and not meager, then Gm has finite index in G, so in this case Vaught's Conjecture for G relative to Gm and a finite set (of representatives of Gm-cosets) is trivial. So the meager case is the nontrivial one.

In [14] we proved Vaught's Conjecture for such G relative to G^- and a finite set, provided that $\mathcal{F}_G$ is a prime field. The main obstacle in the proof when $\mathcal{F}_G$ is not a prime field is the fact that the values of pseudoendomorphisms in G are "fuzzy," that is defined only up to G^-. By a result of Hrushovski [3], the linear dependence over $\mathcal{F}_G$ fully describes the forking dependence of generic elements in G^0. So the naïve approach to describe $G^0(M)$ would be to take a maximal Morley sequence I in $p(M)$ and then $G^0(M)$ should be the projective space over $\mathcal{F}_G$ spanned by I. However, since the values of pseudoendomorphisms are "fuzzy," this is not enough to determine $G^0(M)$. This trouble vanishes in [14] when $\mathcal{F}_G$ is a prime field, since we can choose representatives of elements of $\mathcal{F}_G$ with firmly defined values in G.

Here in the case where $\mathcal{F}_G$ is finite we deal with this trouble introducing a resolving group G' such that G is a homomorphic image of G' and the "fuzzy" forking dependence in G is resolved to dcl-dependence with respect to pre-images in G'. The notion of resolving group seems to be of independent interest.

Now we recall some results from [3]. Again, G is a 0-definable locally modular Abelian group and p is the generic type of G^0. For $a, b \in G$ we write $a =^* b$ if $a - b \in G^-$.

Let $\mathcal{H}$ be the family of $\mathrm{cl}_p(\varnothing)$-definable subgroups H of $G \times G$ such that $G^0 \subseteq \pi_1(H)$ and $(\{0\} \times G) \cap H \subseteq G^-$ (π_1 is the projection to the first coordinate). For $H \in \mathcal{H}$ let $S_H = (\{0\} \times G) \cap H$ and $D_H = \pi_1(H)$. H is the graph of a group homomorphism $f_H : D_H \to G/S_H$, which in turn induces $f'_H : D_H \to G/G^-$ (as $S_H \subseteq G^-$). Notice that $f'_H : G^0 \to (G^0 + G^-)/G^-$.

Let $\mathcal{F} = \mathcal{F}_G = \{f'_H \lceil G^0 : H \in \mathcal{H}\}$. $\mathcal{F}$ is the division ring of $\mathrm{cl}_p(\varnothing)$-definable pseudoendomorphisms of G^0. If $f \in \mathcal{F}$, $H \in \mathcal{H}$, and $f = f'_H \lceil G^0$, we say that f_H and H represent f.

Now we are able to introduce the notion of a resolving group for G. Suppose $f \in \mathcal{F}$. We say that G' is *f-resolving* for G (via g), if the following conditions hold.

(r1) G' is a $\mathrm{cl}_p(\varnothing)$-definable locally modular Abelian group.

(r2) $g : G' \to G$ is a $\mathrm{cl}_p(\varnothing)$-definable group homomorphism with $g[(G')^-] \subseteq G^-$ and the induced mapping $g : (G')^0/(G')^- \to G^0/G^-$ is a group isomorphism.

(r3) Let $f' = g^{-1}fg \in \mathcal{F}_{G'}$. Then f' is represented by some $H' \subseteq G' \times G'$ such that for any $a \in D_{H'}$, if $(a, b) \in H'$ and $(a, c) \in H'$ then $g(b) = g(c)$.

We say that G' is *resolving* for G (via g) if G' is f-resolving for G (via g) for every $f \in \mathcal{F}$.

The idea behind this definition is as follows. By (r3), the rings $\mathcal{F}_G$ and $\mathcal{F}_{G'}$ are isomorphic via the mapping $f \mapsto f'$. Suppose H and f_H represent f. Then the values of f_H in G are defined only up to S_H, so they are "fuzzy." However, by (r3), this fuzziness is resolved in G', via H', representing f. Namely, the values of f'_H are also fuzzy (up to $S_{H'}$); however, their images under g are firm (and they clearly belong to the respective cosets of S_H, that is, the fuzzy values of f_H).

Lemma 3.1 *If $f \in \mathcal{F}$ then there is an f-resolving group G'.*

Proof Suppose f is represented by $H \subseteq G \times G$. Take a generic $a = (a_1, a_2) \in H^0$. Clearly, a is p-simple and $w_p(a) = 1$. So by [3] we can choose an $a' \in \mathrm{acl}(a)$ such that $w_p(a') = 0$ and $\mathrm{stp}(a/a')$ is regular nonorthogonal to p. By [3], $\mathrm{stp}(a/a')$ is a translate of the generic type of $(G')^0$ for some a'-definable connected subgroup G' of H. Let $g : G' \to G$ be the projection to the first coordinate. We will show that G' is f-resolving via g. Clearly, (r1) and (r2) are satisfied. To check (r3) let $f' = g^{-1}fg \in \mathcal{F}_{G'}$ and

$$H' = \{(a, b) \in G' \times G' : a = (a_1, a_2),\ \ b = (b_1, b_2),\ \text{and } a_2 = b_1\}.$$

Clearly, H' is a subgroup of $G' \times G'$. We check that H' represents f'. For this it is enough to see that $(g \times g)(H')$ represents f. However, $(g \times g)(H') = G'$ and clearly G' (as an element of $\mathcal{H}$) represents f, so we are done. It is immediate from the definition of H' that (r3) holds. □

Lemma 3.2 *Suppose $f, f_0 \in \mathcal{F}$, G' is f-resolving for G (via $g : G' \to G$) and G'' is f'_0-resolving for G' (via $g' : G'' \to G'$), where $f'_0 = g^{-1}f_0g$. Then G'' is both f- and f_0-resolving for G (via $g \circ g'$).*

Proof For instance, we check that G'' is f-resolving. Let $f'' = (g')^{-1}f'g'$. Suppose H' represents f' and witnesses (r3) for G'. Then $H'' = (g' \times g')^{-1}(H')$ represents f'' and witnesses (r3) for G''. □

Theorem 3.3 *If $X \subseteq \mathcal{F}$ is finite then there is a group G' and $g : G' \to G$ such that G' is f-resolving via g for every $f \in X$.*

Proof Follows by Lemmas 3.1 and 3.2. □

It would be nice to know if the f-resolving groups for G, $f \in \mathcal{F}$, form an amalgamation class. To be more explicit, suppose $\mathcal{F}_0, \mathcal{F}_1$ are finite subsets of $\mathcal{F}$ and G', G'' are groups such that G' is f-resolving for G (via a g') for every $f \in \mathcal{F}_0$ and G'' is f-resolving for G (via a g'') for every $f \in \mathcal{F}_1$. Does there exist a G^* and homomorphisms $h' : G^* \to G'$ and $h'' : G^* \to G''$ such that $g'h' = g''h''$ and G'' is f-resolving for G (via $g'h'$) for every $f \in \mathcal{F}_0 \cup \mathcal{F}_1$?

Lemma 3.4 *If G is meager and $f \in \mathcal{F}$ then there are $\mathrm{acl}(\varnothing)$-definable G' and g such that G' is f-resolving for G via g.*

Proof We modify the proof of Lemma 3.1. As in Lemma 3.1 we find an a-definable regular group $H_a \subseteq G \times G$ (for some a) which represents f. Assume that $R_\infty(H_a)$ is minimal possible under these restrictions. Let $G' = \bigcap\{H_{a'} : a' \overset{s}{\equiv} a\}$. The proof of [4], Proposition 3.3, shows that G' is definable over $\mathrm{acl}(\varnothing)$ and represents f. Since $R_\infty(H_a)$ is minimal, G' is regular. One sees that G' is f-resolving via the projection to the first coordinate. □

Theorem 3.5 *Assume G is meager and $\mathcal{F}$ is finite. Then there are* $\mathrm{acl}(\varnothing)$*-definable G' and g such that G' is a resolving group for G (via g).*

Proof Follows from Lemmas 3.2 and 3.4. □

Now suppose G and p (the generic type of G^0) are meager, $M \models T$ is countable, and $Q = \Phi(M)$ for some (incomplete) type Φ over $\varnothing$. Also, from now on in this section, "regular" means "meager nonorthogonal to p."

In [14] we were classifying $G(M)$ relative to $G^-(M)$ and then Φ was chosen so that $Q = G^-(M) \cup \mathrm{acl}(\varnothing)$. In this case p was orthogonal to Φ. Here we want to classify $G(M)$ relative to $Gm(M)$, so the natural choice for Φ will be such that $Q = Gm(M) \cup \mathrm{acl}(\varnothing)$. Then p is not orthogonal to Φ; however, Φ and Q are small in some sense, which we make precise now.

We say that $A = \{a_\alpha, \alpha < \gamma\}$ is a *neat construction* if for every $\alpha < \gamma$, either $\mathrm{stp}(a_\alpha/a_{<\alpha})$ is regular and modular or $\mathrm{tp}(a_\alpha/a_{<\alpha})$ has p-weight 0. Here $a_{<\alpha}$ denotes the tuple $\{a_\beta, \beta < \alpha\}$.

We say that a tuple $a \subseteq \mathfrak{C}$ is *neat* if it is contained in a neat construction. We say that $A \subseteq \mathfrak{C}$ is *neat* if every element of A is neat. We say that Φ is *neat* if $\Phi(\mathfrak{C})$ is neat.

Remark 3.6

(1) If $A \subseteq \mathfrak{C}$ is neat then there is a neat construction $B = \{b_\alpha, \alpha < \gamma\}$ such that $\mathrm{acl}(A) \subseteq B$.
(2) If A is neat and $\mathrm{stp}(a)$ is regular nonmodular, then $a \mathop{\smile\!\!\!\!\!\!\;|}\; A$.

Proof (1) Clearly, if $B = \{b_\alpha, \alpha < \gamma\}$ and $C = \{c_\alpha, \alpha < \delta\}$ are neat constructions, then $B \cup C$ is a neat construction of length $\gamma + \delta$ (with the elements of B going first, and then the elements of C).

(2) Let $q = \mathrm{stp}(a)$. By (1) we may assume that $A = \{a_\alpha, \alpha < \gamma\}$ is a neat construction. We prove by induction on $\alpha < \gamma$ that

(a) $b \mathop{\smile\!\!\!\!\!\!\;|}\; a_\alpha (a_{<\alpha})$ for every b realizing q.

By the inductive hypothesis we know that for every b realizing q, $b \mathop{\smile\!\!\!\!\!\!\;|}\; a_{<\alpha}$. So for every b, $\mathrm{stp}(b/a_{<\alpha})$ is regular and nonmodular. Hence (a) follows both when $\mathrm{stp}(a_\alpha/a_{<\alpha})$ is modular and when it has p-weight 0. □

If, in Remark 3.6(2), A is finite and $\mathrm{tp}(a)$ is isolated, then we can prove that $\mathrm{tp}(a/A)$ is isolated. This follows from Proposition 4.2 below. The proof of this proposition relies on the few models assumption and on an analysis of some orbits over neat sets. Specifically, from now on we assume that Φ and Q are neat with $\mathrm{acl}(\varnothing) \subseteq Q$.

Suppose $q \in S(\varnothing)$ is regular and isolated. By Remark 3.6(2) each type in $\mathrm{Tr}(q)$ extends uniquely to a type over Q. Hence $\mathrm{Aut}(Q)$ acts naturally on $\mathrm{Tr}(q)$. We are interested in the orbits of this action. Below we restate some results from [14]. The context in [14] is somewhat different, but the proofs go through.

Lemma 3.7 ([14], Orbit Lemma 2.5) *Assume $r^* \in \mathrm{Tr}(q)$, X is the $\mathrm{Aut}(Q)$-orbit of r^*. Then either X is open in $S(\mathrm{acl}(\varnothing))$ or X is small. In the latter case X is nowhere dense.*

Lemma 3.8 ([14], Theorem 2.6; [1], Lemma 5.2; Basis Lemma) *Let (*) be the following condition.*

(*) *$r \in S(\mathrm{acl}(\varnothing))$ is regular and the $\mathrm{Aut}(Q)$-orbit of r is small.*

Then for every $N \in K_Q$ there is a finite set $B \subseteq N$ of elements realizing types r with (*) such that for any r with (*) realized in N, r has a forking extension over B.

Proofs of Lemmas 3.7 and 3.8 These are the same as in [14]. In [14] we assumed additionally that p is orthogonal to Φ. This enabled us to construct many models by realizing quite freely various meager types orthogonal to Φ (including the modular ones) without having to worry about bad types.

However, if we deal with nonmodular types (which is the case now), then in the many-model constructions in [14] no modular types appear. By Remark 3.6 these constructions are good here, too; that is, they yield many countable models in K_Q. □

Notice that every generic type of G is regular in our sense. Applying Orbit Lemma 3.7 we get the following characterization of Q-isolated generic types of G.

Remark 3.9 Suppose $A \subseteq M$ is finite and $a \in G$ is generic over QA. Then $\mathrm{tp}(a/QA)$ is Q-isolated over A if and only if the $\mathrm{Aut}(Q/A)$-orbit of $\mathrm{stp}(a)$ is open in $S(\mathrm{acl}(\varnothing))$ if and only if this orbit is not small.

Proof If $\mathrm{tp}(a)$ is nonisolated, then clearly $\mathrm{tp}(a/QA)$ is not Q-isolated over A and the orbit of $\mathrm{stp}(a)$ is small (by Theorem 2.2). Suppose $q = \mathrm{tp}(a)$ is isolated. Let X be the $\mathrm{Aut}(Q/A)$-orbit of $\mathrm{stp}(a)$. Working in $T(A)$, by Lemma 3.7, we get that either X is open or X is small and nowhere dense in $S(\mathrm{acl}(\varnothing))$. In the first case $\mathrm{tp}(a/QA)$ is Q-isolated over A; in the second it is not. □

The following remark generalizes [14], Remark 4.1.

Remark 3.10

(1) Suppose $A \subseteq M$ and $r^* \in CL(A) \cap \mathcal{G} \setminus \mathcal{G}m$. Then the set
$$\{r \in r^* + \mathcal{G}m : r \text{ is realized in } M\}$$
is dense in $r^* + \mathcal{G}m$ and for some $a \in M$ depending on A, $\mathrm{stp}(a) \in r^* + \mathcal{G}m$ and $\mathrm{tp}(a/A)$ is isolated.

(2) If a modular type from $\mathcal{G}$ is realized in M, then also the set $\{r \in \mathcal{G}m : r$ is realized in $M\}$ is dense in $\mathcal{G}m$.

Proof (1) By Section 2, $CL(A) \cap \mathcal{G}$ is a subgroup of $\mathcal{G}$ containing $\mathcal{G}m$ as a subgroup of finite index. So $r^* + \mathcal{G}m$ is a relatively open subset of $CL(A) \cap \mathcal{G}$. Suppose $U \subseteq r^* + \mathcal{G}m$ is relatively open. As in [8], Claim 2.14, there is a formula $\varphi(x)$ over A such that $\mathrm{Tr}(\varphi) = CL(A) \cap (r^* + \mathcal{G}m)$. Refining φ (over $A \cup \mathrm{acl}(\varnothing)$), we can assume that $\mathrm{Tr}(\varphi) \subseteq U$. Any element of $\varphi(M)$ realizes a type in U. So some type in U is realized in M. For the last clause we can choose $a \in M$ realizing φ with $\mathrm{tp}(a/A)$ isolated. Clearly, a depends on A.

(2) A similar proof. □

From now on in this section we assume additionally that $\mathcal{M}(G) = 1$ (that is, $\mathcal{G}m$ is finite) and $\Phi(\mathfrak{C}) = Gm \cup \mathrm{acl}(\varnothing)$. Also in this section we say that a is strictly regular if $\mathrm{stp}(a)$ is regular nonorthogonal to p and, moreover, a realizes some p-formula $\varphi(x)$ over $\mathrm{acl}(\varnothing)$ with $\mathcal{M}(P_\varphi) = 1$ (for the definition of p-formulas, see [8] or [14]).

For instance, $G(x)$ is a p-formula and $P_{G(x)} = \mathcal{G}$, so in our case every generic element of G is strictly regular. By Lemma 3.7 (see also [14], Lemma 3.1) if a is strictly regular and $A \subseteq M$ is finite then the $\mathrm{Aut}(Q/A)$-orbit of $\mathrm{stp}(a)$ is either open or finite. In the second case we say that a is Q-finite over A.

The following generalizes, slightly, [14], Theorem 3.2 (the ground level lemma).

Lemma 3.11 *Suppose $Aa \subseteq M$ is a finite independent set of strictly regular elements, a is Q-finite over A but not Q-finite over $\varnothing$. Then either $\mathrm{stp}(a/A)$ is modular or for some strictly regular Q-finite b independent from A, $a \not\perp b(A)$.*

Proof As in Lemmas 3.7 and 3.8, the proof from [14] goes through. It is enough to notice that in the many-model constructions there we never employ modular types. □

Finally we can prove the main result of this section.

Theorem 3.12 *Assume T has few countable models, G is a meager group with $\mathcal{M}(G) = 1$, $\mathcal{F}_G$ is finite, and $Q = Gm(M) \cup \mathrm{acl}(\varnothing)$. Then there is a finite set $C \subseteq M$ such that $G(M)$ is Q-atomic over C. In particular, if there are countably many good pseudotypes over Q then Vaught's Conjecture is true for K_Q.*

Proof The proof is similar to that of [14], Theorem 4.5. $\mathcal{M}(G) = 1$ means that G^0 has finite index in Gm. So replacing G by a generic subgroup of finite index (in which G is interpretable over some parameters) we can assume that $Gm = G^0$ (hence $\mathcal{G}m$ consists of the generic type p of G^0 only). Also, we may choose $\mathrm{acl}(\varnothing)$-definable representatives $f_0, \dots, f_l$ of the elements of $\mathcal{F} = \mathcal{F}_G$ and assume they are defined on all of G, and G is closed under every f_i. We identify $\mathcal{F}$ with $\{f_0, \dots, f_l\}$.

First we choose a Q-finite basis A^* of M: this is a finite independent set of Q-finite strictly regular elements of M such that for every Q-finite strictly regular $b \in M$, $\mathrm{stp}(b)$ is realized in $\mathrm{cl}_p(A^*)$, and A^* is minimal under these requirements. A^* exists by Basis Lemma 3.8.

Next we choose a finite set $E = \{e_0, \dots, e_k\}$ of generic elements of $G(M)$ with the following properties.

(a) $\mathrm{tp}(E)$ is isolated,

(b) $e_0, \dots, e_k$ are pairwise dependent,

(c) $\mathrm{stp}(e_0), \dots, \mathrm{stp}(e_k)$ are pairwise distinct and $CL(e_0) \cap (\mathcal{G} \setminus \mathcal{G}m) = \{\mathrm{stp}(e_0), \dots, \mathrm{stp}(e_k)\}$ (that is, whenever $e \in G \setminus Gm$ is generic and depends on e_0 then $\mathrm{stp}(e) = \mathrm{stp}(e_i)$ for some i).

As in [14], Lemma 4.4, we have the following claim.

(d) There are open neighborhoods V_i of $\mathrm{stp}(e_i), i \leq k$, in $S(\mathrm{acl}(\varnothing))$ and $f_{ij} \in \mathcal{F}$ (with $f_{ii} = \mathrm{id}$) such that if e_i^* realizes a generic type in V_i and $e_j^* =^* e_j + f_{ij}(e_i^* - e_i)$, then $e_j^*, j \leq k$, satisfy (b) and (c).

Choose a finite set A of generic elements of $G(M)$ such that $A \subseteq \mathrm{cl}_p(A^*)$ and if a generic type of G is realized in $\mathrm{cl}_p(A^*) \cap M$ then it is realized in A.

Since the f_is are defined on all of G we have that for every $X \subseteq G$ and $x \in G$, $x \in \mathrm{cl}_p(A^*X)$ if and only if $x \in \mathrm{cl}_p(AX)$.

Next we choose a maximal set $B \subseteq G(M)$ of generic elements such that $e_0 \in B$, the set $A \cup B$ is independent, B is Q-atomic over A, and the set $\{\mathrm{stp}(b) : b \in B\}$ is dense in $\mathcal{G}$. As in [14] we have

(e) every $r \in \mathcal{G}^M$ is realized in $\mathrm{cl}_p(AB)$.

Using (d), as in [14] we get

(f) $G(M) \subseteq \mathcal{F}$-span$(ABEQ)$.

Now let G' be a resolving group for G (via $g : G' \to G$), and by Theorem 3.5 we may assume that both G' and g are $\mathrm{acl}(\varnothing)$-definable. Also, without loss of generality, g is onto. For every $a \in A$, $b \in B$, and $e \in E$, choose $a', b', e' \in G'$ with $g(a') = a$, $g(b') = b$, and $g(e') = e$. Let

$$A' = \{a' : a \in A\}, \quad E' = \{e' : e \in E\}, \quad B' = \{b' : b \in B\}.$$

Because of the choice of G' we have

(g) $G(M) \subseteq \mathrm{acl}(A'B'E'Q)$.

Indeed, let $a \in G(M)$. (f) means that for some $a_i \in A$, $e_j \in E$, $b_k \in B$, and $f_i, f'_j, f''_k \in \mathcal{F}$ we have

$$a - \Big(\sum_i f_i a_i + \sum_j f'_j e_j + \sum_k f''_k b_k\Big) \in Gm.$$

Choose $\mathrm{acl}(\varnothing)$-definable subgroups $H_i, H'_j, H''_k \in G' \times G'$ representing $g^{-1} f_i g$, $g^{-1} f'_j g$, $g^{-1} f''_k g$, respectively, as in the definition of a resolving group. Choose a_i^0, e_j^0, b_k^0 so that $(a'_i, a_i^0) \in H_i$, $(e'_j, e_j^0) \in H'_j$, $(b'_k, b_k^0) \in H''_k$ and let

$$a^* = \sum_i g\big(a_i^0\big) + \sum_j g\big(e_j^0\big) + \sum_k g\big(b_k^0\big).$$

Clearly, $a^* \in \mathrm{acl}(\{a'_i, e'_j, b'_k\}_{i,j,k}) \cap M$ and $a - a^* \in Gm$. Hence $a \in \mathrm{acl}(A'E'B'Q)$, proving (g).

To finish the proof it is enough to show that

(h) B' is Q-atomic over $C = A'E'B'_0$ for some finite $B'_0 \subseteq B'$.

We know that B is Q-atomic over A^*. As in the proof of [14], Theorem 4.5, we have the following claim.

Claim 3.13 *If $a \in \mathrm{cl}_p(A^*) \cap M$, then for some finite $B_0 \subseteq B$, B is Q-atomic over $A^* a B'_0$, where $B'_0 = \{b' : b \in B_0\}$.*

Proof The proof of the corresponding claim in [14] used the ground level lemma. We generalized it suitably in Lemma 3.11. □

By the claim, there is a finite set $B_0 \subseteq B$ such that B is Q-atomic over $C = A'E'B'_0$, where $B'_0 = \{b' : b \in B_0\}$. To finish, we prove that for any finite set C and any $a' \in G'$ generic over C,

$$a' \text{ is } Q\text{-atomic over } C \text{ iff } g(a') \text{ is } Q\text{-atomic over } C.$$

$\Rightarrow$ is clear. For the other direction, suppose a' is not Q-atomic over C. By Lemma 3.7 and Remark 3.9 this means that the $\mathrm{Aut}(Q/C)$-orbit X' of $\mathrm{stp}(a')$ is small; that is, there are finitely many types $r'_i \in X'$ such that any type $r' \in X'$ is not almost orthogonal to some r'_i.

Let X be the $\mathrm{Aut}(Q/C)$-orbit of $\mathrm{stp}(g(a'))$. Clearly, every type $r \in X$ is not almost orthogonal to some $g(r'_i)$. Hence $g(a')$ is not Q-atomic over C. □

Corollary 3.14 *Assume G is a superstable meager group of U-rank ω and $\mathcal{M}$-rank 1, with $\mathcal{F}_G$ finite, and $T = \mathrm{Th}(G)$ has few countable models. Then there are countably many countable models M of T with $G^0(M)$ finite dimensional.*

Proof Without loss of generality, $G^0 = Gm$. Let $n < \omega$. It suffices to show that there are countably many countable models M of T with $G^0(M)$ n-dimensional. We can assume that every such model contains a Morley sequence $A = \{a_0, \dots, a_{n-1}\}$ in the generic type p of G^0, and we work in $T(A)$. So in this new signature we must show that there are countably many countable models of $T(A)$ with $G^0(M)$ 0-dimensional. However, $G^0(M)$ is 0-dimensional if and only if $G^0(M) = G^-(M)$.

Let $Q = G^-(M) \cup \mathrm{acl}(\varnothing)$. Then $\mathrm{Th}(Q)$ is a many-sorted superstable theory of finite rank (since $U(G) = \omega$), and by [2] Vaught's Conjecture holds for $\mathrm{Th}(Q)$ and Q is atomic over a Morley sequence contained in Q^{eq}. By [6] and [7], every $\mathrm{Aut}(Q)$-orbit (called pseudotype there) is τ-stable; hence there are countably many of them. We see that the assumptions of Theorem 3.12 are satisfied; hence we are done. □

I did not manage to remove the assumption that $\mathcal{M}(G) = 1$ and $\mathcal{F}_G$ is finite from Theorem 3.12 and Corollary 3.14. In general, when G is meager and T has few models, then $\mathcal{M}(G)$ is finite and $\mathcal{F}_G$ is a locally finite field. However, when $\mathcal{M}(G) > 1$ then I could not prove the ground level lemma over $Q = Gm(M)$ (this lemma, together with the basis lemma, is the heart of the description of $(\mathcal{G}/\mathcal{G}m)^M$). Also, if $\mathcal{F}_G$ is infinite then possibly there is no resolving group G' (the existence of G' helps us to describe $G(M)$ once we know $(\mathcal{G}/\mathcal{G}m)^M$); maybe one should use an inverse limit of a system of f-resolving groups, $f \in \mathcal{F}$?

4 On Generic Types of a Meager Group

In this section we assume that G is a meager group and p is the generic type of G^0. Our goal is to describe $(\mathcal{G}/\mathcal{G}m)^M$, that is, the set of $\mathcal{G}m$-cosets in $\mathcal{G}$ containing a type realized in M, where M is a countable model of T.

Notice that $(\mathcal{G}/\mathcal{G}m)^M$ determines $(G/Gm)(M)$. Also, when G is a locally modular group which is not meager, then $\mathcal{G}/\mathcal{G}m$ is finite, so there are finitely many possibilities for $(\mathcal{G}/\mathcal{G}m)^M$. Hence again the meager case is the nontrivial one.

When G is meager, $\mathcal{G}/\mathcal{G}m$ is a profinite group of power $2^{\aleph_0}$ and $(\mathcal{G}/\mathcal{G}m)^M$ differs from a subgroup of $\mathcal{G}/\mathcal{G}m$ at most by 0. ($0 \in (\mathcal{G}/\mathcal{G}m)^M$ if and only if some modular generic type of G is realized in M.)

Our results fall short of characterizing $\mathcal{G}^M$, the set of generic types of G realized in M. But it is not known yet how to describe even $(\mathcal{G}m)^M$. I did not manage to describe the set $(\mathcal{G}/\mathcal{G}m)^M$ relative to $Gm(M)$. The best I could do was a description of $(\mathcal{G}/\mathcal{G}m)^M$ over $G^-(M)$ (or over $\varnothing$). Again we assume that $Q = \Phi(M)$ where Φ is a countable disjunction of formulas over $\varnothing$ and M is countable.

In order to describe $(\mathcal{G}/\mathcal{G}m)^M$ relative to Q we must generalize the ground level lemma ([14], Theorem 3.4) again, this time considering regular types of $\mathcal{M}$-rank possibly > 0. I managed to do this assuming that Φ is orthogonal to p, meaning that p is orthogonal to any complete type extending Φ (in Lemma 3.11 this assumption is weakened, but we deal with $\mathcal{M}$-rank 0 types there).

So in this section we assume that Φ is orthogonal to p and $\mathrm{acl}(\varnothing) \subseteq Q$. The main examples of Φ and Q one should have in mind are $Q = G^-(M) \cup \mathrm{acl}(\varnothing)$ and $Q = \mathrm{acl}(\varnothing)$ (we add $\mathrm{acl}(\varnothing)$ to Q for convenience). Here we say that a is regular if $r = \mathrm{stp}(a)$ is regular nonorthogonal to p. We say that a is Q-small over A if the $\mathrm{Aut}(Q)$-orbit of r over A is small. Since Φ is orthogonal to p, each type in $\mathrm{Tr}(\mathrm{tp}(a))$ extends uniquely to a type in $S(Q)$; hence $\mathrm{Aut}(Q)$ acts naturally on $\mathrm{Tr}(\mathrm{tp}(a))$. By Orbit Lemma 3.7 and Section 2, if a is regular, then the $\mathrm{Aut}(Q)$-orbit of $r = \mathrm{stp}(a)$ is either open or small and in the latter case it is nowhere dense in $S(\mathrm{acl}(\varnothing))$. By Remark 3.9 this orbit is open if and only if $r|Q$ is Q-isolated.

We need some additional facts about $\mathrm{Aut}(Q)$-orbits. Assume a is regular and X is the $\mathrm{Aut}(Q)$-orbit of $\mathrm{stp}(a)$. For any $r \in X$ we have $r \vdash r|Q$, so we can identify X with a subset of $S(Q)$ (which is the $\mathrm{Aut}(Q)$-orbit of $\mathrm{tp}(a/Q)$, or a pseudotype in the sense of [6]).

We say that $q \in S(Q)$ is τ-stable (and τ-based on $c \subseteq Q$) if the $\mathrm{Aut}(Q/c)$-orbit of q is co-meager in its closure [7]. Also, $q \in S(Q)$ is called *good* if q is realized in some model in K_Q. In [7] I formulated the τ-stability conjecture saying that under the few models assumption every good type $q \in S(Q)$ is τ-stable. I proved this conjecture in several cases [13].[1]

Lemma 4.1 *Assume $N \in K_Q$, X is the* $\mathrm{Aut}(Q)$ *orbit of* $\mathrm{stp}(a)$ *for some regular a, and $A \subseteq N$ is finite.*

(1) *For every $r \in \mathrm{cl}(X)$ there is a model $M \in K_Q$ containing A and realizing r.*
(2) $\mathrm{cl}(X)$ *is* $\mathrm{Aut}(Q)$*-invariant.*
(3) *There is an* $\mathrm{Aut}(Q)$*-orbit $X' \subseteq \mathrm{cl}(X)$ such that X' is co-meager in* $\mathrm{cl}(X)$.
(4) *If X is open, then the open* $\mathrm{Aut}(Q/A)$*-orbits are dense in X.*

Proof (1) Clearly, every $r \in \mathrm{cl}(X)$ is regular, nonorthogonal to p, hence orthogonal to Φ. It follows that for any $b \models r|QA$, the type $\Phi(x) \cup \{x \neq c : c \in Q\}$ is nonisolated over QAb. Hence there is a model $M \in K_Q$ containing A and b.

(2) is obvious.

(3) We can identify X with a subset of $S(Q)$. So we are done by (1) and [7], Lemma 2.1.

(4) T has few countable models, so by (1), $\mathrm{cl}(X)$ is a union of $< 2^{\aleph_0}$-many $\mathrm{Aut}(Q/A)$-orbits, and they are either open or small. If $Y \subseteq X$ is a small $\mathrm{Aut}(Q/A)$-orbit, then $Y' = \mathrm{cl}(Y)$ is closed nowhere dense and $\mathrm{Aut}(Q/A)$-invariant. By (3), Y' contains a co-meager $\mathrm{Aut}(Q/A)$-orbit Y'', and by [7], Corollary 2.5, there are countably many such orbits Y'' (since they correspond to good τ-stable pseudotypes, whose Scott height is $\leq \mathrm{SH}(Q) + 1$).

So there are also countably many nowhere dense sets of the form $\mathrm{cl}(Y)$, where $Y \subseteq X$ is a small $\mathrm{Aut}(Q/A)$-orbit. Hence the union of such orbits forms a meager subset of X. It follows that the open $\mathrm{Aut}(Q/A)$-orbits are dense in X. □

When all the orbits in question are finite, the next proposition is trivial and holds also for $Q = Gm(M)$. Without this we need the assumption that Φ is orthogonal to p and we must use the few models assumption.

Proposition 4.2 *Assume that every good type in $S(Q)$ is τ-stable. Assume a, b, c are regular.*

(1) *If b is Q-small and a is Q-small over b, then a is Q-small.*
(2) *If a is Q-small over b and b is Q-small over c, then a is Q-small over c.*

Proof (1) Suppose a is not Q-small. Let X and Y be the Aut(Q)-orbits of stp(a) and stp(b), respectively, and let X_b be the Aut(Q/b)-orbit of stp(a). So X is open and X_b, Y are small (hence, nowhere dense). By assumption tp(b/Q) is τ-stable, τ-based on some finite $c \subseteq Q$. Clearly, the Aut(Q/c)-orbit of stp(a) is open and contained in X, so without loss of generality, we can absorb c into the signature and assume additionally that Y is co-meager in cl(Y).

For b' realizing a type in Y let $X_{b'}$ be the Aut(Q)-conjugate of X_b over b' (that is, $X_{b'} = h(X_b)$ for any $h \in$ Aut(Q) with $h(b) = b'$; the choice of h does not matter).

We can assume (extending the signature by an element of acl$(\varnothing)$) that all types in Y are not almost orthogonal. Hence also, all types in cl(Y) are such. Hence,

(a) for any $r \in$ cl(Y), $r|b$ is modular.

Clearly, $X = \bigcup\{X_{b'} : b'$ realizes a type in $Y\}$ and if $b' \overset{s}{\equiv} b$ then $X_{b'} = X_b$ (since then we can choose $h \in$ Aut(Q) with $h(b) = b'$ and $h\lceil Q = \mathrm{id}_Q$).

Let b^* realize a type in Y. By Lemma 4.1(4) there is an open Aut(Q/b^*)-orbit $X' \subseteq X$. Similarly, there is a (relatively) nonmeager Aut(Q/b^*)-orbit $Y' \subseteq Y$ (here we use the fact that Y is co-meager in cl(Y)). Without loss of generality, stp$(b) \in Y'$ and stp$(a) \in X'$. Naming b^*, by (a) we may assume that $Y = Y'$, $X = X'$, and every type in cl(Y) is modular.

The main point of the proof will be construction of a model in K_Q admitting many automorphisms. In this construction we use ideas from [12], particularly that of a flat Morley sequence.

Choose a countable set $B = \{b_n, n < \omega\}$ of independent elements realizing types in Y such that for every n, the Aut$(Q/b_{<n})$-orbit of stp(b_n) is not meager in Y and the set $\{$stp$(b_n) : n < \omega\}$ is dense in Y (here $b_{<n} = \{b_k : k < n\}$). In other words, B is a flat Morley sequence in Y (more precisely, we identify Y with a subset of $S(Q)$; then Y becomes (in the notation from [12]) the Aut(Q)-orbit $o(q)$ of the τ-stable type $q =$ tp(b/Q) and B is a flat Morley sequence in $o(q)$).

By [12], Proposition 4.5, the choice of B is unique up to Aut(Q); in particular, by [12], Lemma 4.4, we have

(b) for all $k, l < \omega$ there is an $f \in$ Aut(Q) with $f[B] = B$ and $f(b_k) = b_l$,
(c) if B' is another countable flat Morley sequence in $o(q)$, then some $f \in$ Aut(Q) maps B onto B'.

Now we prove that

(d) $\bigcup_{n<\omega} X_{b_n}$ is dense in X.

Indeed, for every $E \in FE(\varnothing)$ and an E-class α meeting X, the set $\{r \in Y :$ for $b' \models r$, α meets $X_{b'}\}$ has nonempty interior in Y; hence, by denseness of $\{$stp$(b_n) : n < \omega\}$, X_{b_n} meets α for some n.

Let $A = Q \cup B$. $\mathrm{Aut}(\mathfrak{C}/\{A\})$ acts on $S(A)$ and for $r \in S(A)$ let $o(r)$ denote the orbit of r under this action. We say that $r \in S(A)$ is τ-isolated if $o(r)$ is not meager (this is simply a generalization of the notion of Q-isolation, introduced in [6]). Similarly as in [12], Proposition 3.4, we shall prove that

(e) the τ-isolated types are dense in $S(A)$.

So let $\varphi(x, c) \in L(A)$, where $c \subseteq A$, and we may assume that φ has no forking extension over A and that $c \subseteq Q \cup b_{<k}$ for some $k < \omega$. Also, we may assume that $S(Qb_{<k}) \cap [\varphi(x, c)]$ meets just one nonmeager $\mathrm{Aut}(Q/b_{<k})$-orbit. Let

$$S = \{r \in S(A) \cap [\varphi(x, c)] : \text{ for every } n \geq k, r\lceil Qb_{<n} \text{ is Q-isolated over } b_{<n}\}.$$

To prove (e) it is enough to show that S is co-meager in $S(A) \cap [\varphi(x, c)]$ and that all types in S are conjugate over $\{A\}$.

For every $n \geq k$ let $f_n : S(A) \cap [\varphi(x, c)] \to S(Qb_{<n}) \cap [\varphi(x, c)]$ be restriction and let $S_n = \{r \in S(Qb_{<n}) \cap [\varphi(x, c)] : r$ is Q-isolated over $b_{<n}\}$. By our assumptions on φ, f_n is open and continuous and S_n is co-meager in $S(Qb_{<n}) \cap [\varphi(x, c)]$. It follows that the set $S = \bigcap_n f_n^{-1}(S_n)$ is co-meager in $S(A) \cap [\varphi(x, c)]$.

Now let $\mathrm{tp}(d/A), \mathrm{tp}(d'/A) \in S$. We want to find $f \in \mathrm{Aut}(\mathfrak{C}/\{A\})$ with $f(d) = d'$.

Let $B_{\geq k} = \{b_n : n \geq k\}$ and let $Z = \{r|Qb_{<k}d : r \in Y\}$. Clearly, Z is co-meager in $\mathrm{cl}(Z)$ and Z is a union of some $\mathrm{Aut}(Q/b_{<k}d)$-orbits; say $Z_l, l < \omega$, are all the orbits over $b_{<k}d$ contained in Z that are nonmeager in Z. Clearly, $\bigcup_{l<\omega} Z_l$ is dense in Z and by [12], $B_{\geq k}$ is a flat Morley sequence in $\bigcup_l Z_l$ (over $b_{<k}d$).

Defining analogously Z' and Z'_l for d' in place of d we get that $B_{\geq k}$ is a flat Morley sequence in $\bigcup_l Z'_l$ over $b_{<k}d'$. Now $S(Qb_{<k}) \cap [\varphi(x, c)]$ meets just one nonmeager pseudotype over $b_{<k}$; hence, for some $g \in \mathrm{Aut}(Q/b_{<k})$, we have $g(d') = d$. Hence, $g(\{Z'_l, l < \omega\}) = \{Z_l, l < \omega\}$ and $g(B_{\geq k})$ is a flat Morley sequence in $\bigcup_l Z_l$ over $b_{<k}d$. By [12], Proposition 4.5, there is $h \in \mathrm{Aut}(Q/b_{<k}d)$ mapping $g(B_{\geq k})$ onto $B_{\geq k}$. We see that $f = h \circ g \in \mathrm{Aut}(\mathfrak{C}/\{A\})$ and $f(d') = d$. This proves (e).

By (e) and [12], there is a countable model N containing A, which is τ-atomic over A, and such an N is unique up to $\mathrm{Aut}(\mathfrak{C}/\{A\})$. In particular, $N \in K_Q$ and N has the following property:

(f) for all $n, k < \omega$ there is an automorphism of N mapping b_n to b_k.

Indeed, N is τ-atomic over A also in the expanded language $\mathcal{L}(b_n)$ and $\mathcal{L}(b_k)$. By (b) there is an $h_0 \in \mathrm{Aut}(\mathfrak{C})$ mapping b_n to b_k and A onto A. $h_0(N)$ is also τ-atomic over A, in $\mathcal{L}(b_k)$. By uniqueness of countable τ-atomic models, for some $h_1 \in \mathrm{Aut}(\mathfrak{C})$ preserving A setwise, with $h_1(b_k) = b_k$, h_1 maps $h_0(N)$ onto N. We see that $h_1 \circ h_0$ is an automorphism of N mapping b_n to b_k.

Choose a type Φ' so that $\Phi'(\mathfrak{C}) = \Phi(\mathfrak{C}) \cup \mathrm{cl}(Y)(\mathfrak{C})$ and let $Q' = \Phi'(N)$. Q' is a neat set, because p is orthogonal to Φ and every type in $\mathrm{cl}(Y)$ is modular. Let $C = \{f(b_0) : f \in \mathrm{Aut}(Q')\}$ and $X_N = \bigcup\{X_{b'} : b' \in C\}$. We see that $B \subseteq C$ and by (d) and (f) we have

(g) X_N is $\mathrm{Aut}(Q')$-invariant and dense in X and every $\mathrm{Aut}(Q')$-orbit contained in X_N meets $X_{b'}$ for every $b' \in C$.

X_N is meager (as a countable union of nowhere dense sets $X_{b'}$), so every $\mathrm{Aut}(Q')$-orbit contained in X_N is small (by Lemma 3.7). Choose $b' \in C$ and $r^0, \ldots, r^l \in X_{b'}$ (for some $l < \omega$) such that for every $r \in X_{b'}$, $r \not\perp^{a} r^i$ for some i. Let X_i be the

Aut(Q')-orbit of r^i. For every $i \leq l$ choose $r^i_j \in X_i$, $j \leq l_i$ such that for every $r \in X_i$, $r \not\perp^{a} r^i_j$ for some j.

Let $R = \{r^i_j : i \leq l,\ j \leq l_i\}$ and for $r \in R$ let $X_r = CL(r) \cap X_N$. By the meager forking assumption (see Section 2), $CL(r) \cap X$ is nowhere dense; hence, also X_r is nowhere dense. We see that $X_N = \bigcup_{r \in R} X_r$ is nowhere dense, contradicting (g).

(2) follows from (1). □

The next corollary improves Remark 3.6(2).

Corollary 4.3 *Under the assumptions of Remark 3.6, if A is finite and* tp(a) *is isolated, then* tp(a/A) *is isolated.*

Proof In Proposition 4.2, set $\Phi(x)$ so that $\Phi(\mathfrak{C}) = \mathrm{acl}(\varnothing)$. In particular, by [8], for $Q = \mathrm{acl}(\varnothing)$ every type in $S(Q)$ is τ-stable, so the assumption of Proposition 4.2 is satisfied (in fact, for $Q = \mathrm{acl}(\varnothing)$ the proof of Proposition 4.2 is easier and we do not need there the τ-stability assumption). By Remark 3.6(1), A is a subset of a neat construction. By superstability we may assume that $A = \{a_k, k < n\}$ is a neat construction. We know that $a \mathop{\smile\!\!\!\!\!\!|}\; A$. So it suffices to prove by induction on $k \leq n$ that tp$(b/a_{<k})$ is isolated for every b realizing tp(a). This follows from Proposition 4.2. □

Our next goal is the following generalization of the ground level lemma ([14], Theorem 3.2).

Lemma 4.4 *Assume every good type in $S(Q)$ is τ-stable. Suppose Aa is a finite independent set of regular elements, a is Q-small over A but not Q-small over $\varnothing$. Then either* stp(a/A) *is modular or for some Q-small regular b independent from A, we have that $a \not\mathop{\smile\!\!\!\!\!\!|}\; b(A)$.*

Proof The proof follows that of [14], Theorem 3.2. By Proposition 4.2, if $b \in A$ and b is Q-small over $A \setminus \{b\}$, then a is Q-small over $A \setminus \{b\}$. So minimizing A we can assume that no $b \in A$ is Q-small over $A \setminus \{b\}$ and that a is not Q-small over any proper subset of A.

Let $c \in A$ and $B = A \setminus \{c\}$. Let X be the Aut(Q/B)-orbit of stp(c), Y the Aut(Q/B)-orbit of stp(a), and Y_c the Aut(Q/Bc)-orbit of stp(a). X and Y are open, while Y_c is small, so expanding the signature by an element of acl$(\varnothing)$ we can assume that all types in Y_c are not almost orthogonal and X and Y are clopen.

For any c' realizing a type in X the set $s_{c'} = CL(c') \cap X$ is closed nowhere dense; hence it may be regarded as a nonisolated type over acl$(\varnothing)$. Let $\mathfrak{X} = \{s_{c'} : c'$ realizes a type in $X\}$. $\mathfrak{X}$ is a partition of X into closed Aut(Q)-conjugate nowhere dense sets; hence the topology on X induces a compact Hausdorff topology on $\mathfrak{X}$. Notice that c' realizes $s_{c'}$. Also, by the meager forking assumption, for any set D, $s_{c'}$ has a forking extension over D if and only if $s_{c'}$ is isolated over D.

If C is a finite B-independent set of realizations of types in X, then X splits into $< 2^{\aleph_0}$-many Aut(Q/BC)-orbits; some of them are open (the Q-isolated ones), the others being small. Hence also $\mathfrak{X}$ splits into Aut(Q/BC)-orbits. The orbits in $\mathfrak{X}$ corresponding to small orbits in X are finite; the orbits in $\mathfrak{X}$ corresponding to open orbits in X are open. Similarly as in [11], Lemma 2.1, we have the following claim.

Claim 4.5 *There are finitely many finite* Aut(Q/BC)*-orbits in $\mathfrak{X}$.*

Similarly, for a' realizing a type in Y we define $s_{a'}$ as $CL(a') \cap Y$ and regard it as a nonisolated type over acl($\varnothing$). We define $\mathfrak{Y}$ as the set $\{s_{a'} : a' \text{ realizes a type in } Y\}$ and endow it with the induced topology. As above, $\mathfrak{Y}$ splits into Aut(Q/BC)-orbits, finitely many of which are finite, the rest being open.

In particular, for some open $U \subseteq \mathfrak{Y}$, s_a is the only element of the (finite!) Aut(Q/Bc)-orbit of s_a lying in U and no other finite Aut(Q/Bc)-orbit meets U. Thus extending the signature by an element of acl($\varnothing$), we can replace $\mathfrak{Y}$ by this neighborhood and assume that there is a unique finite Aut(Q/Bc)-orbit in $\mathfrak{Y}$, having moreover size 1.

The same is true for any c' realizing a type in X (since X is an Aut(Q/B)-orbit). So we can define for such c' the nonisolated type $r_{c'} \in \mathfrak{Y}$ as the only element of the only finite Aut(Q/Bc')-orbit in $\mathfrak{Y}$. Notice that for c', c'' realizing types in X, if $s_{c'} = s_{c''}$ then $r_{c'} = r_{c''}$.

Indeed, by Proposition 4.2(2), the union of the small Aut(Q/Bc')-orbits in Y equals the union of the small Aut(Q/Bc'')-orbits in Y; hence the finite Aut(Q/Bc')-orbit in $\mathfrak{Y}$ is the same as the finite Aut(Q/Bc'')-orbit in $\mathfrak{Y}$.

A similar argument yields the following "exchange property" of Q-smallness.

(a) If d, d' are regular and not Q-small and d' is Q-small over d, then d is Q-small over d'.

X splits into infinitely, but $< 2^{\aleph_0}$-many Aut(Q)-orbits over Bc. Since by Lemma 3.7 they are small or open (and the corresponding orbits in $\mathfrak{X}$ are finite or open), the open ones are dense in X. So we can choose open Aut(Q/Bc)-orbits $X_n \subseteq X$, $n < \omega$, so that they converge (topologically) to stp(c). Hence, the corresponding open Aut(Q/Bc)-orbits $\mathfrak{X}_n \subseteq \mathfrak{X}$ converge to s_c. cl($\mathfrak{X}_n$) $\setminus \mathfrak{X}_n$ is a nowhere dense union of some Aut(Q/Bc)-orbits, hence is finite (by the claim). Since by the claim there are finitely many finite orbits in $\mathfrak{X}$, discarding some $\mathfrak{X}_n$s we can assume that all the $\mathfrak{X}_n$s and X_ns are clopen.

Using (a) as in [14], we can assume that whenever $I \subseteq \omega$ is finite, $j \in \omega \setminus I$, and c_i realizes a type in X_i, $i \in I$, then there is no finite Aut(Q)-orbit over $Bc\{c_i, i \in I\}$ in $\mathfrak{X}_j$. As in [14] for some $n < \omega$ we find

(b) $c_1, \ldots, c_n$ realizing types in X, a_i realizing r_{c_i}, $i \leq n$ with $\{a_i, c_i, i \leq n\}$ being Bc-independent, and c_{n+1} realizing a type in X with an open Aut(Q)-orbit over $Bc\{a_i c_i, i \leq n\}$ and $r_{c_{n+1}}$ isolated over acl($\varnothing$)$\cup Bc\{a_i c_i, i \leq n\}c_{n+1}$ (equivalently: $r_{c_{n+1}}$ has a forking extension over this set).

From this point on the proof is the same as in [14], Theorem 3.2, but easier, since we do not have to find a p-formula true of b. Essentially, by (b), for some a_{n+1} realizing $r_{c_{n+1}}$,

$$c_{n+1} \mathop{\smile\!\!\!\!\!\!|}\ Bcc_{\leq n}a_{\leq n} \text{ and } a_{n+1}c_{n+1} \not\mathop{\smile\!\!\!\!\!\!|}\ Bcc_{\leq n}a_{\leq n}.$$

We can assume that $Bc_{n+1}a_{n+1}$ and Aa are Aut(Q)-conjugate, so it suffices to find b satisfying our demands with respect to the set $A' = Bc_{n+1}$ and $a' = a_{n+1}$. After some minimization, essentially $b = Cb(a_{n+1}c_{n+1}/Bcc_{\leq n}a_{\leq n})$ is good. ☐

Using Lemma 4.4 we can classify the sets $(\mathcal{G}/\mathcal{G}m)^N$, $N \in K_Q$, as follows. We say that $A \subseteq M$ is a Q-small basis of M if

(b1) A is an independent set of regular Q-small elements,
(b2) for every Q-small regular $a \in N$, stp(a) is realized in cl$_p(A)$, and
(b3) A is minimal under these restrictions.

Notice that, in a Q-small basis, at most one element realizes a modular type over $\mathrm{acl}(\varnothing)$. By Lemma 3.8 we have the following.

Remark 4.6 M contains a finite Q-small basis. All Q-small bases of M have the same size.

Theorem 4.7 *Assume every good type in $S(Q)$ is τ-stable, $N \in K_Q$ is countable, and A is a Q-small basis of N. Then there is a countable $N' \in K_Q$ Q-atomic over A such that $(\mathcal{G}/\mathcal{G}m)^N = (\mathcal{G}/\mathcal{G}m)^{N'}$. Moreover, up to* $\mathrm{Aut}(Q)$ *there are countably many possibilities for $(\mathcal{G}/\mathcal{G}m)^N$.*

Proof Similar to that of [14], Theorem 4.3. By Remark 4.6, A is finite. So by Lemma 2.1 we can find a countable $N'' \in K_Q$ which is Q-atomic over A. Clearly, A is a Q-small basis of N''. We choose $B \subseteq G(N)$ ($B'' \subseteq G(N'')$, respectively) as a maximal set of regular elements such that

(a) $A \cup B$ ($A \cup B''$, respectively) is independent,
(b) B (B'', respectively) is Q-atomic over A, and
(c) the set $\{\mathrm{stp}(b) : b \in B\}$ ($\{\mathrm{stp}(b) : b \in B''\}$, respectively) is dense in $\mathcal{G}$.

Using Ground Level Lemma 4.4, we prove as in [14] that for every $r \in \mathcal{G}$,

(d) $(r + \mathcal{G}m)$ contains a type realized in N (N'', respectively) iff r is realized in $\mathrm{cl}_p(AB)$ ($\mathrm{cl}_p(AB'')$, respectively).

As in the proof of uniqueness of a countable Q-atomic model [6] we find an $f \in \mathrm{Aut}(Q/A)$ with $f(B'') = B$. Let $N' = f(N'')$. So $(\mathcal{G}/\mathcal{G}m)^N = (\mathcal{G}/\mathcal{G}m)^{N'}$ and N' is Q-atomic over A.

For the last clause notice that if $A = \{a_k, k < n\}$ and $A^* = \{a^*_k, k < n\}$ is a Q-small basis of some countable $N^* \in K_Q$ such that for every $k < n$ we have $\mathrm{stp}(a_k) \not\perp^a \mathrm{stp}(a^*_k)$, then $(\mathcal{G}/\mathcal{G}m)^N$ and $(\mathcal{G}/\mathcal{G}m)^{N^*}$ are $\mathrm{Aut}(Q)$-conjugate.

Indeed, we can assume that $N \mathop{\smile\!\!\!\!\!\!\mid} N^*(Q)$. Suppose $B^* \subseteq N^*$ is chosen in the same way as $B \subseteq N$ above. Then by Proposition 4.2, B is Q-atomic over AA^* and B^* is Q-atomic over AA^*. Hence there is an $f' \in \mathrm{Aut}(Q/AA^*)$ mapping B^* onto B (f' is found in the same way as f above). By (d) we get that $(\mathcal{G}/\mathcal{G}m)^N = (\mathcal{G}/\mathcal{G}m)^{f'(N^*)}$.

So up to $\mathrm{Aut}(Q)$, $(\mathcal{G}/\mathcal{G}m)^N$ depends only on the set of $\not\perp^a$-classes of $\mathrm{stp}(a_k)$, $k < n$. Since p is meager, the division ring $\mathcal{F}_G$ is countable. Hence in the set of Q-small regular types in $S(\mathrm{acl}(\varnothing))$ there are at most countably many $\not\perp^a$-classes (since by Lemma 3.8 these classes can be embedded into a finite-dimensional projective space over $\mathcal{F}_G$). So we see that up to $\mathrm{Aut}(Q)$, there are at most countably many possibilities for $(\mathcal{G}/\mathcal{G}m)^N$. □

Corollary 4.8 *Assume G is a* 0*-definable locally modular Abelian group. Then up to isomorphism there are countably many sets $(\mathcal{G}/\mathcal{G}m)^N$, where N is a countable model of T.*

Proof By [8], for $Q = \mathrm{acl}(\varnothing)$, every type over Q is τ-stable, so we can apply the previous theorem. In fact, we need the τ-stability assumption only in Proposition 4.2, and, as indicated in the proof of Corollary 4.3 in case of $Q = \mathrm{acl}(\varnothing)$, this assumption may be waived. □

The results of this paper indicate that the description of $Gm(M)$ relative to $G^-(M)$ may be the crucial part for a proof of Vaught's Conjecture for a meager or (more generally) locally modular group G. Indeed, in Section 3 we described $G(M)$ relative to $Gm(M)$ provided that $\mathcal{M}(G) = 1$ and $\mathcal{F}_G$ is finite; the proof was similar as in the weakly minimal case. We finish by stating some problems. We always assume T has few countable models.

Problem 4.9 Describe $Gm(M)$, or even $G^0(M)$, relative to $G^-(M)$.

Problem 4.10 Describe $\mathcal{G}m^M$ relative to $G^-(M)$ (this is weaker than the previous item).

Problem 4.11 Describe $G(M)$ relative to $Gm(M)$ when $\mathcal{M}(G) > 1$ or $\mathcal{F}_G$ is infinite.

Note

1. In general, however, the conjecture is false. Hjorth pointed out a counterexample to me.

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