Notre Dame Journal of Formal Logic
Volume 47, Number 4, 2006

Filters on Computable Posets

Steffen Lempp and Carl Mummert

Abstract We explore the problem of constructing maximal and unbounded filters on computable posets. We obtain both computability results and reverse mathematics results. A maximal filter is one that does not extend to a larger filter. We show that every computable poset has a Δ^0_2 maximal filter, and there is a computable poset with no Π^0_1 or Σ^0_1 maximal filter. There is a computable poset on which every maximal filter is Turing complete. We obtain the reverse mathematics result that the principle "every countable poset has a maximal filter" is equivalent to ACA_0 over RCA_0. An unbounded filter is a filter which achieves each of its lower bounds in the poset. We show that every computable poset has a Σ^0_1 unbounded filter, and there is a computable poset with no Π^0_1 unbounded filter. We show that there is a computable poset on which every unbounded filter is Turing complete, and the principle "every countable poset has an unbounded filter" is equivalent to ACA_0 over RCA_0. We obtain additional reverse mathematics results related to extending arbitrary filters to unbounded filters and forming the upward closures of subsets of computable posets.

1 Introduction

In this paper, we study maximal and unbounded filters on computable posets. We obtain computability results and reverse mathematics results regarding the existence of these filters.

We use the following terminology. A *poset* is a set P with a reflexive, antisymmetric, transitive relation $\preceq$. A poset $\langle P, \preceq\rangle$ is *computable* if P is a computable subset of $\mathbb{N}$ and $\preceq$ is a computable binary relation on P. A *filter* is a subset F of a poset such that F is upward closed (if $p \in F$ and $p \preceq q$ then $q \in F$) and for all $p, q \in F$ there is an $r \in F$ with $r \preceq p$ and $r \preceq q$. The entire poset is thus a filter if every pair of elements is compatible. A filter is *maximal* if it is not contained in a strictly larger filter. A filter F is *unbounded* if there is no $p \notin F$ with $p \preceq q$ for all $q \in F$.

Received February 16, 2006; accepted October 16, 2006; printed December 28, 2006
2000 Mathematics Subject Classification: Primary, 03D, 03B30; Secondary, 06
Keywords: computable poset, filter, reverse mathematics

In Section 2, we study maximal filters. We show that every computable poset has a Δ^0_2 maximal filter. This result is optimal: there is a computable poset with no Σ^0_1 or Π^0_1 maximal filter. There is also a computable poset P such that any maximal filter on P is Turing complete. We obtain a reverse mathematics result: the principle that every countable poset has a maximal filter is equivalent to ACA_0 over RCA_0.

In Section 3, we study unbounded filters. We show that every computable poset has a Σ^0_1 unbounded filter, and there is a computable poset with no Π^0_1 unbounded filter. There is a computable poset P such that every unbounded filter on P is Turing complete. We obtain two reverse mathematics results. The principle that every countable poset has an unbounded filter is equivalent to ACA_0 over RCA_0. We define enumerated filters, which are analogous to Σ^0_1 filters, and show that the principle "Every sequence of enumerated filters on countable posets extends to a sequence of unbounded enumerated filters" is equivalent to ACA_0 over RCA_0.

In Section 4, we show that the upward closure of a computable subset of a computable poset may be Turing complete. The principle that every subset of a countable poset has an upward closure is equivalent to ACA_0 over RCA_0.

There has been significant previous research on the computability aspects of linear and partial orders. Downey [1] gives a thorough description of many results. We note that Mummert [2] has shown that there is a computable poset with a computable filter F such that the complete Σ^1_1 set is one-one reducible to any extension of F to a maximal filter.

The following results appeared in the Ph.D. thesis of Mummert [3]: Corollaries 2.2 and 4.2 and Theorems 3.1, 3.5, and 3.6. The remaining results are due to both authors.

1.1 Notation We use the following standard notation from computability theory. W_e denotes the Σ^0_1 set with index e and $W_{e,s}$ denotes the subset of W_e which is enumerated in s or fewer steps. $0'$ denotes the canonical Σ^0_1 complete set. The symbol $\leq_{wtt}$ denotes weak (bounded) truth table reducibility of subsets of $\mathbb{N}$. Readers unfamiliar with these concepts may consult the texts by Rogers [4] or Soare [6].

Reverse mathematics is a program in mathematical logic which classifies theorems based on the set-existence (comprehension) axioms required to prove the theorems. This classification is made using subsystems of second-order arithmetic. In this paper, we use two subsystems of second-order arithmetic: RCA_0, which contains Δ^0_1 comprehension and Σ^0_1 induction, and ACA_0, which contains arithmetical comprehension and arithmetical induction. A complete definition of these subsystems, along with a complete description of the goals of reverse mathematics, is given by Simpson [5].

2 Maximal Filters

A filter on a computable poset is maximal if it is not strictly contained in another filter on the poset.

Theorem 2.1 *Every computable poset has a Δ^0_2 maximal filter.*

Proof Let $P = \{p_0, p_1, \ldots\}$ be a computable poset. Begin by forming the oracle $A = \{\langle p, q\rangle \in P \times P \mid \exists r(r \preceq p \wedge r \preceq q)\}$. This oracle is clearly Σ^0_1. Now we define a maximal filter $G = \{q_i \mid i \in \mathbb{N}\}$ on P inductively. At stage 0, let $q_0 = p_0$. At stage $i + 1$, we will add two elements q_{2i+1} and q_{2i+2} to G. First we consider element $p_{i+1} \in P$. If $\langle p_{i+1}, q_{2i} \rangle \in A$ then we may effectively let q_{2i+2} be a common extension of q_{2i} and p_{i+1} and let $q_{2i+1} = p_{i+1}$. If $\langle p_{i+1}, q_{2i} \rangle \notin A$, let q_{2i+1} and q_{2i+2} both be q_{2i}. It is not hard to show that G is a maximal filter on P. If element p_i is added to G then it is added no later than stage i; thus G is computable from A, so G is Δ^0_2. □

Corollary 2.2 ACA_0 *proves that every countable poset has a maximal filter.*

Proof The previous proof may be formalized in ACA_0 using Σ^0_1 comprehension to form A. □

Theorem 2.3 *There is a computable poset P such that $0' \leq_{wtt} F$ for any maximal filter F on P.*

Proof Let K be any Σ^0_1 set. For each $k \in \mathbb{N}$, we construct a poset P_k inductively. P_k begins with two incompatible descending sequences. These sequences will eventually become compatible if $k \in K$; otherwise they will become two incompatible infinite descending sequences. If the sequences do not merge, then exactly one of the two maximal elements of the sequences will appear in any maximal filter, whereas if the sequences do merge then both maximal elements must appear in the unique maximal filter.

Formally, we let $P_k = \mathbb{N}$ and define the order $\preceq_k$ with the following rules:

$$\begin{aligned} 2m &\preceq_k 2n && \text{if } n \leq m, \\ 2m+1 &\preceq_k 2n+1 && \text{if } n \leq m, \\ 2m &\preceq_k 2n+1 && \text{if } n \leq m \text{ and } k \in K_m, \\ 2(m+1)+1 &\preceq_k 2n && \text{if } n \leq m \text{ and } 2m+2 \preceq_k 2m+1. \end{aligned}$$

As usual, K_m denotes the set of numbers that enter K in no more than m steps of its canonical enumeration. Clearly, $\preceq_k$ is a computable partial order. If $k \notin K$ then $2m \perp 2n+1$ for all n, m, and no maximal filter can contain both 0 and 1. Otherwise, for all sufficiently large m, n we have $2m \preceq 2n+1$ and $2m+1 \preceq 2n$, every two elements are compatible, and the unique maximal filter contains 0 and 1.

The construction above is uniform in the sense that we may uniformly compute the order $\preceq_k$ from k. We may thus form the product poset P, which we now describe. The elements of P are finite sequences of natural numbers. We order P by putting $\bar{a} \preceq \bar{b}$ if and only if $|\bar{a}| \geq |\bar{b}|$ and $a_i \preceq_i b_i$ for all $i \leq |\bar{b}|$. Clearly, the order on P is computable, and for any $k \in \mathbb{N}$ and any maximal filter F on P the set $\pi_k F = \{n \in \mathbb{N} \mid \exists \bar{a} \in F(|\bar{a}| \geq k \wedge n = a_k)\}$ is a maximal filter on P_k.

Note that for each k exactly one of the following options holds.

1. $0 \in \pi_k F$ or $1 \in \pi_k F$, but not both.
2. $0 \in \pi_k F$ and $1 \in \pi_k F$.

Beginning with $k = 0$ and proceeding inductively, we may effectively determine for each k which of the options holds using no more than 2 queries for each new k. For example, we first determine whether 0 or 1 is in $\pi_0 F$. Assume that $0 \in \pi_0 F$. Then we can determine whether $1 \in \pi_1 F$ by asking whether $\langle 0, 1 \rangle \in F$.

If the first option above holds for k then $k \notin K$. If the second option holds, then $k \in K$. Thus $K \leq_{wtt} G$ for any maximal filter G on P. The theorem follows by taking $K = 0'$. □

Corollary 2.4 *The principle that every countable poset has a maximal filter is equivalent to* ACA_0 *over* RCA_0.

Proof The proof of Theorem 2.3, relativized to an arbitrary oracle, may be formalized in RCA_0. □

Theorem 2.5 *There is a computable poset with no* Σ^0_1 *or* Π^0_1 *maximal filter.*

Proof We informally view our poset P as an infinite product of posets $\langle P_e \mid e \in \mathbb{N}\rangle$ and $\langle Q_e \mid e \in \mathbb{N}\rangle$. The construction of P_e ensures that W_e is not a maximal filter on P, and the construction of Q_e ensures that the complement of W_e is not a maximal filter on P.

For each e, the poset P_e is built as follows. We first construct an infinite descending sequence $\langle a_i \mid i \in \mathbb{N}\rangle$. Now we wait until an element a_i enters W_e. If no element a_i enters, then W_e is not a maximal filter on P_e. Whenever an element a_{i+1} enters W_e, we add an element b_i such that $b_i \prec a_i$ and $b_i \perp a_{i+1}$. We then extend $\prec$ so that $b_i \prec b_j$ if $i > j$ and so that $\prec$ remains transitive. Then we return to waiting. If no element of the form b_i is ever added to W_e, then the infinite descending sequence $\langle b_i \rangle$ generates a filter properly including W_e. If W_e ever includes any element of the form b_i, we stop adding any new elements of the form b_j to the poset. In this case, let i be the largest number such that b_i was added to the poset; we know a_{i+1} is in W_e, and $b_i \perp a_j$ for all $j \geq i+1$ because we will add no more elements of the form b_j. Thus W_e is not a maximal filter on P_e.

For each e, the poset Q_e is built as follows. We construct two infinite descending sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ as in the proof of Theorem 2.3. We wait to see whether W_e ever includes a_0 or b_0. If it never does, we keep the infinite descending chains incompatible, which means that the complement of W_e is not a filter on Q_e. If W_e ever includes a_0 or b_0, we cause the two chains to eventually be compatible, which means that the complement of W_e is not a maximal filter on Q_e.

To construct the poset P, we view each natural number as a code for a finite sequence of natural numbers, and thus view $P = \mathbb{N}$ as a product poset with one coordinate for each natural number. The order on coordinate $2i$ of P is similar to the order on poset P_i, and the order on coordinate $2i+1$ of P is similar to the order on Q_i. We watch the effects of each Σ^0_1 or Π^0_1 subset on the appropriate coordinate of P and use the descriptions of P_e and Q_e above to shape the order in that coordinate. Because the elements of P are finite sequences and the order relations on P_e and Q_e are uniformly computable, the order on P will be computable. □

3 Unbounded Filters

A filter F on a poset P is unbounded if there is no $p \in P \setminus F$ such that $p \preceq q$ for all $q \in F$.

Theorem 3.1 *Every computable poset has a* Σ^0_1 *unbounded filter.*

Proof Let $P = \langle p_i \mid i \in \mathbb{N}\rangle$ be a computable poset. We describe an enumeration of an unbounded filter on P. At stage 0, let $q_0 = p_0$. At stage $n+1$, if $p_{i+1} \leq q_i$ then let $q_{i+1} = p_{i+1}$; otherwise, let $q_{i+1} = q_i$. The upward closure of $\{q_i \mid i \in \mathbb{N}\}$ is easily seen to be a Σ^0_1 unbounded filter on P. □

Theorem 3.2 *There is a computable poset P such that $0' \leq_{wtt} F$ for any unbounded filter F on P.*

Proof We use the computable posets $\langle P_k, \preceq_k \rangle$ from the proof of Theorem 2.3. We form a computable poset P whose elements are sequences of the form $\langle n, a_0, a_1, \ldots, a_n \rangle$ where $n \in \mathbb{N}$ and $a_k \in P_k$ for each $k \leq n$. We order P by putting $\langle n, a_0, \ldots, a_n \rangle \prec \langle m, b_0, \ldots, b_m \rangle$ when $n > m$ and $a_i \prec_i b_i$ for each $i \leq m$. Clearly, P is a computable poset; we call this the uniform product of the posets $\langle P_k \rangle$.

We now show that the projection of any unbounded filter on P to coordinate $k+1$ is an unbounded filter on P_k. Clearly, any unbounded filter on P is generated by an infinite strictly descending sequence $\langle r_i \rangle$ on P. Let p_i denote the element of P_k in coordinate $k+1$ of r_i whenever this coordinate exists. The order relation on P ensures that $\langle p_i \rangle$ is an infinite strictly descending sequence on P_k. Any such sequence determines an unbounded filter on P_k.

Note that every unbounded filter on P_k is maximal; thus we have shown that any unbounded filter on P uniformly computes a maximal filter on P_k for each k. The remainder of the proof is similar to Theorem 2.3. □

We obtain a corollary by formalizing the previous proof in RCA_0.

Corollary 3.3 *The principle that every countable poset has an unbounded filter is equivalent to* ACA_0 *over* RCA_0.

The next theorem is proved by applying the uniform product technique of Theorem 3.2 to the sequence of posets Q_e in the proof of Theorem 2.5.

Theorem 3.4 *There is a computable poset with no Π^0_1 unbounded filter.*

In RCA_0, we define an *enumerated filter* on a countable poset $\langle P, \preceq \rangle$ to be a function F from $\mathbb{N}$ to P whose range is a filter on P. That is, if p is in the range of F and $p \preceq q$ then q is in the range of F, and for all p, q in the range of F there is an r in the range of F such that $r \preceq p$ and $r \preceq q$. Note that a filter on P is definable by a Σ^0_1 formula if and only if RCA_0 is able to form the corresponding enumerated filter. We view an enumerated filter as a code for the set of poset elements it determines, writing $p \in F$ for an enumerated filter F if p is in the range of F. We say that an enumerated filter is unbounded if there is no $q \in P$ such that $q \prec p$ whenever $p \in F$. By formalizing the proof of Theorem 3.1, we may prove in RCA_0 that every countable poset has an unbounded enumerated filter.

Theorem 3.5 RCA_0 *proves that every enumerated filter on a countable poset is contained in an unbounded enumerated filter on the poset.*

Proof Let F be an enumerated filter on a countable poset P. If F is unbounded then we are done. If F is bounded, let p be a lower bound for F and construct an unbounded enumerated filter containing p as in the proof of Theorem 3.1. □

The proof of Theorem 3.5 is nonuniform in the sense that the noncomputable choice of whether a given filter is unbounded must be made. The next theorem suggests that no uniform proof of Theorem 3.5 is possible.

Theorem 3.6 *The following are equivalent over* RCA_0.

1. ACA_0.

2. *If $\langle P_i \mid i \in \mathbb{N}\rangle$ is a sequence of countable posets and $\langle F_i \mid i \in \mathbb{N}\rangle$ is such that each F_i is an enumerated filter on P_i then there is a sequence $\langle G_i \mid i \in \mathbb{N}\rangle$ such that G_i is an enumerated unbounded filter on P_i extending F_i for each $i \in \mathbb{N}$.*

Proof It is not difficult to prove (2) in ACA_0. To prove that (2) implies ACA_0, we work in RCA_0. Let $f : \mathbb{N} \to \mathbb{N}$ be given; we will show that the range of f exists, which implies ACA_0. For each $e \in \mathbb{N}$, form a poset $\langle P_e, \preceq_e\rangle$ as follows. P_e has an infinite descending sequence $\langle a_i^e \mid i \in \mathbb{N}\rangle$ and one additional element b^e. We let $b^e \preceq_e a_i^e$ if there is no $j \leq i$ such that $f(j) = e$. For all i we have $a_i^e \not\preceq_e b^e$. It is clear that $\preceq_e$ is a partial order on P_e, and the sequence of posets $\langle\langle P_e, \preceq_e\rangle \mid e \in \mathbb{N}\rangle$ may be formed in RCA_0.

For each $e \in \mathbb{N}$ let $F_e = \{a_i^e \mid i \in \mathbb{N}\}$; the sequence $\langle F_e \mid e \in \mathbb{N}\rangle$ may be formed in RCA_0 as well. Apply (2) to form a sequence $\langle G_e \mid e \in \mathbb{N}\rangle$ such that each G_e is an enumerated filter on P_e and $F_e \subseteq G_e$. Note that $b^e \in G_e$ if and only if there is no $i \in \mathbb{N}$ such that $f(i) = e$. There is thus a Π^0_1 formula which tells whether e is in the range of f. Because the range of f has a trivial Σ^0_1 definition, we have shown the range of f is definable by a Δ^0_1 formula, which means that we may form this set in RCA_0. □

4 Upward Closures

The upward closure of a set $F \subseteq P$ is the set $\{q \in P \mid \exists p \in F(p \preceq q)\}$.

Theorem 4.1 *There is a computable poset P with a computable linearly ordered subset F such that $0'$ is one-one reducible to the upward closure of F.*

Proof Let W_e be a Σ^0_1 complete set. We let P have an infinite descending sequence $\langle a_i \mid i \in \mathbb{N}\rangle$ and infinitely many pairwise incompatible elements $\{b_j \mid j \in \mathbb{N}\}$. We put $a_i \preceq b_j$, if $j \in W_{e,i}$, and $a_i \not\preceq b_j$ otherwise. We always have $b_j \not\preceq a_i$.

The poset just described is computable. Let A denote the computable subset $\{a_i \mid i \in \mathbb{N}\}$. Clearly, $j \in W_e$ if and only if b_j is in the upward closure of A. □

Corollary 4.2 *The following are equivalent over* RCA_0.

1. ACA_0.
2. *Every subset of a countable poset has an upward closure.*
3. *Every linearly ordered subset of a countable poset has an upward closure.*

Proof Because the upward closure of any set $A \subseteq P$ is Σ^0_1 definable from A, ACA_0 proves (2). Clearly, (2) implies (3). The proof of Theorem 4.1 may be formalized in RCA_0 to show that (3) implies ACA_0. □

References

[1] Downey, R. G., "Computability theory and linear orderings," pp. 823–976 in *Handbook of Recursive Mathematics, Vol. 2*, edited by Yu. L. Ershov, vol. 139 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1998. Zbl 0941.03045. MR 1673590. 480

[2] Mummert, C., "Reverse mathematics of MF spaces," submitted for publication. 480

[3] Mummert, C., *On the Reverse Mathematics of General Topology*, Ph.D. thesis, Pennsylvania State University, 2005. 480

[4] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, 2d edition, The MIT Press, Cambridge, 1987. MR 886890. 480

[5] Simpson, S. G., *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR 1723993. 480

[6] Soare, R. I., *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987. Zbl 0667.03030. Zbl 0623.03042. MR 882921. 480

Acknowledgments

The first author's research was partially supported by NSF grant DMS-0140120. The second author's research was partially supported by a VIGRE graduate traineeship at the Pennsylvania State University under NSF grant DMS-9810759.

Department of Mathematics
University of Wisconsin
Madison WI 53706-1388
lempp@math.wisc.edu

Department of Mathematics
University of Michigan
Ann Arbor MI 48109-1043
mummertc@umich.edu