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Continu′ous Time Goes by* Russell

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Abstract Russell and Walker proposed different ways of constructing instants from events. For an explanation of "time as a continuum," Thomason favored Walker's construction. The present article shows that Russell's construction fares as well. To this end, a mathematical characterization problem is solved which corresponds to the characterization problem that Thomason solved with regard to Walker's construction. It is shown how to characterize those event structures (formally, interval orders) which, through Russell's construction of instants, become linear orders isomorphic to a given (or, deriving, to *some*—nontrivial ordered) real interval. As tools, separate characterizations for each of resulting (i) Dedekind completeness, (ii) separability, (iii) plurality of elements, (iv) existence of certain endpoints are provided. Denseness is characterized to replace Russell's erroneous attempt. Somewhat minimal nonconstructive principles needed are exhibited, and some alternative approaches are surveyed.

* The song "As Time Goes By" by H. Hupfeld, played in the movie *Casablanca* [39]

1 Introduction and Summary

The problem the present paper is concerned with derives from Russell's ([36], Lecture IV) attempt mathematically to construct "instants" of time from "events." Thomason's [40] paper starts almost the same way. According to Thomason, however, *Russell*'s construction has the disadvantage that it is *difficult to see* what assumptions about the temporal relationships among events will ensure that the instants constructed comprise a continuum, isomorphic to the real numbers. Thomason [40] thus presents an alternative way—due to the mathematician-physicist *A. G. Walker*[1] [42]—of constructing instants of time out of events. Thomason shows

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of some conditions on the temporal relationships of events that they *do* ensure instants of time, as constructed according to *Walker*, comprising a "continuum."[2] He then concludes that Walker's theory offers, *as Russell's does not*, a plausible explanation of time as a continuum. The present paper shows (besides some by-products) how to single out those relations on sets of events which become order-isomorphic to the real numbers through *Russell*'s construction. More generally, it shows how to tell from such a relation whether the resulting order is in a class (closed under isomorphisms) of *intervals* of the real numbers[3]—an interval being open in each, one, or no direction, ordered by the relation of being less.[4] At least one of the characterizations presented might be considered a "refutation" of Thomason's claim denying explanatory power of Russell's theory.[5] The *keys* to these characterizations are (i) a sufficient condition on events and their temporal relationships for Russellian time being dense,[6] due to N. Wiener ([46], pp. 446–47) or to Russell;[7] (ii) a characterization, due to Russell, of those events that get first or, respectively, last instants by his construction. The sufficient condition according to (i) is modified here to obtain a *necessary* and sufficient one.

The main results are stated in Section 7. The sections which precede Section 7 merely explain the notions used in the latter. In proving claims afterward, I keep books on what nonconstructive choice or maximality principles (the very axiom of choice or something weaker) I use. It will even be shown that one of the characterizing conditions can be used as an *alternative* to any such principle in the relevant context. Some proofs merely replace existing *Principia*-notation proofs and thus may be helpful to at least some readers. The last section comments on some remaining aspects of continuumlikeness of time.

2 Relations, Linearity, and Real Intervals

I will usually write $x\ R\ y$ instead of (the ordered pair) $\langle x, y\rangle \in R$. Considering some $R_1, \ldots, R_n$, I will write $x_1\ R_1\ x_2\ R_2\ x_3 \ldots x_n\ R_n\ x_{n+1}$ meaning that $x_1\ R_1\ x_2$, $x_2\ R_2\ x_3, \ldots$ *and* $x_n\ R_n\ x_{n+1}$.

When X, R are sets such that $R \subseteq X \times X$, R is a *binary relation on* X; $\langle X, R\rangle$ is called a *related set* then. If R is *not* a subset of $X \times X$, $\langle X, R\rangle$ is shorthand for $\langle X, R \cap (X \times X)\rangle$.[8] Sometimes I will abuse language by talking of elements and cardinality of related sets $\langle X, R\rangle$ as if I would talk of X. Related sets $\langle X, R\rangle$, $\langle X', R'\rangle$ are *isomorphic* if there is a one-to-one map φ from X *onto* X' such that for any $x, y \in X$

$$x\ R\ y \quad \text{if and only if} \quad \varphi(x)\ R'\ \varphi(y).$$

A *least element* in a related set $\langle X, R\rangle$ is an $x \in X$ such that $x\ R\ y$ for any *other* $y \in X$. If *instead* $y\ R\ x$, then x is a *greatest* element.

By a *(strict) linear order on some set* Y I understand a transitive binary relation R on Y such that for any $x, y \in Y$ *exactly one* of $x\ R\ y$, $y\ R\ x$, and $x = y$ holds. In this case, I will call any related set $\langle X, R\rangle$ such that $X \subseteq Y$ a *linearly ordered set*, or, for short, a *loset*.

$\langle \mathbf{R}, <\rangle$ will denote the set of all real numbers linearly ordered by the binary relation of being less. (In doubt, $< \subseteq \mathbf{R} \times \mathbf{R}$.) By a *real interval* I will understand a set $I \subseteq \mathbf{R}$ such that

$$\text{if} \quad s, t \in I \ \text{ and } \ s < r < t \quad \text{then} \quad r \in I.$$

I am going to call a real interval *open* if it contains neither a least nor a greatest element, *half-open* if it contains a least or a greatest element but not both, and *compact* if it contains both a least and a greatest element (which may coincide). By a *nontrivial* real interval I understand a real interval having more than one element—which is the same as having the cardinality of $\mathbf{R}$. Observe that being nontrivial is the same as being nonvoid in case of an *open* real interval and as having distinct borders in the case of a *compact* real interval, while *all half-open* intervals are nontrivial.

3 Some Events, Some Russellian Instants

For the remaining, I *fix* some nonvoid set E of "events" and a binary relation $P \subseteq E \times E$ on it, the philosophical meaning of which is to be 'wholly precedes'. Proofs of what is claimed on the few lines following are indicated in the first two subsections of Section 8.

An *antichain (in* $\langle E, P\rangle$*)* is a subset A of E such that for any *two* $a, b \in A$ *neither* $a \ P \ b$ *nor* $b \ P \ a$. A *maximal* antichain is an antichain that cannot be extended to another antichain by adding any nonvoid set of events. By the axiom of choice, any antichain can be extended to a maximal antichain, and as the empty set *is* an antichain, there is at least *some* maximal antichain. For the same time that $\langle E, P\rangle$ is *fixed*, let L be just the set of maximal antichains in $\langle E, P\rangle$. By Russell's proposal,[9] the elements of L are *instants of time* as "constructed" from those "events" presented in E and from their temporal relationships encoded by P. (Therefore, it will be informal variables like s, t that range in L, while $A, B, \ldots$ will range over *arbitrary* subsets of E.)

The "order of time" then is defined to be

$$T := \{\, \langle s, t\rangle \in L \times L \mid a \ P \ b \text{ for some } a \in s \text{ and some } b \in t \,\}.$$

In "order notation" and "philosophical diction," $s \ T \ t$ (only) for "instants" s, t in L just if some event a in s 'wholly precedes' some event b in t.

I furthermore *fix* that P be *irreflexive* (i.e., no event precede itself) *and* that for any events a, b, c, d,

$$\text{if} \quad a \ P \ b \text{ and } c \ P \ d \quad \text{then} \quad a \ P \ d \text{ or } c \ P \ b. \tag{I}$$

($\langle E, P\rangle$ is now an *interval order* in the sense of [14] and [15], and I may use results from Wiener [46],[10] who calls a P satisfying an equivalent pair of conditions a 'relation of complete sequence', as well as from Russell [37], who uses a triple of conditions equivalent to each of the previously mentioned pairs.) Reading (I) as 'if a wholly precedes b and c wholly precedes d, but c does *not* wholly precede b, then a wholly precedes d' may exhibit (I) to be a "self-evident" feature of P if the latter is read as 'wholly precedes'.[11] This would render $\langle E, P\rangle$ quite a good starting point for *philosophically justifying* instants of time. In fact, in the situation of the hypothesis of the proposed reading, (i) *either* b wholly precedes c, and then by "self-evident" transitivity of 'wholly precedes' a wholly precedes d; (ii) *or* b and c have "something in common" which must be preceded by a and precede d, and this might be considered some "evidence" for a wholly preceding d.[12]

By (I) and irreflexivity of P, the latter relation is transitive. Moreover, (I) ensures that T is a linear order on L ([46], pp. 445–46). In this respect, T is an adequate mathematical reconstruction of temporal precedence between instants of time. Some aspects of adequacy, however, remain to be considered.

4 Core Task

I am going to propose a few characterization problems to be solved, indicating them as "questions." First compare the characterization problem that Thomason [40] deals with. He explains how Walker would construct, instead of $\langle L, T \rangle$, *another* time ordering $\langle L', T' \rangle$ from $\langle E, P \rangle$.[13] He then presents a necessary and sufficient condition on $\langle E, P \rangle$ for $\langle L', T' \rangle$ being isomorphic to $\langle \mathbf{R}, < \rangle$. The analogous problem for $\langle L, T \rangle$ he considers "difficult."

Question 1 *When, in terms of $\langle E, P \rangle$, is $\langle L, T \rangle$ isomorphic to $\langle \mathbf{R}, < \rangle$?*

Here, I am using 'when ...' to abridge something like 'What conditions on $\langle E, P \rangle$ are necessary and sufficient for $\langle E, P \rangle$ being isomorphic to $\langle \mathbf{R}, < \rangle$?' and similarly below. These questions are, admittedly, not perfectly precise, but I am not attempting to explicate the notion of 'characterization' in this paper; it will suffice to present convincing solutions to the problems, however "ill-posed" the latter may be. We will encounter solutions for other characterization problems by conditions which could be formalized as first-order sentences in a language interpretable in $\langle E, P \rangle$. It may be no surprise in connection with "the continuum," however, that the main characterizing conditions are not first-order but stipulate a countable set of objects from E having a certain property (cf. [29], Theorem 2.1).

Variants of Question 1 may be interesting from a physical, mathematical, or perhaps even philosophical point of view. First, one might really want an explanation of "time as a continuum," but one might disagree with Thomason as to what a *continuum is*. Indeed, the purportedly well-known order-type of "*the* continuum" that Cantor characterized in his celebrated [9] was actually that of the *closed* (i.e., compact) unit interval, and another notion of "continua," entailing topological compactness as well, was prominent in a branch of early topology, so-called curve theory ([27]; [32], p. 56; [33], p. 213). So (as some readers conclude immediately and others will see soon) a continuum may be—and has been—considered something essentially different to a structure comprising all the real numbers. Unlike the real numbers, it might have a first point, a last point, or both.

On the other hand, the question of what *a* or *the continuum* is might be considered too "theoretical," namely, "only" of mathematical or even philosophical interest. It might be considered more important whether the construction renders time as it is used in *physics*. Indeed, physical textbooks seem to postulate that instants of time are just the same as real numbers. However, from the viewpoint of physical, relativistic cosmology (of General Relativity, that is) which prevails nowadays, astronomical observations indicate that not *every* real number corresponds to an instant of time (in the usual way): a "Big Bang" is believed to have *started* time, and a final collapse of the universe is reckoned to *end* time.[14]

However, this reasoning may only lead to considering bounded open real intervals instead of all the real numbers, and this switch does not change anything at least order-theoretically or topologically. First or last instants which would make a difference seem to play no role in physics—you usually encounter open time intervals in textbooks; even the Big Bang has no starting instant of time.[15] Yet, *some* physicist might one day leave the herd.

So there may be some, if only little, reason to consider more general versions of Question 1. The additional answers will not need extra mathematical effort. From the results the reader may choose what she likes most. The reader may like to know something about "continua," and she may conceive of continua so that two of them are isomorphic to each other (like the open real intervals), or so that there are two or more isomorphism types of continua. For example, a continuum might be viewed to be anything order-isomorphic to some nontrivial real interval. Consider the following Question-Scheme.

Question-Scheme 1* *When, in terms of* $\langle E, P\rangle$, *is* $\langle L, T\rangle$ *isomorphic to* $\langle I, <\rangle$*?*

This "scheme" produces one question for each real interval I. Question 1 is that example where $I = \mathbf{R}$. The "scheme" represents readers with somewhat very "narrow" conceptions of continuumlikeness. As an example of a *broad* conception, its answers enable answering the following question.

Question 1$^+$ *When, in terms of* $\langle E, P\rangle$, *is* $\langle L, T\rangle$ *isomorphic to some* $\langle I, <\rangle$ *where* I *is a nontrivial real interval?*

5 Via Completeness and Dense Subsets

The following is well known.[16]

Fact 1 A related set is isomorphic to $\langle \mathbf{R}, <\rangle$ if and only if it is a *nonvoid*, *complete*, and *separable loset* having neither a *least* nor a *greatest* element.

To understand this, recall some definitions. Let $\langle X, R\rangle$ be some related set again.

A *lower bound* of some $Y \subseteq X$ is an $x \in X$ which is a least element of $\{x\} \cup Y$ (x may be an element of Y). A *greatest lower bound* of $Y \subseteq X$ is a greatest element of the set of all lower bounds of Y. By analogy, an *upper bound* of Y is a greatest element of the union of its singleton with Y, and a *least upper bound* is a least element of the set of all upper bounds. (If $\langle X, R\rangle$ is a loset, Y has at most *one* greatest lower or, respectively, least upper bound, of course.)

Now $\langle X, R\rangle$ is *complete* if every nonvoid subset of X having a *lower* bound has a *greatest* such and every subset of X having an *upper* bound has a *least* such.

A set Y is *dense* in $\langle X, R\rangle$ if $Y \subseteq X$ and for any *two* $x, z \in X$ such that $x\ R\ z$ there is some $y \in Y$ such that $x\ R\ y\ R\ z$.

Finally, $\langle X, R\rangle$ is *separable* if there is a *countable*[17] subset of X being dense in $\langle X, R\rangle$.[18]

In view of Fact 1, and since $\langle L, T\rangle$ is a nonvoid loset anyway (by Section 3), we approach our goal by moving from Question 1 to the following.

Question 1′ *When, in terms of* $\langle E, P\rangle$, *is* $\langle L, T\rangle$ *complete and separable without least or greatest element?*

Question 1′ can be split into the following questions which will be dealt with separately from each other.

Question 2 *When, in terms of* $\langle E, P\rangle$, *is* $\langle L, T\rangle$ *complete?*

Question 3 *When, in terms of* $\langle E, P\rangle$, *is* $\langle L, T\rangle$ *separable?*

Question 4a *When, in terms of* $\langle E, P\rangle$, *has* $\langle L, T\rangle$ *a least element?*

Question 4b *When, in terms of* $\langle E, P\rangle$, *has* $\langle L, T\rangle$ *a greatest element?*

To deal with Question-Scheme 1*, Fact 1 can be generalized to arbitrary real intervals—taking some subtleties (as observed in Section 2) into account.

Fact 1* Let I be a real interval.

(a) Assume I is open. If I is empty, then $\langle I, <\rangle = \langle \varnothing, \varnothing\rangle$, and this is the only related set isomorphic to $\langle I, <\rangle$. Otherwise a related set is isomorphic to $\langle I, <\rangle$ if and only if it is a nonvoid, complete, and separable loset having neither a least nor a greatest element.

(b) If I contains a least element, but no greatest (the other way round, respectively), a related set is isomorphic to $\langle I, <\rangle$ if and only if it is a complete and separable loset with[19] a least (greatest, respectively) and without greatest (least, respectively) element.

(c) Assume I is compact.[20] If I contains just one element, then a related set is isomorphic to $\langle I, <\rangle$ if and only if it is $\langle \{x\}, \varnothing\rangle$ for some object x. Otherwise a related set is isomorphic to $\langle I, <\rangle$ if and only if it is a complete and separable loset with a least and a greatest element where the latter do *not coincide*.[21]

Thus there are four types (with respect to order-isomorphisms) of nontrivial real intervals: open, half-open (two types), and compact-with-more-than-one-element. Answers to Questions 2, 3, 4a, and 4b suffice to recognize the first three types for $\langle L, T\rangle$ (corresponding to cases (a) and (b) of Fact 1*) from looking at $\langle E, P\rangle$. For recognizing the fourth type, it suffices to deal with the following final question.

Question 5 *When, in terms of $\langle E, P\rangle$, has L more than one element?*

An obvious "answer-scheme" to Question-Scheme 1* arises, following the lines of Fact 1*. An answer to Question 1^+ derives which may be considered a consequence of the following fact, which is entailed by Fact 1*.

Fact 1^+ A related set is isomorphic to $\langle I, <\rangle$ for some nontrivial real interval I if and only if it is a complete and separable loset having more than one element.

6 Pivotal Derived Notions

I am going to introduce further relations on the set E of events in order to state some conditions more succinctly than I could without them. At the same time, some visualizing possible situations may be in order so that it is easier to understand what I mean.

Assume *for a moment* that E is a three-element set $\{a, b, c\}$ and $P = \{\langle a, b\rangle\}$. This situation is visualized by Figure 1.[22]

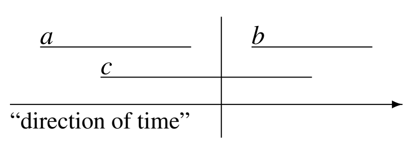

Figure 1 Three events

$a\ P\ b$ is visualized by arranging horizontal strokes representing a and b so that a *vertical* stroke can be filled in right-hand to the horizontal stroke representing a and left-hand to the horizontal stroke which represents b.

By contrast, no vertical stroke would have two horizontal strokes on different sides such that one of them would represent a and the other would represent c. This holds for b in place of a as well. Rather, c is "overlapping" a as well as b.

(By the way, $\{a, c\}$ and $\{b, c\}$ are *antichains*, *maximal* antichains, in fact, and the only ones. So they form the set L of instants of time, and we have $\{a, c\}\ T\ \{b, c\}$.)

Furthermore, c "begins earlier than" b, "witnessed" by a overlapping c but wholly preceding b. Similarly, a "ends earlier than" c, "witnessed" by b overlapping c but being wholly preceded by a.

The above 'moment' (specializing $\langle E, P\rangle$) is over, and I generalize the situation by further definitions of binary relations on E representing the relationships observed above.

$$S := \{\langle a, b\rangle \in E \times E \mid \text{neither } a\ P\ b \text{ nor } b\ P\ a\};$$

$$SP := \{\langle c, b\rangle \in E \times E \mid c\ S\ a\ P\ b \text{ for some } a\};^{23}$$

$$PS := \{\langle a, c\rangle \in E \times E \mid a\ P\ b\ S\ c \text{ for some } b\}.^{24}$$

(Visualize by help of Figure 1.) Now $a\ S\ b$ is to mean that a *overlaps* b, the other way round, or just that a and b overlap, however you like;[25] $c\ SP\ b$ is to mean that c *begins earlier than* b; and $a\ PS\ c$ is to mean that a *ends* earlier than c.[26]

An event a will be called an SP*-minimal* element of a subset A of E, if $a \in A$ and there is no $b \in A$ such that $b\ SP\ a$. "Dually,"[27] an event a will be called a PS*-maximal* element of A, if $a \in A$ and there is no $b \in A$ such that $a\ PS\ b$.[28]

The relations on E defined just before make it easier to define the notion of 'having a first (last, respectively) instant' in one line, which in [37] plays an important role for the question of the existence of instants. This notion is vital for formulating answers to *all my* questions but one as well. To enable the reader to make sense of my following definitions, I precede them with an outline of Russell's [37] discovery in terms of his philosophical interpretation.

Call an event b a *contemporary* of some event a whenever they overlap (purely mathematically, $a\ S\ b$). Call b an *initial* contempory of a if *additionally* a does not begin earlier than b (*not* $a\ SP\ b$). Now Russell found out that an event e has a first instant (i.e., there is a T-least maximal antichain containing e) if and only if (keep the following in mind for a moment), *whenever e begins earlier than some event a, this a is wholly preceded by some initial contemporary of e*. (In this case, the set of initial contemporaries of e *is* that first instant.)

To make mathematical use of the notion of 'having a first instant', I introduce a symbol denoting its extension (as far as E is concerned):

$$\exists_- := \{\, e \in E \mid \text{whenever } e\ SP\ a,\ e\ (S \setminus SP)\ b\ P\ a \ \text{ for some } b\,\}.$$

So $e \in \exists_-$ "philosophically" means that e has a first instant.

Dually,

$$\exists_+ := \{\, e \in E \mid \text{whenever } a\ PS\ e,\ a\ P\ b\ (S \setminus PS)\ e \ \text{ for some } b\,\}$$

contains all events having a *last* instant.

(However, I am not quite sure about whether it was really *Russell* who found that $e \in \exists_-$ if and only if e has a first instant and about what actually *Wiener* contributed to the result.)[29]

7 Solutions

Nonconstructive[30] assumptions used As a whole, the ensuing answers to the questions asked in Sections 4 and 5 assume the axiom of choice or, at least, the principle of countable choice[31] and that every antichain (in our $\langle E, P\rangle$) of at most 2 elements extends to a *maximal* antichain. Theorem 7.1, however, needs the latter assumption for singletons in place of antichains only, and nonvoidness of L (Section 3) merely means existence of a maximal antichain with *no* regard to what elements of E it should contain. I call $\langle E, P\rangle$ *n-maximizing* if every antichain of at most n elements extends to a maximal antichain; the assumptions mentioned before refer to this for $n = 2, 1, 0$, respectively.[32] Moreover, the condition of Theorem 7.3 characterizing separability of $\langle L, T\rangle$ implies that $\langle E, P\rangle$ is 2-maximizing and thus (whenever assumed) removes any reliance on a nonconstructive principle *as far as existence of maximal antichains is concerned*; in particular, no maximizingness assumption is involved in sufficiency of the characterization of real intervals. Now, I believe that '*$\langle E, P\rangle$ is 2-maximizing*' or any of some other consequences of the axiom of choice discussed in Subsection 8.2 below does *not* imply the principle of countable choice. In case this belief is correct, it is important to note that the latter principle is used for both directions of Theorem 7.3. Under the previous separability assumption on $\langle E, P\rangle$ it remains the only nonconstructive principle underlying one direction of Theorem 7.3 and Corollary 7.6. I name these assumptions and the directions in which they are used in parentheses near each statement below.

Question 5 is most easily answered, so start with the following theorem.

Theorem 7.1 *L has more than one element if and only if $P \neq \varnothing$.*

(This uses just that $\langle E, P\rangle$ is 1-maximizing for 'if'.)

Answers to Questions 4a and 4b are slightly more difficult.

Theorem 7.2

(a) *$\langle L, T\rangle$ has a least element if and only if E has an SP-minimal element contained in $\exists_-$* ("having a first instant", that is).

(b) (dually to the previous) *$\langle L, T\rangle$ has a greatest element if and only if E has a PS-maximal element contained in $\exists_+$* ("having a *last* instant," that is).

(Both (a) and (b) use that $\langle E, P\rangle$ is 2-maximizing for 'only if'.)

In the following, situations like $a\ P\ c\ S\ d\ P\ b$ are mentioned repeatedly. Such a situation is illustrated in Figure 2. ($a\ P\ b$ follows by (I).)

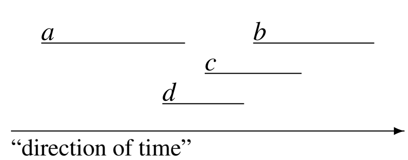

Figure 2 Antichain $\{c, d\}$ "in gap" of chain $\{a, b\}$

In such a situation, $\{c, d\}$ is an *antichain* "in the gap" of the chain $\{a, b\}$ (where a *chain* is a subset of E forming a loset when P is restricted to it). If E were just $\{a, b, c, d\}$, L would be $\{\{a, d\}, \{c, d\}, \{b, c\}\}$, and T would do $\{a, d\}\ T\ \{c, d\}\ T\ \{b, c\}$, so $\{c, d\}$ would be an "instant" T-between "all" the instants containing a in one direction and "all" the instants containing b in the other. This enables answering Question 3 as follows.

Theorem 7.3 *$\langle L, T\rangle$ is separable if and only if there is a* countable *subset D of E such that for all $\langle a, b\rangle \in P$ "at the same time"*

(i) *if $a \in \exists_+$ and $b \in \exists_-$, then there are $c, d \in D$ such that $a\ P\ c\ S\ d\ P\ b$;*
(ii) *if $a \notin \exists_+$, then whenever $e\ PS\ a$, there are $c, d \in D$ such that $e\ P\ c\ S\ d\ P\ b$;*
(iii) (dually to the previous condition) *if $b \notin \exists_-$, then whenever $b\ SP\ e$, there are $c, d \in D$ such that $a\ P\ d\ S\ c\ P\ e$.*

(This uses countable choice in both directions and '$\langle E, P\rangle$ is 2-maximizing' for 'only if'. In proving the theorem, I will generalize it to infinite cardinalities of dense subsets, then using the full axiom of choice.)

Here is one by-product announced in the beginning.

Corollary 7.4 *$\langle L, T\rangle$ is* dense *(i.e., L is dense in $\langle L, T\rangle$) if and only if for all $\langle a, b\rangle \in P$ such that $a \in \exists_+$ and $b \in \exists_-$ there are $c, d \in E$ such that $a\ P\ c\ S\ d\ P\ b$* (compare first condition in the previous theorem).

(This uses that $\langle E, P\rangle$ is 2-maximizing for 'only if'.) The criterion "in terms of" $\langle E, P\rangle$ for $\langle L, T\rangle$ being dense given here is a weakened version of a condition stated by Russell in a footnote of [36], Lecture IV, and by Wiener in [46], pp. 446–47,[33] namely,

$$\text{for all } \langle a, b\rangle \in P, \text{ there are } c, d \in E \text{ such that } a\ P\ c\ S\ d\ P\ b. \tag{III}$$

This, of course, is only a *sufficient*, not a necessary condition for $\langle L, T\rangle$ being dense.[34] Wiener in [46], pp. 447–49, however, uses an *additional* premise to prove that $\langle L, T\rangle$ is dense, namely, $\exists_- = E$ (in my terms, meaning 'every event has a first instant' as indicated in Section 6 above). Thus he does *not* consider (III) sufficient as I do. But Wiener uses $\exists_- = E$ only to prove that the postulated two-element antichain "in the gap" of a two-element chain extends to a maximal antichain. By contrast, the present paper assumes maximal-antichain extendibility of *any* two-element antichain ($\langle E, P\rangle$ *2-maximizing*) from the start, so the crucial step in deriving denseness of $\langle L, T\rangle$ from (III) as mentioned goes *without* invoking Wiener's (in fact, Russell's)[35] additional premise $\exists_- = E$. In [37], Russell presents a condition on $\langle E, P\rangle$ *necessary* for $\langle L, T\rangle$ being dense and a pair of conditions he claims to be *sufficient*. A simple counterexample presented by Anderson ([2], pp. 256–57) shows that this claim is wrong.[36] As a remedy, Anderson suggests

$$\text{for all } \langle a, b\rangle \in P, \text{ there is } c \in E \text{ such that } a\ P\ c\ P\ b$$

as an alternative to the conjunction of (III) and $\exists_- = E$ for entailing that $\langle L, T\rangle$ is dense—obviously not realizing that under his [2], p. 255, assumption of the axiom of choice (which entails that $\langle E, P\rangle$ is 2-maximizing; see Subsection 8.2 below) (III) *alone* is a condition entailing that $\langle L, T\rangle$ is dense and a (much) weaker one than his own proposal. Thus, to my knowledge, the statement of the previous corollary is new.[37]

To tackle Question 2, define for every $A \subseteq E$

$$U(A) := \{\, e \in E \mid a\ P\ e \text{ for all } a \in A \,\};$$
$$H(A) := \{\, e \in E \mid e\ P\ b \text{ for all } b \in U(A) \,\}.$$

$U(A)$ comprises all those events that are wholly preceded by *all* events of A. $H(A)$, by contrast, is some kind of "hull" of A collecting not only all those events that do not end later than some event from A (so, of course, A is a subset of $H(A)$), but collects even those events that *do* end later than all events from A, but only "nonuniformly."[38] $\langle H(A), U(A)\rangle$ is a pair of maximal sets of events such that all the events of the first set wholly precede all the events of the second set. Every pair with this property is $\langle H(A), U(A)\rangle$ for some $A \subseteq E$, and the property generalizes the notion of a Dedekind cut (in the sense of [34], Definition 2.22, for example).[39]

My answer to Question 2 then is the following.

Theorem 7.5 *$\langle L, T\rangle$ is complete if and only if every $A \subseteq E$ satisfies one of the following conditions:*

(i) *A or $U(A)$ is empty;*
(ii) *$H(A)$ has a PS-maximal element contained in $\exists_+$;*
(iii) *$U(A)$ has an SP-minimal element contained in $\exists_-$;*
(iv) *for every $a \in H(A)$ there is some c neither in $U(A)$ nor in $H(A)$ such that $a\ P\ c$, and for every $b \in U(A)$ there is some d neither in $U(A)$ nor in $H(A)$ such that $d\ P\ b$.*

(That $\langle E, P\rangle$ is 2-maximizing is used for 'only if' again.) To get an idea of what goes on in cases (ii) and (iii), the reader may look at Theorem 7.2. If one of them applies, the "boundary" of one of $U(A)$ and $H(A)$ "generates" a bound as desired for both of them, unless the events outside of $U(A)$ and $H(A)$ according to case (iv) form a maximal antichain that is a bound as desired.[40]

The previous theorems solve the problems formulated in Section 4 as explained in Section 5. This may still not be perfectly clear, so I give two examples in the ensuing corollary. Its first part deals with one class of instances of Question-Scheme 1*—the one Thomason seems to be interested in exclusively, namely, (up to isomorphism) just Question 1. The second part deals with Question 1^+ which appears interesting to me.

Corollary 7.6

(a) *$\langle L, T\rangle$ is isomorphic to $\langle I, <\rangle$ for some nonvoid open real interval I ($I = \mathbf{R}$, e.g.) if and only if E neither has an SP-minimal element contained in $\exists_-$ nor a PS-maximal element contained in $\exists_+$, there is a countable subset D of E according to Theorem 7.3, and every $A \subseteq E$ satisfies one of the cases of Theorem 7.5.*
(b) *$\langle L, T\rangle$ is isomorphic to $\langle I, <\rangle$ for some nontrivial real interval I if and only if there is a countable subset D of E according to Theorem 7.3, every $A \subseteq E$ satisfies one of the cases of Theorem 7.5, and (if E has an SP-minimal element in $\exists_-$ and a PS-maximal element in $\exists_+$)*[41] *$P \neq \varnothing$.*

(Both (a) and (b) use countable choice and, for 'only if', that $\langle E, P\rangle$ is 2-maximizing.)

8 Proofs

8.1 "Constructivity" as opposed to choice principles This subsection starts elaborating the remarks beginning Section 7 on what the paper particularly assumes. I explain what I claimed there on avoidability of choice and similar principles as well as the loose use of terms like 'constructive'.

The axiom of choice—AC for short—seems, in general, not to be questioned in mathematics nowadays and is considered part of standard (axiomatic) set theory.[42] On the other hand, when it found broad mathematical attention for the first time about a hundred years ago,[43] adversaries included Peano, Borel, Lebesgue, and Baire— see [23], p. 103, or, for more, [28]. Russell only temporarily was convinced of it ([2], p. 255) and considered special assumptions on events to guarantee existence of instants without its use (see Subsection 8.2). In the meantime, there have, at least, always been mathematicians who liked to point it out when a proof used AC or similar nonconstructive existence claims. Moreover, there is still some interest in finding proofs only using *constructive* existence claims, even with an intuitionistic intent—here I am alluding to *constructive mathematics* (see, e.g., [6] or [38]). Subsections 8.1 and 8.2 address such interests.

In axiomatic set theory, everybody knows what "set theory without the axiom of choice" is. I designate this part of set theory by ZF.[44] Subsections 8.1 and 8.2 tell something about which claims of the paper hold 'by means of ZF only'. The latter phrase will return to stress this.

Whenever the paper claims that something holds "constructively" or that a proof is "constructive," this just means that it is provable "by means of ZF only." This terminology is definitely not correct considering what may be understood by constructivity in mathematics.[45] I consider it justified, however, by the fact that purportedly "constructive" proofs avoid use of (ZF-)equivalents of AC ([35], p. xiv]) or of "weak versions" of it (other choice, maximality, or extension principles, some of which will be reviewed below) by presenting "definitions" the only free variables of which stand in place of things which exist by the hypothesis of the implication to be proved.[46] This is to indicate how a *formal*[47] proof could work.[48] Thus, when the hypothesis of a theorem asserts existence of some x such that . . . and claims existence of a corresponding y such that . . . , a "definition" of such a y "in terms of x"[49] suffices for "constructivity"—for holding "by means of ZF only," that is. There is no need to specify such an x (by any "construction" or "definition").[50]

According to the beginning of Section 7, certain claims of the paper hold by means of ZF only. Like everywhere in set theory, the corresponding *metamathematical* claims will not be checked in a strict manner. The informal proofs only implicitly indicate their correctness. The specifically metamathematical aspects of the claims of the paper are addressed in an intermediately explicit manner in Subsection 8.2, in parts of Subsection 8.6, and in the rest of the present subsection as follows.

Corollary 7.6 rests on (some of) Facts 1, 1*, and 1^+, and so do similar claims I alluded to. My corresponding metamathematical claims rely on the following.

Meta-Fact Facts 1, 1*, and 1^+ hold by means of ZF only.

For those "facts" (on the "object level"), I am referring to proofs to be found in the literature and to obvious variants of such proofs. For my corresponding metamathematical claim, one has to check that respective proofs can be carried out in ZF.

Proof of Meta-Fact In each case, an isomorphism as wanted arises as a straightforward (indeed, unique) extension of an isomorphism of countable dense subchains which have no ends, obviously involving nothing more than ZF. That the construction of such a "core" isomorphism as well can be carried out by means of ZF only is to my knowledge most explicit in [30], pp. 32–33, and in [5], p. 200.[51] □

8.2 Existence of Russellian instants of time This subsection reviews some metamathematical facts relating the basic assumption of the paper (according to the beginning of Section 7) that $\langle E, P\rangle$ is 2-*maximizing* to AC and some of its consequences.[52]

Russell, doubting AC, proposed to avoid it by $E = \exists_{-}$ in [36] (like [46], see Section 7 above) and by '$\langle E, P\rangle$ is 2-maximizing' in [37] (see note 36) in order to guarantee existence of enough instants.[53] These pains and possibilities of avoiding AC by first- or second-order constraints on structures of the special kind considered make "ZF only" more interesting to me than merely widespread curiosity about AC as mentioned in Subsection 8.1 above would have done.

In Subsection 8.6, I show that the characterizing condition of Theorem 7.3 is a similar condition on the particular structure under consideration which avoids AC in guaranteeing existence of enough instants. The present subsection deals with one further condition of this kind as well as with consequences of AC which, by contrast, generalize over *all* structures of the given kind.

Generalizing from $\langle E, P\rangle$, a related set $\langle X, R\rangle$ is an *interval order* whenever R is irreflexive and for all $\langle x, x'\rangle, \langle y, y'\rangle \in R$ either $x\ R\ y'$ or $y\ R\ x'$. The reader will need no further help in generalizing to what an *antichain* in $\langle X, R\rangle$ is. $\langle X, R\rangle$ is *maximizing* if any antichain in $\langle X, R\rangle$ extends to a maximal one (equally clear). It is then, of course, n-maximizing for our earlier cases $n = 0, 1, 2$. Let MAIO be the assumption that in every interval order there is a maximal antichain, and let MAIO$'$ be the assumption that *every* interval order is maximizing.

Proposition 8.1 *By means of* ZF *only,*

(a) MAIO$'$ *is equivalent to* MAIO*;*
(b) $\langle X, R\rangle$ *is maximizing provided X can be well-ordered.*

The second part restates what Russell knew ([37] and as reported by [2], pp. 254–55). Accordingly, AC (being equivalent to the well-ordering principle) implies MAIO and thus (by the first part) MAIO$'$. By contrast to the proceeding presented here, MAIO$'$ derives from Zorn's Lemma in an entirely elementary way (see [2], pp. 255–56, deriving another variant of MAIO and MAIO$'$). The reason for my detour is that for philosophical reasons one might assume that the interval order $\langle E, P\rangle$ has a *countable* underlying set E; then the following conclusion from the proposition might be of interest.

Corollary 8.2 *By means of* ZF *only, each interval order the underlying set of which is countable is maximizing.*

Remember the Henkin completeness proof for countable first-order languages goes without AC in contrast to the case of uncountable languages? An application will follow in Subsection 8.6.

Of course, Corollary 8.2 holds for every *ordinal* cardinal number (for every "aleph", see Subsection 8.6 below) in place of ω; but, at the moment, I cannot imagine other infinite cardinals than ω which *for some philosophical reasons might be considered bounding the cardinality of the underlying set of events* and, at the same time, is well-ordered by means of ZF only.[54]

Proof of Proposition 8.1 Let $\langle X, R\rangle$ be some interval order and $Y \subseteq X$ an antichain.

(a) Let $Y' := \{\, y \in X \mid \{y\} \cup Y \text{ is an antichain}\,\}$ (so $Y \subseteq Y'$). Then $\langle Y', R\rangle$ is an interval order which by assumption has a maximal antichain A. By maximality of A

and the definition of Y', Y must be a subset of A. This proves one direction of the claim; the other follows trivially from the fact that $\varnothing$ trivially is an antichain.

(b) If X can be well-ordered, there is a one-to-one map φ from some ordinal α onto $X \setminus Y$. Define $\langle Y'_\beta, Y_\beta\rangle$ for $\beta \in \alpha$ by $Y'_0 := Y$, $Y_\beta := Y'_\beta \cup \{\varphi(\beta)\}$ if this is an antichain and $Y_\beta := Y'_\beta$ otherwise, $Y'_{\beta+1} := Y_\beta$ for $\beta + 1 \in \alpha$, and $Y'_\beta := \bigcup_{\gamma\in\beta} Y_\gamma$ for limit ordinals $\beta \in \alpha$. Now Y is a subset of $\bigcup_{\beta\in\alpha} Y_\beta$, and the latter is easily seen to be a maximal antichain. □

Now I am going to consider two more ZF-consequences of AC which might seem relevant for $\langle E, P\rangle$ 2-*maximizing*, in order to indicate that those assumptions about 'maximizing' are rather weak compared to AC. Compare MAIO to Kurepa's principle '*in every poset [partially ordered set, i.e.] there is a maximal antichain*' which, assuming ZF, is equivalent to AC according to [12], pp. 61–62. Observe that an interval order $\langle X, R\rangle$ is rendered a poset when R is replaced by its union with the diagonal of $X \times X$. Since this would be a *special* kind of poset, I believe that Kurepa's principle—whence neither AC—cannot be derived from ZF+MAIO alone.

By [12] (pp. 128, 131) the assumption BPI that in every Boolean algebra there is a prime ideal (which is, as is well known, derivable from ZF+AC and equivalent to existence of a maximal ultrafilter) does *not*, together with ZF only, imply AC. As, assuming ZF, MAIO is equivalent to the assumption that there is some somehow *special* ultrafilter in every Boolean algebra of some *special* kind (I cannot go into more detail here), I believe that, assuming nothing but ZF, BPI and MAIO do not imply each other. This belief implies that MAIO cannot be derived from ZF alone.

By the previous content of the subsection I feel justified in assuming that $\langle E, P\rangle$ is 2-maximizing. That is, the reader is invited to choose his favorite from the justifications offered—maybe believing in AC (maybe the version most properly named so and vigorously advocated by Anderson and Gödel according to [2], p. 255), maybe believing MAIO, maybe assuming E (a set of 'events of which we are conscious', [36], p. 121) is countable and forgetting about all uncountable interval orders.

Some further notation introduced in Subsection 8.4 will indicate the role that assumptions on existence of instants play in the proofs.

Final remarks: Alternative constructions (i) Walker's ([42], (7)) construction of instants of time and their ordering from $\langle E, P\rangle$ yields a related set $\langle L', T'\rangle$ where L' is the set of all triples $\langle A, B, E \setminus (A \cup B)\rangle$ (rendered as $\langle A, E \setminus (A \cup B), B\rangle$ in Thomason's [40], Proposition 2) such that A, B are nonvoid subsets of E, $a\ P\ b$ for all $a \in A$ and all $b \in B$, and finally for all $c \in E \setminus (A \cup B)$ there are $a \in A$ and $b \in B$ such that $a\ S\ c\ S\ b$, and where T' is the set of those ordered pairs $\langle\langle A, B, C\rangle, \langle A', B', C'\rangle\rangle$ for which A is a *proper* subset of A'. By Thomason's [40], Proposition 3 (and because $\langle E, P, SP, PS\rangle$ is an event ordering according to [40], Definition 1), $\langle L', T'\rangle$ is a complete loset. Obviously, *this* 'construction of time instants from events' works *by means of* ZF *only*.[55]

(ii) Fishburn [15], pp. 23, 25, 29 (proof of Theorem 6), and other authors offer other "constructions of time instants from events" working by means of ZF only—without mentioning *time*, however.[56]

(iii) I have invented other constructions myself, but they have not been published yet.

(iv) Hamblin [16] seems to present yet another construction. At the end of his paper, however, he notes that his construction is just Walker's, as it is presented in (i) above—at least according to someone telling him so at the conference where Hamblin presented his paper. However again, that someone was wrong in this respect, and Hamblin ought not to have believed that person. Hamblin's construction yields only *some* of Walker's instants (up to isomorphism)—it does not, in general, yield a *complete* loset, as Walker's construction does (see Subsection 9.2).[57]

8.3 Derived relations, witnesses, and linearity of time Define

$$PSP := \{\, \langle a, b\rangle \mid a \;\; P \;\; c \;\; S \;\; d \;\; P \;\; b \text{ for some } c, d\,\}.$$

This is, of course, some double composite of relations in two equivalent and well-known ways. The situation visualized in Figure 2 is just a case of $a \;\; PSP \;\; b$, and PSP proves to be closely related with denseness. But it even is closely related with being an interval order (I am adding something more soon needed).

Lemma 8.3

(a) $PSP \subseteq P$. *(*W*)*

(b) *S is reflexive.*

(c) *P is transitive.*

(d) *SP and PS are irreflexive.*

Proof (a) If $a \;\; P \;\; b \;\; S \;\; c \;\; P \;\; d$, then $a \;\; P \;\; d$ follows from the definition of S and from (I) by propositional logic.

(b) This is an immediate consequence of the definition of S together with P being irreflexive.

(c) If $a \;\; P \;\; b$, $c \;\; P \;\; d$, and $b = c$, $a \;\; P \;\; b \;\; S \;\; c \;\; P \;\; d$ by reflexivity of S, then $a \;\; P \;\; d$ follows from (W).

(d) Both $e \;\; S \;\; c \;\; P \;\; e$ and $e \;\; P \;\; c \;\; S \;\; e$ would contradict the definition of S. □

In fact, irreflexivity of P and (W) (Wiener's [46] assumptions) could have been assumed *instead* of irreflexivity of P and (I).[58]

I am going to write '*$s \;\; T \;\; t$ with witnesses a, b*' if $s \;\; T \;\; t$, $a \in s$, $b \in t$, and $a \;\; P \;\; b$. By definition of T, $s \;\; T \;\; t$ is equivalent to there being witnesses according to this convention.

For better understanding some proofs below, it may be helpful to visualize witnesses of instants as horizontal strokes crossing vertical strokes representing these instants. For example, Figure 3 shows witnesses a, c of $s \;\; T \;\; t$ and witnesses, d, b of $t \;\; T \;\; u$. This situation forces c, d to overlap and a to 'wholly precede' b (apart from vertical strokes, it is the same situation as in Figure 2).

Now I am ready for a proof of what I claimed in Section 3.

Proposition 8.4 *$\langle L, T\rangle$ is a nonvoid loset.* (*0)

('(*0)' indicates, for later comparison, that '$\langle E, P\rangle$ is 0-*maximizing*' has been invoked—fitting notation introduced in Subsection 8.4.) That $\langle L, T\rangle$ is a loset is proved in Wiener's [46], pp. 445–46, but, alas, using *Principia*-notation. I am merely indicating what is going on to those who prefer to shun *Principia*-notation.

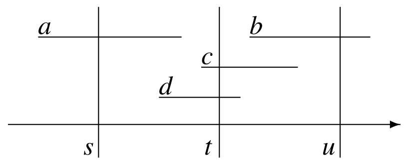

Figure 3 Witnesses for the order of instants.

Proof That L is nonvoid was decided in Subsection 8.2. $s\ T\ s$ is excluded by the definitions of T and of an antichain—this is irreflexivity. If $s, t \in L$ do not equal, (by maximality of antichains s, t) some element a of s does not overlap some element b of t. So $a\ P\ b$ or $b\ P\ a$, hence $s\ T\ t$ or $t\ T\ s$—this is connectedness.[59] Finally, let $s\ T\ t$ with witnesses a, c and $t\ T\ u$ with witnesses d, b; by $c, d \in t$ we have $a\ P\ c\ S\ d\ P\ b$, so (W) yields $a\ P\ b$; this is transitivity of T, and, in the upshot, T linearly orders L (my original definition of linearity was redundant).[60] □

Russell [37] wrongly sketched a proof of why T should be transitive assuming another set of assumptions on $\langle E, P\rangle$.[61]

8.4 Further notation and Theorem 7.1 In the rest of the paper, I will write 'iff' for 'if and only if', and |Z| will, as usual, denote the cardinality of the set Z.[62]

Concerning existence assumptions, I will, moreover, abbreviate '$\langle E, P\rangle$ is 2-*maximizing*' by '(*)'; as well, '(*)' will indicate that (*) is invoked. '(*1)' indicates that only '1-maximizing' instead of '2-maximizing' is used (cf. '(*0)' with Proposition 8.4). '(*⇐)' indicates that only the 'if', '(*⇒)' that only the 'only if' of an 'iff' is affected—analogues apply for (*1). '(−*)' will emphasize that a part of a statement has been invoked which does *not* involve any of these existence assumptions.

For $R \subseteq X \times Y$, let me define a number of "lifts." For any set Z, let $\mathcal{P}(Z)$ be its power set; then

$$R! := \{\, \langle x, Y'\rangle \in X \times \mathcal{P}(Y) \mid \{x\} \times Y' \subseteq R \,\};$$
$$R\text{¡} := \{\, \langle x, Y'\rangle \in X \times \mathcal{P}(Y) \mid (\{x\} \times Y') \cap R \neq \varnothing \,\}.$$
[63]

$!R$ and $\text{¡}R$ are analogously defined as subsets of $\mathcal{P}(X) \times Y$ such that $X'\ !R\ y$ means $X' \times \{y\} \subseteq R$ and $X'\ \text{¡}R\ y$ means $(X' \times \{y\}) \cap R \neq \varnothing$. By these definitions, $!(R\text{¡})$ indicates that existential quantification occurs in the scope of universal quantification, but where parentheses do not matter I omit them, as with $!R!$ or $R\text{¡¡}$. As an example, now $T = \text{¡}P\text{¡} \cap (L \times L)$.

Furthermore, I let $Ry := \{x \in X \mid x\ R\ y\}$ for $y \in Y$, and dually $xR := \{\, y \in Y \mid x\ R\ y \,\}$ for $x \in X$. Thus, for example,

$$R\text{¡}Y' = \{\, x \in X \mid (\{x\} \times Y') \cap R \neq \varnothing \,\}$$

if $Y' \subseteq Y$.

For any subset A of E, let

$$\Lambda(A) := \{\, t \in L \mid A \subseteq t \,\} \ \text{ and } \ \mathrm{V}(A) := \bigcup_{a \in A} \Lambda(\{a\}).$$

Thus $\Lambda(\varnothing) = L = \mathrm{V}(E)$, and $\Lambda(A) \neq \varnothing$ for any antichain A if and only if $\langle E, P\rangle$ is maximizing (Subsection 8.2). For $A \subseteq B \subseteq E$ we have $\Lambda(B) \subseteq \Lambda(A)$ and $\mathrm{V}(A) \subseteq \mathrm{V}(B)$. From the definition of V and the irreflexivity of T we have the following lemma.

Lemma 8.5 *Suppose $A \,!P!\, B$, $A' \subseteq A$, and $B' \subseteq B$. Then $\mathrm{V}(A') \,!T!\, \mathrm{V}(B')$ and $\mathrm{V}(A') \cap \mathrm{V}(B') = \varnothing$.*

I will omit braces in the arguments of Λ and V, so by (*), $\Lambda(c, d) \neq \varnothing$ whenever $c\ S\ d$, and by (*1), $\mathrm{V}(e) = \Lambda(e) \neq \varnothing$ for any event e.

At this point, the proof of Theorem 7.1 ($|L| > 1$ iff $P \neq \varnothing$) seems convenient.

Proof of Theorem 7.1 Assume $P = \varnothing$. Then E is an antichain and, as it is the greatest subset of E, a maximal one. Every other antichain properly extends to the antichain E and thus is *not* maximal. So $L = \{E\}$ and therefore $|L| = 1$.

Now assume there is $\langle a, b\rangle \in P$. $\mathrm{V}(a)(= \Lambda(a))$ and $\mathrm{V}(b)$ are both nonvoid (*1) and, by Lemma 8.5, disjunct. Hence, $a \neq b$ and $|L| \geq |\{a, b\}| > 1$. □

For an explanation of the notion of *duality*, cf. note 27.[64] For a formal language, of course, one could easily *prove* that any formal proof of some statement about $\langle E, P\rangle$ (using perhaps formalizations of (W) and the irreflexivity of P as axioms—note that they would be "self-dual") dualizes to a proof of the dualized statement. Without a formal language, in the sequel I will omit proofs of dual statements just because readers could easily assure themselves of them by *informally* dualizing the informal arguments presented explicitly (cf. [3], p. 13).

8.5 First and last instants; Theorem 7.2 This subsection is concerned with the notions of having a first or, respectively, a last instant, and proves Theorem 7.2. A *first instant of* a subset A of E is a T-least element of $\mathrm{V}(A)$; dually, a *last instant of* A is a T-greatest element of $\mathrm{V}(A)$. By omitting braces of singletons, it is then clear what a first or last instant of some event is. As T is a linear order (and hence asymmetric), first and last instants are unique, of course.

The statements concerning first instants I present will always be duals of the statements concerning last instants. Therefore, I each time only prove one of these two cases.

Lemma 8.6 *Let $A \subseteq E$ and $t \in L$.*

(a) *t is first (last) instant of A iff $t \in \mathrm{V}(A)$ and not $\mathrm{V}(A)$ ¡T t (not t T¡ $\mathrm{V}(A)$).*
(b) *$\mathrm{V}(A)$ ¡T t iff A ¡SP¡ t, and t T¡ $\mathrm{V}(A)$ iff t ¡PS¡ A.* (*⇐)

Proof (a) One direction follows from asymmetry—which follows from transitivity and irreflexivity, the other from connectedness of linear orders.

(b) Assume there is $s \in \mathrm{V}(A)$ such that $s\ T\ t$ with witnesses c, b. Pick a from $A \cap s$. Then $s \in \Lambda(a, c)$ and therefore $a\ S\ c\ P\ b$. Now assume $a\ SP\ b$ for some $a \in A$ and some $b \in t$. Then $a\ S\ c\ P\ b$ for some c. Choose s from $\Lambda(a, c)$ (*). Then $s\ T\ t$ by $c\ P\ b$, and $s \in \Lambda(a) \subseteq \mathrm{V}(A)$. □

In Section 6 I said that, as Russell found out,[65] $e \in \exists_-$ if and only if e has a first instant. One direction is proved in Wiener's [46], the other in Russell's [37]. For readers who find it hard to decipher the *Principia*-notation (and who prefer my notation), below I outline a proof in a different style. To this aim, for $e \in E$ I define $t_-(e) := e(S \setminus SP)$ and (dually) $t_+(e) := (S \setminus PS)e$.

Lemma 8.7 *Suppose $e \in E$.*

(a) *$t_-(e)$ and $t_+(e)$ are antichains having e as an element.*
(b) *$e \in \exists_-$ iff $t_-(e)$ (¡P)! eSP, and $e \in \exists_+$ iff PSe !(P¡) $t_+(e)$.*
(c) *$e \in \exists_-$ iff $t_-(e) \in L$, and $e \in \exists_+$ iff $t_+(e) \in L$.*

Proof (a) $e \in t_-(e) \cap t_+(e)$ follows from reflexivity of S and irreflexivity of SP and PS. Were $a, b \in t_-(e)$ and $a\ P\ b$, then $e\ S\ a\ P\ b$ and hence $e\ SP\ b$, contradicting $b \in t_-(e)$.

(b) These are merely restatements of my original definitions of $\exists_-$ and $\exists_+$ using the notation introduced in the meantime.

(c) Assume $e \in \exists_-$. By (a) the goal is to demonstrate that $t_-(e)$ is *maximal* as an antichain. For reductio, assume there is an $a \in (S!\ t_-(e)) \setminus t_-(e)$. So $e\ S\ a$ because of $a\ S!\ t_-(e)$; so from $a \notin t_-(e)$ follows $e\ SP\ a$. By (b) the latter yields $t_-(e)$ ¡$P\ a$, and this contradicts the assumption $a\ S!\ t_-(e)$. Now assume $t_-(e) \in L$ and $e\ SP\ a$. By (b) the goal is to demonstrate that $t_-(e)$ ¡$P\ a$. For reductio, assume $a\ (P \cup S)!\ t_-(e)$. But $a\ P\ b \in t_-(e)$ would imply $e\ SP\ a\ P\ b$, and, by transitivity of P, $e\ SP\ b$, contradicting $b \in t_-(e)$. So $a\ S!\ t_-(e)$, contradicting maximality of $t_-(e)$ as an antichain, since by $e\ SP\ a$ the event a is no element of $t_-(e)$. □

Proposition 8.8 *Suppose $e \in E$ and $t \in L$.*

(a) *t is first instant of e iff $t = t_-(e) \in L$, and t is last instant of e iff $t = t_+(e) \in L$.* (*1 $\Rightarrow$)
(b) *$e \in \exists_-$ iff e has a first instant, and $e \in \exists_+$ iff e has a last instant.* (*1 $\Leftarrow$)

Proof If t is first instant of e, then, by Lemma 8.6 (*1),[66] $t \subseteq t_-(e)$. As t is a maximal antichain and $t_-(e)$ is at least an antichain by Lemma 8.7(a), even $t = t_-(e)$. Now $t_-(e) \in L$, and by Lemma 8.7(c) $e \in \exists_-$. This was one direction of both parts of the proposition, but the remaining directions should be treated separately.

(a) Assume $t_-(e) \in L$. By Lemma 8.7(a) then even $t_-(e) \in \Lambda(e)$. Let $t' \neq t$ be another element of $\Lambda(e)$. Thus, (as a maximal antichain cannot be a subset of a different one) there is $b \in t' \setminus t_-(e)$. Then $b\ S\ e$ and $e\ SP\ b$, and from the latter by Lemma 8.7(b) follows $t_-(e)$ ¡$P\ b$. Hence $t_-(e)\ T\ t'$, and, in the upshot, $t_-(e)$ is first instant of e.

(b) Assume $e \in \exists_-$. Then by Lemma 8.7(c), $t_-(e) \in L$, and by the previous, $t_-(e)$ is first instant of e. □

As the characterizing notions of Theorem 7.2 will go on to play an important role, I introduce symbols for them:

$$\exists^*_- := \{\, A \subseteq E \mid \text{ an } SP\text{-minimal element of } A \text{ is in } \exists_- \,\};$$
$$\exists^*_+ := \{\, A \subseteq E \mid \text{ a } PS\text{-maximal element of } A \text{ is in } \exists_+ \,\}.$$

Proposition 8.9 *Suppose $A \subseteq E$ and $t \in L$.*

(a) *t is first instant of A iff $t = t_-(e) \in L$ for some SP-minimal element e of A, and t is last instant of A iff $t = t_+(e) \in L$ for some PS-maximal element e of A.* (*⇒)

(b) *$A \in \exists^*_-$ iff A has a first instant, and $A \in \exists^*_+$ iff A has a last instant.* (*⇐)

Proof (I am proceeding similarly as for Proposition 8.8.) If t is first instant of A, then by Lemma 7.3(*) there must be an SP-minimal element a of A such that $a \in t$. As t cannot be a first instant of A without being a first instant of a, by Proposition 8.8, t must be $t_-(a) \in L$. Hence by Lemma 8.7(c), $a \in \exists_-$ and, finally, $A \in \exists^*_-$. For the remaining, proceed as follows.

(a) Assume $t_-(e) \in L$ for some SP-minimal element e of A. By Lemma 8.7(a) then $t_-(e) \in \mathrm{V}(A)$. Let $t' \neq t$ be another element of $\mathrm{V}(A)$. Thus, there is a $b \in (t' \cap A) \setminus t_-(e)$. $b\ P\ e$ by reflexivity of S and SP-minimality of e would contradict $b \in A$. Thus $e(S \cup P)b$, hence $e\ SP\ b$ and finally $t_-(e)\ T\ t'$ as in the proof of Proposition 8.8.

(b) Assume $A \in \exists^*_-$. So there is an SP-minimal element e of A in $\exists_-$. Then by Lemma 8.7(c), $t_-(e) \in L$, and by the previous, $t_-(e)$ is first instant of A.

□

Proof of Theorem 7.2 Theorem 7.2 is that instance of Proposition 8.9(b) (*⇐) (with 'iff' reversed!) where $A = E$. □

8.6 Dense subsets; Theorem 7.3, Corollary 7.4 For any subset D of E write

$$P_D := \{\, \langle a, b\rangle \mid a\ P\ c\ S\ d\ P\ b \text{ for some } c, d \in D \,\}.$$

(Think of "P as witnessed by D.")

Call D *quasi-dense* if, for all $\langle a, b\rangle \in P$ at the same time,

(i) $a\ P_D\ b$ if $a \in \exists_+$ and $b \in \exists_-$;
(ii) $PSa\ !P_D\ b$ if $a \notin \exists_+$;
(iii) (dually to the previous condition) $a\ P_D!\ bSP$ if $b \notin \exists_-$.[67]

Existence of a *countable* quasi-dense subset of E will in this subsection turn out to be one of the characterizing conditions. Being able to formulate this condition, I can now state what I earlier tried to announce about existence of instants.

Theorem 8.10 *By means of* ZF *only, if a countable subset of E is quasi-dense, $\langle E, P\rangle$ is maximizing.*

I need some lemma for this.

Lemma 8.11 *Let D be a quasi-dense subset of E, and let $\{a, b\}\ !S!\ B \subseteq E$.*

(a) *If $a\ P\ c\ S\ c'\ P\ b$, then $\{c, c'\}\ !S!\ B$.*
(b) *If $a \in \exists_+$ and $b \in \exists_-$, then there is $c \in D$ such that $a\ P\ c\ S!\ B$.*
(c) *If $a \notin \exists_+$, there is $c \in D$ such that $B\ !S\ c\ P\ b$.*

Proof (a) Additionally to the hypotheses, assume $d \in B$, so $a\ S\ d\ S\ b$. $c\ P\ d$ would by $c\ P\ d\ S\ a\ P\ c$ and by (W) contradict irreflexivity of P. $d\ P\ c$, by $c'\ P\ b\ S\ d\ P\ c$ and (W), would contradict $c'\ S\ c$. Therefore, $c\ S\ d$ and, in general, $c\ S!\ B$. A dual reasoning yields $c'\ S!\ B$.

(b) In the situation hypothesized there are $c, c' \in D$ such that $a\ P\ c\ S\ c'\ P\ b$, so the claim follows from (a).

(c) If $a \neq \exists_+$, assume $d \in B$, so $a\ S\ d\ S\ b$ and *not* $d\ PS\ a$, since otherwise $d\ P\ b$ by (W). By definition of $\exists_+$, there is $a' \in PSa$ such that *not* $a'\ P\ d$. Since *not* $d\ PS\ a$ and P is transitive (Lemma 8.3), neither $d\ P\ a'$. Therefore, $a'\ S\ d\ S\ b$. Moreover, there are $c, c' \in D$ such that $a'\ P\ c'\ S\ c\ P\ b$, so the claim follows from (a). □

Proof of Theorem 8.10 Assume $D \subseteq E$ is countable and quasi-dense and A is some antichain. Let $A' := \{\, e \in E \mid e\ S!\ A \,\}$. Since A is an antichain, $A \subseteq A'$. If A' is an antichain, it is a maximal one and we are ready. If not, there are $a, b \in A'$ such that $a\ P\ b$. By Lemma 8.11 and some supplement of dual reasoning, then $D' := \{\, d \in D \mid d\ S!\ A \,\}$ turns out to be nonvoid. $\langle D', P\rangle$ is an interval order, where D' is countable, and has, by Corollary 8.2 and by means of ZF only, a maximal antichain D_0. Let $A_0 := \{\, e \in E \mid e\ S!\ (A \cup D_0) \,\}$. Since $D_0 \subseteq D'$, we have $A\ !S!\ D_0$, and definition of A_0 yields $A \subseteq A_0$. I have to show that A_0 is an antichain; definition of A_0 will then make clear that A_0 is a *maximal* antichain extending A.

To show that A_0 is an antichain, assume $a, b \in A_0$ and, for reductio, $a\ P\ b$. By Lemma 8.11 and some dual supplement, there is then some $c \in D$ such that $a\ P\ c$ or $c\ P\ b$ and $c\ S!\ (A \cup D_0)$. Therefore, $c \in D'$ and, since D_0 is a maximal antichain of $\langle D', P\rangle$, even $c \in D_0$. So $a\ S\ c\ S\ b$ by definition of A_0, contradicting $a\ P\ c$ or $c\ P\ b$. □

I am now inserting some auxiliary facts that will be needed in the following.

Lemma 8.12 *Suppose $a, b \in E$.*

(a) $\Lambda(a)\ !T!\ \Lambda(c, d)\ !T!\ \Lambda(b)$ *whenever* $a\ P\ c\ S\ d\ P\ b$.

(b) *Suppose, additionally,* $t \in L$. *If* $\mathrm{V}(a)\ !T\ t$, *then* $a\ P¡\ t$ *and* $a\ (PS)!\ t$; *if* $t\ T!\ \mathrm{V}(b)$, *then* $t\ ¡P\ b$ *and* $t\ !(SP)\ b$.[68]

Proof (a) follows immediately from the definitions of Λ and T.

(b) Assume $\mathrm{V}(a)\ !T\ t$. Then $c\ P\ a$ for *no* $c \in t$ by asymmetry of T. $a\ S!\ t$ would imply $a \in t$, contradicting the assumption and irreflexivity of T. So there is some $c \in t \cap aP$ (i.e., $a\ P\ c \in t$), and then $a\ P\ c\ S!\ t$ which implies $a\ (PS)!\ t$. This proves the first statement; the other one follows dually. □

Theorem 7.3 says that $\langle L, T\rangle$ is separable if and only if E has a countable quasi-dense subset. More generally I can prove the following.

Proposition 8.13 *Let κ be an infinite cardinal.*[69] *L has a subset of cardinality at most κ being dense in $\langle L, T\rangle$ iff E has a quasi-dense subset of cardinality at most κ.*[70] (*)

Proof Assume D is quasi-dense and $|D| \leq \kappa$. For every $\langle c, d\rangle \in S \cap (D \times D)$ choose $t_{c,d}$ from $\Lambda(c, d)$ (*).[71] Let $M := \{\, t_{c,d} \mid \langle c, d\rangle \in S \cap (D \times D) \,\}$. By $\kappa^2 = \kappa$ and κ-choice (see below present proof), this is a set such that $|M| \leq \kappa$; it remains to be shown that M is dense in $\langle L, T\rangle$. So suppose $s\ T\ t$. (i) If there are witnesses $a \in \exists_+$ and $b \in \exists_-$, by case (i) of the definition of quasi-denseness there are $c, d \in D$ such that $a\ P\ c\ S\ d\ P\ b$. By Lemma 8.12(a), this yields $s\ T\ t_{c,d}\ T\ t$. (ii) If there are only witnesses a, b of $s\ T\ t$ such that $a \notin \exists_+$, alas, choose any one such pair $\langle a, b\rangle$. By Proposition 8.8, s is no last instant of a, and by Lemma 8.6, there is some $e \in s \cap PSa$. By case (ii) of the definition of quasi-denseness there are $c, d \in D$ such that $e\ P\ c\ S\ d\ P\ b$, and this yields $s\ T\ t_{c,d}\ T\ t$ as before. (iii) The proof of the direction at stake is completed by dual treatment of the previous case dualized.

Now assume M is dense in $\langle L, T\rangle$ and $|M| \le \kappa$. For each $\langle s, t\rangle \in T \cap (M \times M)$ choose witnesses $d_{s,t}, c_{s,t}$. Let $D := \bigcup_{\langle s,t\rangle \in T\cap(M\times M)}\{c_{s,t}, d_{s,t}\}$. By $\kappa^2 = \kappa$ and κ-choice, this is a set such that $|D| \le \kappa$, and it remains to be shown that D is quasi-dense. So suppose $a\ P\ b$. (i) If $a \in \exists_+$ and $b \in \exists_-$, by Lemmas 8.5 and 8.7, $t_+(a)\ T\ t_-(b)$, and by iterated application of denseness of M one can choose $s, t, u \in M$ such that $t_+(a)\ T\ s\ T\ t\ T\ u\ T\ t_-(b)$. It remains to be shown that $a\ P\ c_{s,t}\ S\ d_{t,u}\ P\ b$. By their choice, $c_{s,t}$ and $d_{t,u}$ are elements of t, so $c_{s,t}\ S\ d_{t,u}$. Furthermore, $a\ PS\ d_{s,t}$ by Lemma 8.12(b) and because of $\mathrm{V}(a)\ !T\ s$ (definition of last instant). By choices $d_{s,t}\ P\ c_{s,t}$, so $a\ P\ c_{s,t}$ by (W). $d_{t,u}\ P\ b$ is proved dually. (ii) If $a \notin \exists_+$, definition of $\exists_+$ yields e, c such that $e\ P\ c\ S\ a$. Pick s from $\Lambda(a, c)$ (*). This s is an instant of a, but no last one, so there is $u \in \Lambda(a) \cap sT$. Denseness yields a t such that $s\ T\ t\ T\ u$. That $c_{s,t}\ S\ d_{t,u}$ obtains as before. $e\ P\ c\ S\ d_{s,t}\ P\ c_{s,t}$ (by choices and by (W)) yields $e\ P\ c_{s,t}$, and $d_{t,u}\ P\ c_{t,u}\ S\ a\ P\ b$ yields $d_{t,u}\ P\ b$. Thus, $e\ P\ c_{s,t}\ S\ d_{t,u}\ P\ b$. (iii) works dually to the previous case. □

Having discussed nonconstructive existence assumptions guaranteeing the existence of maximal antichains, I should note that, given κ, the proof uses some weaker version of the axiom of choice, namely, that there is a choice function on sets of cardinality κ (“κ-choice”)—at least on such sets mentioned in the proof. Moreover, both ways round of the proof use $\kappa^2 = \kappa$. This is entirely constructive if κ is a *well-ordered* (*ordinal*) cardinal, an “*aleph*”—see [24], p. 90, 2.32–34(i); pp. 96–97, 3.20–23; p. 98, 3.29(viii). If your knowledge on cardinals derives from [22], p. 27, you do not know of any other cardinals, and the only problem is κ-choice. Questioning AC, however, κ may be something different, as defined, for example, in [24], p. 83. In this case, the statement that $\kappa^2 = \kappa$ for all infinite cardinals κ is equivalent to the full axiom of choice—see [24], p. 164, 1.14.

Now the proposition is somewhat vague about κ, and what it needs depends on how it is understood. Fortunately, Theorem 7.3 is a special case of the proposition looking much brighter.

Proof of Theorem 7.3 Theorem 7.3 is that special case of Proposition 8.13 where $\kappa = \omega$ (*⇒). □

By [24] (p. 90, 2.32, 2.33(ii); pp. 96–97, 3.20–23), $\omega^2 = \omega$ by means of ZF only. So the only “parts” of AC the theorem needs are the principle of *countable* choice and (*). While the latter assumption follows, by Theorem 8.10, from the hypothesis that a countable quasi-dense subset exists, it is even needed for the second part of Proposition 8.13. Considering the facts reported in [24] (pp. 167–69, 2.1, 2.3, 2.7) and [12] (pp. 100–101, 155, 159–60), I do not expect that the principle of countable choice and any maximizingness assumption (from (*0) up to MAIO) imply each other. Therefore, although the previous proposition made heavy use of AC, it still makes sense to track how small a part of it is needed for the central characterization results of the present paper.

Now I turn to Corollary 7.4.

Lemma 8.14 *If $\langle L, T\rangle$ is dense, then*

$$P \cap (\exists_+ \times \exists_-) \subseteq PSP. \tag{D}$$

Proof Assume $\langle L, T\rangle$ is dense. Consider $\langle a, b\rangle \in P \cap (\exists_+ \times \exists_-)$. So by Lemma 8.7 and Proposition 8.8(b) (−*), $t_+(a)$ is last instant of a and $t_-(b)$ is first instant of b. From $a\ P\ b$ follows $t_+(a)\ T\ t_-(b)$. So denseness of $\langle L, T\rangle$ requires a $t \in L$ such that $t_+(a)\ T\ t\ T\ t_-(b)$. "Lastness" and "firstness" imply $\mathrm{V}(a)\ !T\ t\ T!\ \mathrm{V}(b)$, so by Lemma 8.12(b) there are $c, d \in t$ such that $a\ P\ c\ S\ d\ P\ b$. □

Corollary 8.15 *Suppose E is infinite. Then L has a subset of at most the cardinality of E dense in $\langle L, T\rangle$ if and only if* (D) *holds* (*⇐). *So if E is countable, $\langle L, T\rangle$ is separable if and only if* (D) (−*).

Proof Assume L has a subset of at most the cardinality of E dense in $\langle L, T\rangle$. Then, trivially, $\langle L, T\rangle$ is dense, and (D) holds by Lemma 8.14.

If (D) holds, E can be shown to be a quasi-dense subset of E, so the other way statement follows from Proposition 8.13(*) (however, for *countable* E, remember note 71 for avoiding(*)). Indeed, (D) is simply condition (i) of the definition of quasi-denseness in the case of $D = E$. If $a \notin \exists_+$, consider $e\ PS\ a$. There is an e' such that $e\ P\ e'\ S\ a$, but because of $a \notin \exists_+$ we have $e'\ PS\ a$ again. (W) and $a\ P\ b$ together yield $e\ P\ e'\ S\ e'\ P\ b$; so condition (ii) of the definition of quasi-denseness holds, and condition (iii) obtains dually. □

Corollary 7.4 says that $\langle L, T\rangle$ is dense if and only if (D) holds.

Proof of Corollary 7.4 If $\langle L, T\rangle$ is dense, (D) again holds simply by Lemma 8.14. Assuming (D) for the other way round, the proof of Corollary 8.15 shows that E is quasi-dense, and the proof of Proposition 8.13 shows how to construct a subset of L dense in $\langle L, T\rangle$— forget about its cardinality—from a quasi-dense subset of E (*) (but cf. note 71 for avoiding (*)). Thus one sees that $\langle L, T\rangle$ is dense. (One could as well have directly applied Corollary 8.15 having reasoned that (D)— if $P \neq \varnothing$—or events outside of $\exists_- \cap \exists_+$ force E to be infinite, while $\langle L, T\rangle$ trivially is dense in case of $P = \varnothing$ by Theorem 7.1.) □

8.7 Completeness, Theorem 7.5 Condition (iv) of Theorem 7.5 needs some elucidation; to abridge it I therefore define

$$\begin{aligned}\exists^* := \{\, \langle A, B\rangle \mid A, B \subseteq E,&\\ &A\ !(P\text{¡})\ (E \setminus (A \cup B)),\ \text{and}\\ &\qquad (E \setminus (A \cup B))\ (\text{¡}P)!\ B\,\}.\end{aligned}$$

Theorem 7.5 can now be restated as follows.

Theorem 8.16 *$\langle L, T\rangle$ is complete if and only if every $D \subseteq E$ satisfies one of the following conditions: (i) D or $U(D)$ is empty; (ii) $H(D) \in \exists^*_+$; (iii) $U(D) \in \exists^*_-$; (iv) $H(D)\ \exists^*\ U(D)$.*

Its proof seems to require quite a number of steps.

Lemma 8.17 *If d is an SP-minimal element of $D \subseteq E$, then $P!D = Pd$; if d is a PS-maximal element of D, then $D!P = dP$.*

Proof Assume d is an SP-minimal element of $D \subseteq E$. $P!D \subseteq Pd$ simply follows from $d \in D$. As for the other way round, assume for reductio $a\ P\ d$ and *not* $a\ P\ e$ for some $e \in D$. Then $e\ P\ a$ or $e\ S\ a$. In the first case, transitivity of P (Lemma 8.3) yields $e\ P\ d$, and reflexivity of S, furthermore, yields $e\ S\ e\ P\ d$, contradicting SP-minimality of d. In the second case, $e\ S\ a\ P\ d$ directly contradicts SP-minimality of d. The rest is duality. $\square$

Recall that, by the notation introduced in this section, $U(D) = D!P$ and $H(D) = P!U(D)$ for every $D \subseteq E$. It may be helpful to compare the following lemma with Propositions 8.8 and 8.9.

Lemma 8.18 *Suppose $D \subseteq E$. Let $A := H(D), B := U(D)$, and finally $C := E \setminus (A \cup B)$. Then*

(a) *$P¡A \subseteq A$, and $B¡P \subseteq B$;*
(b) *C is an antichain;*
(c) *$A \exists^* B$ iff $C \in L$;*
(d) *for all $t \in L$: $\mathrm{V}(A)\ !T\ t\ T!\ \mathrm{V}(B)$ iff $t = C \in L$;*
(e) *if $A \exists^* B$, then $A \notin \exists^*_+$ and $B \notin \exists^*_-$.*

Proof (a) follows easily from transitivity of P and from the definitions of A, B.

(b) For reductio, assume there are $c, d \in C$ such that $c\ P\ d$. Since $d \notin B$, there is $a \in D$ such that *not* $a\ P\ d$, that is, $d\ P\ a$ or $d\ S\ a$. In the first case, $d\ P!\ B$ by transitivity of P, so $d \in A$ contradicting $d \in C$. In the second case, for every $b \in B$ one gets $c\ P\ d\ S\ a\ P\ b$, so $c\ P\ b$ by (W), and, in the upshot, $c\ P!\ B$. Therefore $c \in A$ contradicting $c \in C$.

(c) By (b), I just have to show that $A \exists^* B$ is equivalent to C being *maximal* as an antichain. But being *not* maximal for C is the same as $e\ S!\ C$ for some $e \in E \setminus C$, which directly contradicts $A \exists^* B$. For the other way round, if *not* $A \exists^* B$, then *not* $a\ P\ c$ for some $a \in A$ and every $c \in C$, *or not* $c\ P\ b$ for some $b \in B$ and every $c \in C$. Now, each of $c\ P\ a$ and $b\ P\ c$ by (a) would exclude $c \in C$, so, in fact, $a\ S!\ C$ or $C\ !S\ b$, and $C \cup \{a\}$ or $C \cup \{b\}$ is an antichain properly extending C.

(d) If $t \in L$ and $\mathrm{V}(A)\ !T\ t\ T!\ \mathrm{V}(B)$, then by Lemma 8.12(b) and irreflexivity of PS and SP we have $t \subseteq C$. Since t is a maximal antichain, $t = C$ follows from (b). If, for the other way round, $C \in L$, then by (c) $A \exists^* B$, and from this $\mathrm{V}(A)\ !T\ C\ T!\ \mathrm{V}(B)$ follows easily.

(e) Assume $A \exists^* B$. I am going to show a little more than claimed, namely, that neither A can have a PS-maximal element nor B can have an SP-minimal element. For reductio, assume B has an SP-minimal element b. Allowed to by $A \exists^* B$, pick c from $C \cap Pb$. By Lemma 8.17 now $c\ P!\ B$, so $c \in A$ contradicting $c \in C$. Assuming that A has a PS-maximal element nearly dually leads to a contradiction—only recognize additionally $D \subseteq A$.[72] $\square$

Lemma 8.19 *Let M be a nonvoid subset of L having no T-greatest element and satisfying $N := M\ !T \neq \emptyset$. Let $D := P¡¡M$, and let then A, B, C be as in Lemma 8.18. Then*

(a) *$M \subseteq \mathrm{V}(D) \subseteq \mathrm{V}(A)$;*
(b) *if t is an element of N, but no T-least one, then $t \in \mathrm{V}(B)$;*
(c) *$\mathrm{V}(B) \subseteq N$;*
(d) *$N = \mathrm{V}(B)$, or $N = \{C\} \cup \mathrm{V}(B)$, or $N = \{t\} \cup \mathrm{V}(B)$ for some $t \in \mathrm{V}(A)$;*
(e) *if $A \in \exists^*_+$, then $N \cap \mathrm{V}(A) \neq \emptyset$.*

Proof (a) Fix $t \in M$. t is no T-greatest element of M, so choose $t' \in M$ such that $t\ T\ t'$ with witnesses a, a'. So $a\ P\ a' \in t' \in M$, hence $a\ P$¡ $t' \in M$ and $a\ P$¡¡ M, that is, $a \in D$, and from $a \in t$ follows $t \in \mathrm{V}(D)$. The inclusion $\mathrm{V}(D) \subseteq \mathrm{V}(A)$ is simply a consequence of $D \subseteq A$.

(b) Fix $t', t \in N$ such that $t'\ T\ t$ with witnesses b', b, and be $a \in D$. So choose s from M having an element c such that $a\ P\ c$. From $t' \in N$ follows $s\ T\ t'$ with witnesses d, d'. Thus $a\ P\ c\ S\ d\ P\ d'\ S\ b'\ P\ b$, and from this $a\ P\ b$ by double application of (W). In the upshot, $D\ P!\ b$, so $b \in B$ and $t \in \mathrm{V}(B)$.

(c) By (a) and Lemma 8.5, we have$M\ !T!\ \mathrm{V}(B)$, so $\mathrm{V}(B) \subseteq N$.

(d) Assume $\mathrm{V}(B) \neq N \neq \{C\} \cup \mathrm{V}(B)$. By (c), (b), and $N \neq \mathrm{V}(B)$, there is a T-least element t of N outside of $\mathrm{V}(B)$ such that $N = \{t\} \cup \mathrm{V}(B)$. By T-leastness and being outside of $\mathrm{V}(B)$, $t\ T!\ \mathrm{V}(B)$. By $N \neq \{C\} \cup \mathrm{V}(B)$ we have $t \neq C$, hence by Lemma 8.18(d) *not* $\mathrm{V}(A)\ !T\ t$. Thus, $t \in \mathrm{V}(A)$ (and we are ready) *or* $t\ T$¡ $\mathrm{V}(A)$. In the latter case, however, by Lemma 8.6(b)($-$*) there are $d \in t$ and $a \in A$ such that $d\ \ PS\ \ a$. By the definition of A, for every $b \in B$ we have $a\ \ P\ \ b$, therefore $d\ PSP\ b$ and, by (W), $d\ \ P\ \ b$. Thus $d \in A$ and $t \in \Lambda(a) \subseteq \mathrm{V}(A)$ as in the first case. (In fact, the latter case is impossible since (T¡ $\mathrm{V}(A)$) !(T¡) M—but I do not need this.)

(e) If $A \in \exists^*_+$, by Proposition 8.9($-$*) $\mathrm{V}(A)$ has a T-greatest element t. I have to show $t \in N$. For reductio, assume $t \notin N$. Then $t \in M \cup (T$¡ $M) = T$¡ M, so there is $t' \in M$ such that $t\ T\ t'$. Since t is last instant of A, $\mathrm{V}(A)\ !T\ t'$, and by (a) $M\ !T\ t'$, contradicting irreflexivity of T. □

Proof of Theorem 7.5 Let M, N, D, A, B, C be as in Lemma 8.19. Since M, having no T-greatest element, contains no upper bound of itself (with respect to T), N is the set of upper bounds of M. Thus for completeness of $\langle L, T\rangle$ I have to show that N has a T-least element if D satisfies one of the conditions (i) through (iv) of the theorem.[73] (i) By Lemma 8.19(a), since M is nonvoid, D *cannot* be empty. If, instead, $B = \varnothing$, then $\mathrm{V}(B) = \varnothing$ as well, and by Lemma 8.19(b) N has only one—T-least—element. (ii) If $A \in \exists^*_+$, by Lemma 8.18(e) and (c) $C \notin L$, in particular, $C \notin N$. On the other hand, $N \cap \mathrm{V}(A) \neq \varnothing$ by Lemma 8.19(e). By Lemma 8.5, $\mathrm{V}(A) \cap \mathrm{V}(B) = \varnothing$, so $N \neq \mathrm{V}(B)$. Therefore, and as $C \notin N$, by Lemma 8.19(d), $N = \{t\} \cup \mathrm{V}(B)$ for some $t \in \mathrm{V}(A)$. By Lemma 8.5 this t is T-least element of N. (iii) If $B \in \exists^*_-$, like before $C \notin N$; furthermore, by Proposition 8.9($-$*), $\mathrm{V}(B)$ has a T-least element. So by Lemma 8.19(d), $N = \{t\} \cup \mathrm{V}(B)$ for some $t \in \mathrm{V}(A)$—in which case $t \in \mathrm{V}(A)$ is a T-least element of N as before—or $N = \mathrm{V}(B)$—in which case the mentioned T-least element of $\mathrm{V}(B)$ is T-least in N. (iv) If $A\ \exists^*\ B$, by Lemma 8.18(c) and (d) together with Lemma 8.19(a), $C \in N$. By Lemma 8.18(d) together with irreflexivity of T, moreover, $C \notin \mathrm{V}(A) \cup \mathrm{V}(B)$. So by Lemma 8.19(d) $N = \{C\} \cup \mathrm{V}(B)$, and again by Lemma 8.18(d) C is T-least element of N.

Now assume $\langle L, T\rangle$ is complete, and let D be some nonvoid subset of E such that $B := U(D) \neq \varnothing$, $B \notin \exists^*_-$, and $A := H(D) \notin \exists^*_+$. I have to show that $A\ \exists^*\ B$. Because of $\varnothing \neq D \subseteq A$ we have $\mathrm{V}(A) \neq \varnothing$, and from $B \neq \varnothing$ follows $\mathrm{V}(B) \neq \varnothing$. By Lemma 8.5 we have $\mathrm{V}(A)\ !T!\ \mathrm{V}(B)$, so $\mathrm{V}(A)$ has some upper bound with respect to T. By completeness of $\langle L, T\rangle$, it must have a *least* upper bound t. According to Proposition 8.9(*), $\mathrm{V}(A)$ has no T-greatest and $\mathrm{V}(B)$ has no T-least element. So $\mathrm{V}(A)\ !T\ t\ T!\ \mathrm{V}(B)$. Now $A\ \exists^*\ B$ follows from Lemma 8.18(d) and (c). □

Corollary 7.6 only puts Theorems 7.1 through 7.5 as well as Facts 1 and 1^+ together and thus is clear. (For avoiding (*) in the 'if' direction, remember facts like Theorem 8.10 from Subsection 8.6. Need of countable choice is due to Theorem 7.3.)

9 Philosophical and Historical Remarks

9.1 Russell on continuity of time and on conscious events required In [36], Lecture V, Russell acknowledged that time should be "continuous" in some sense. Though, he did not investigate continuity in the sense of being isomorphic to the real numbers or some real interval. Instead, he only discusses *denseness* of time—calling it 'compactness'[74]—and maintains this were the only "philosophically mattering" among all the mathematical implications of continuity ([36], Lecture V). (Considering *Principia Mathematica* [44], $*275$, he must have been fully aware of what continuumlikeness is in the *mathematical* sense.) Accordingly, [36], Lecture IV, and [37] only present conditions on the ordering of events concerning *denseness* of time and nothing else coming as close to continuity as this. Russell completely ignores the question of *separability*—as the discussion of change and motion usually does Neither, Dedekind-continuity is at stake.

I have no idea why Russell's distinction of which ingredients of continuumlikeness matter philosophically and which do not should be right. Like Thomason, I have dealt with the remaining ingredients, but I cannot discuss here in which respect they really matter.

Recall that the set of *rational* numbers, for example, linearly ordered by $<$, forms, like that of the *real* numbers, a *dense* loset which, in contrast to the real numbers, however, is *not* complete—so Russell seems to hold that a *countable* set of instants of time would "philosophically" do, in particular, to explain motion to someone frightened by Zeno's paradoxes.

Now Russell asked in [36], Lecture V, "Is there, in actual empirical fact, any sufficient reason to believe the world of sense continuous?" Since he was aware of no better characterization of event structures giving rise to density, he tried to decide the case of private time by recurring to (III).

Having replaced Russell's Condition (III), which is sufficient but not necessary for density, by my *characterization* of density in Corollary 7.4, one might ask whether the situation has improved for Russell by that technical result.

Russell ([36], Lecture V) concluded from his (III)[75] that the number of events conscious to one being should be infinite in any finite period of time. Unfortunately, Russell seems never to have made explicit what a finite period of time could be.

Unfortunately enough, given any instants s, t such that $s < t$, (infinitely iterated application of) density requires that infinitely many events start after s and before t.[76] But this follows directly from the definitions of density and of Russell's instants; no characterization result is involved.

Russell goes on by saying "If this is to be the case in the world of one man's sense-data, and if each sense-datum is to have not less than a certain finite temporal extension, it will be necessary to assume that we always have an infinite number of sense-data simultaneous with any given sense-datum."

Here, the difficulty is to understand 'not less than a certain finite temporal extension'. However, if all events simultaneous with one event e are simultaneous with each other, they comprise a single instant, the only one at which e is. In this case, e could be no sense-datum as required by Russell, since it would last for one instant only and thus have *no* temporal extension at all. Therefore, for a given sense-datum e there must be a preceding b, both simultaneous with e, and infinitely frequent application of (III) will verify Russell's prophecy of an infinity of events simultaneous with e.

Now, I must admit that the characterization by Corollary 7.4 does *not* yield any alleviation for Russell. Where infinitely many contemporaries of e do not come in in the way of (III), they come in as members of infinite chains of contemporaries of contemporaries of e which have no last or first instant ([37] explains the latter phenomenon).[77]

One might rescue continuous private time by "possible" (private) events, which one *could have arranged* just to yield "possible" density. Kamp ([19], p. 376) seems to think of the same solution.[78]

Russell, however, felt "apparently forced to conclude that the space of sense-data is not continuous, but that does not prevent us from admitting that sense-data have parts which are not sense-data and that the space of these parts may be continuous. The logical analysis we have been considering provides the apparatus for dealing with the various hypotheses, and the empirical decision between them is a problem for the psychologist." I wonder about the psychologist, but I leave that aside. I conjecture that Russell's last words on parts refer to Whitehead's treatment of space as outlined by Russell's [36], Lecture IV, before. Concerning *time*, Russell might as well have conceded that there are enough *events* giving rise to continuity without being *sense-data*.

Of course, density raises the same problem within the *Walker* framework according to Thomason's [40]. Indeed, along these lines denseness of instants is equivalent to the condition that, *if* $a \ S \ b$, *then* $c \ P \ d$ *for some events* c, d *both of which are simultaneous with both* a *and* b ([40], Proposition 5). Applying this rule to $e \ S \ e$ yields infinitely many contemporaries of e.

In the remainder of [36] Lecture V, however, Russell seems to convince himself that no serious philosophical problem is involved by the possibility that continuity could not appear already "in the world of one man's sense-data." Rather, mathematical and physical time in the received sense may be found by "*logical*" analysis (maybe using results of the present paper), and this suffices for solving any problem lingering in the literature.

9.2 Relative merits of Russell's vs. Walker's construction As mentioned in the beginning, Thomason ([40], p. 95) concludes, "Thus Walker's theory offers, as Russell's does not, a plausible explanation of time as a continuum." I object: Additionally to nonvoidness of E, irreflexivity of P, and condition (I), *continuumlikeness* of Walker's construction as well as of Russell's construction imposes further restrictions on $\langle E, P\rangle$. A 'plausible explanation of time as a continuum' requires an explanation of why these restrictions should obtain.[79] For both constructions this is, I think, not easy. (By contrast, Thomason ([40], p. 95) considers the empirical conditions that would yield Walker instants as needed 'highly plausible'—see below after (i) and (ii).) However, concerning that part of a 'plausible explanation of time as a continuum' only which concerns conditions on $\langle E, P\rangle$ necessary and sufficient for continuumlikeness of the constructed time, Walker's construction does *not* fare better than Russell's. This I hope to demonstrate by the ensuing two mathematical examples. They show that both Walker's and Russell's construction may fail or succeed in bringing about continuumlikeness, especially denseness—depending on what kind of event structure we actually face.

(i) $\langle E, P\rangle$ might (up to isomorphism) consist of those open real intervals having rational boundaries, endowed with total precedence in the natural way. This is a case where Thomason's characterizing conditions are satisfied and (according to [40], Theorem 7) the construction $\langle L', T'\rangle$ due to Walker/Thomason is isomorphic to $\langle \mathbf{R}, <\rangle$—so we have continuumlikeness. Concerning *Russell*'s construction $\langle L, T\rangle$, however, every rational number "doubles"—so $\langle L, T\rangle$ is *not dense* and, consequently, not isomorphic to $\langle \mathbf{R}, <\rangle$ ([4], Theorem I.4.2.14 and [8], p. 246).

(ii) Let $\langle E, P\rangle$ (up to isomorphism) consist of all *compact* real intervals endowed with total precedence in the natural way. Then Thomason's [40], Definition 3, denseness condition (as stated near the end of Subsection 9.1) is not satisfied, every real number "doubles," and $\langle L', T'\rangle$ is very far from being dense ([40], Proposition 5). Russell's $\langle L, T\rangle$, on the other hand, is isomorphic to $\langle \mathbf{R}, <\rangle$, that is, continuumlike.

Admittedly, it may be considered an advantage for Walker's construction that it yields completeness ("Dedekind-continuity") "from the start," that is, without any restriction on $\langle E, P\rangle$ (as opposed to Theorem 7.5). (Indeed, the countability conditions given by Thomason and in the present paper hold with about equal probabilities in our world, so Theorem 7.5 may render time being a continuum less plausible from Russell's view than from Thomason's. Still, however, Russell's and Walker's/Thomason's constructions "fare as well" inasmuch as we now know assumptions ensuring that time is a continuum for both ways of constructing instants of time.)

Russell, being skeptical about the axiom of choice and about existence of instants along his own lines (recall from Subsection 8.2), might have been pleased about Walker's construction which needs ZF only. He might, however, have criticized that Walker's construction in general yields an instant "at" which an event occurs ([36], Lecture IV) only under another special condition —see [40], Theorem 8.

While the competition is undecided so far, a decision seems to come from category theory, as Thomason [41] maintains. However, I do not agree with all of that and hope to publish alternative views. Recall from the end of Subsection 8.2 that there are further rivals.[80]

Notes

1. For hints on Walker's legacy in physics, see note 14.

2. Indeed, the conditions he presents are *necessary* as well. Kleinknecht [21] does something very similar, but presents a pair of conditions which is only *sufficient*, not necessary. (I additionally comment on [21] in an unpublished version of the present paper.)

3. A real interval as defined in Section 2, that is.

4. This extends Thomason's and Walker's scope to something of which admittedly physicists will hardly acknowledge any use—see Section 4.

5. Subsection 9.2 discusses this statement.

6. As defined in Corollary 7.4 below.

7. I have had some troubles in distinguishing Russell's from Wiener's credits. According to the footnote of [46], p. 441, Wiener investigated the matter on Russell's suggestion. Indeed, at that time Wiener was a student under Russell at Cambridge University ([26], pp. 45–56; [13]). Thus, while Wiener explicitly attributes the definition of "instants" under consideration here and another notion to Russell, it is no surprise when further credits are difficult to track. Moreover, the first edition of Russell's [36] appeared in the same year as Wiener's [46].

8. Requiring $R \subseteq X \times X$ for a "related set" $\langle X, R\rangle$ is common practice; see, for example, [15], pp. 2–3, yet it does not really matter what R contains besides elements of $X \times X$ (cf. [22], p. 14). Sometimes I will use 'binary' and 'on' just to prevent fear of tricks. The difference that $R \subseteq X \times X$ makes is exemplified by the question whether $\{\, \langle \{\varnothing\}, R\rangle \mid R \text{ some set} \,\}$ (i.e., the class of those $\langle X, R\rangle$ where $X = \{\varnothing\}$) has exactly two elements or is a proper class.

9. For distinguishing Russell's from Whitehead's credits concerning constructing "points" and "instants," see [2], pp. 252–53.

10. Wiener's paper is summarized in modern notation by [13].

11. This is, in only a slightly different situation, held by [2], p. 252; also cf. [25], p. 181.

12. See [25], p. 181. For further interpretations or applications of interval orders, see [15], Section 2.1, pp. 19–21. In *philosophically justifying* instants of time, however, it is difficult to say what it *is* that b and c "have in common" in the second case without committing a *petitio principii*. I am afraid the only convincing motivations for (I) refer to how interval structures are represented by "point structures" like the present $\langle L, T\rangle$. Search the internet for "Robertson Walker". Arthur Geoffrey Walker was born July 17, 1909 and died March 31, 2001.

13. The end of Subsection 8.2 here defines $\langle L', T'\rangle$ exactly.

14. This, however, stems from the special role time is given in the mathematical rendering of how the universe seems to behave, and talking of "time" perhaps only makes sense when spacetime is a "product" of time and space in some special sense (I do not expect spacetime singularities of black holes that are formed eventually or that dissolve by a reasoning of Hawking [17] would deprive the picture of *one* time line for all "point" events of spacetime of use). Of course, in some "metaphysical" view "Time" could have existed earlier than the Big Bang and could since then somehow "interplay" with "mathematical-physical time." Such a "metaphysical everexisting Time" would be of no empirical value, only, maybe, of some aesthetical one—if someone's taste behaves appropriately. What is most important here: such a "metaphysical Time outside of physical time" could not be reconstructed from "empirical" events! "Empirical" events occur after the Big Bang and before a final collapse and will therefore only detect the instants of—bounded—*physical* time. Indeed, Walker's [42] approach (presented by [40]) accompanied his presentation [43] of a mathematical model of the whole development of the universe according to General Relativity. The spacetimes according to this mathematical model are nowadays called 'Robertson-*Walker* spacetimes' (cf. [31], pp. 341–51, for example).

15. See note 14.

16. According to [40], p. 94 (where denseness is redundant, namely, following from existence of a dense subset); see, for example, [34], pp. 33–37, where instead of Exercise 2.29 it suffices to think of the set of the rational numbers as a countable set dense in $\langle \mathbf{R}, < \rangle$.

17. By my understanding here, 'countable' does *not* imply 'infinite'. So some questions of cardinality can be discussed separately in the sequel.

18. See [34], Definition 2.28, for the topological reason for choosing the term "separable".

19. Thus "nonvoid" is not needed.

20. My definition slightly differed from the usual topological one, and $I \neq \varnothing$ is implied here.

21. According to [34], p. 40, the nontrivial cases may be proved essentially the same way as Fact 1. Alternatively, one could *derive* them from Fact 1 by observing what is preserved when one removes or adds least or greatest elements. The enumeration of order types in [34], p. 40, is imprecise, if not incomplete.

22. See [40], p. 88.

23. If such an a exists, it is an element of E by the definition of a related set or, sufficing as well, by the definition of S.

24. Of course, SP and PS are just the *compositions* of the binary relations P and S and vice versa, respectively, in the sense of [15], p. 3.

25. Note: $a \ S \ b$ if and only if $\{a, b\}$ is an antichain, and a subset A of E is an antichain if and only if $a \ S \ b$ for all $a, b \in A$.

26. I am merely reporting some notation and terminology introduced by Russell in [37]. $\langle E, P, SP, PS \rangle$ is now an 'event ordering' in the sense of Thomason [40], Definition 1.

27. The notion of 'duality' can be made rather precise. I could have made it *quite* precise if I had introduced a formal language. Without, just think of the *dual* of a "statement" (or a "condition") as the result of interchanging the event symbols on both sides of P, if P is the *only* relation symbol (besides $=$) occurring and if conjunctions are "written out" instead of using "chain notation." *Using* "chain notation" for conjunctions, consider a one-term conjunction a "conjunction chain" as well; then the dual is the result of reversing all "conjunction chains"—still if no symbols for derived notions occur. Otherwise, S remains unchanged, while SP is replaced by PS and vice versa, and 'SP-minimal' is replaced by 'PS-maximal' and vice versa. Finally the symbol $\exists_-$ introduced soon has to be replaced by $\exists_+$ and vice versa. (Cf. [3], p. 13.)

28. SP-minimal and PS-maximal elements may exist because, by Lemma 8.3, SP and (dually) PS are irreflexive.

29. Russell once claimed the result was his own, but his references indicate the possibility of his only having *conjectured* it, while it was *Wiener* who found the *proof*. For distinguishing Russell's from Wiener's credits in general, see note 7.

30. What 'constructive' and 'nonconstructive' mean here and in the sequel is rendered more precise in Subsection 8.1. For a first approximation, "constructivity" avoids the axiom of choice and any weak variant of it.

31. See [24], p. 167, 2.1. However, this principle even has been accepted in (parts of) constructive mathematics [38].

32. Choose an SP-maximal and a PS-minimal element from an antichain to see that $\langle E, P\rangle$ is n-maximizing for every nonnegative integer n as soon as it is 2-maximizing.

33. Both called—according to the terminology of *Principia Mathematica* [44], $*270$—'compact' what nowadays is called 'dense' (or 'dense-in-itself'). According to *this* meaning, compactness of time would imply that there is a very first and a very last instant.

34. To see that (III) is not a necessary condition for $\langle L, T\rangle$ being dense, let for every pair z_1, z_2 of integers $I(z_1, z_2)$ be the real interval
$$\{\, r \in \mathbf{R} \mid z_1 2^{z_2} - 2^{z_2 - 1} < r \leq z_1 2^{z_2} \,\}.$$
Then let E be the set of all these intervals and let P be the relation on E naturally deriving from $<$. Now, for example, $I(1, 0), I(1, 1)$ are P-neighbors, but $\langle L, T\rangle$ turns out to be isomorphic to $\langle \mathbf{R}, <\rangle$ (map each real number r to the set of "events" containing r) and so is dense.

35. Wiener ([46], p. 447) attributes the condition to Russell.

36. More formally/precisely than in Anderson [2], $\langle E, P\rangle$ is a counterexample to Russell's [37] claim when E is the set of integers and
$$T = \{\, \langle a, b\rangle \in E \times E \mid a + 1 < b \,\}$$
(so $S = \{\, \langle a, b\rangle \in E \times E \mid a - b \in \{-1, 0, 1\} \,\}$).

37. [13], p. 170, says that Theorem 3.4 of [15] "provides access to conditions that are necessary and sufficient, but the matter is rather technical and we shall not pursue it here." I estimate Corollary 7.4 and its proof much simpler than that theorem; therefore, it seems unlikely that [13] could have meant the condition of my corollary.

38. That is, $b \in H(A)$ if $a\ P\ c\ S\ b$ for all $a \in A$ and some c "depending" on a, without there being a "single" c such that $a\ P\ c\ S\ b$ for all $a \in A$. It is also easy to see that the operator H is idempotent and monotone.

39. By [34], Lemma 2.23, for losets completeness is equivalent to Dedekind completeness, where the latter means that one of the members of every Dedekind cut contains a boundary of itself. Actually, these pairs together with the complement of the union of their members form just the triples that are instants of time in the sense of *Walker*'s construction according to [40] and [42].

40. The formulation was inspired by the definition of a "complex open gap" in [15], p. 42.

41. Otherwise $P \neq \varnothing$ holds anyway.

42. Accordingly, [2], pp. 254–56, and [40], p. 87, use it for a "modern" treatment of instants along Russell's lines.

43. However, there was a "prehistory" of a quarter of a century—see [28], Chapter 1.

44. I prefer to consider ZF a purely formal theory, that is, a certain set of formulas of some formal language of first-order predicate logic with identity, defined by a finite list of axioms and axiom-schemes as in [11], pp. 3–13 or [22], pp. xv–xvi. In the present paper, however, I cannot distinguish this formal theory from the body of informal theorems and proofs which could be formalized in "proper" ZF according to common practice (cf. [22], pp. 1–2).

45. Constructive mathematics typically rejects the principle of tertium non datur which is not questioned in classical mathematics—see [38]. I will not check whether any proof invokes this principle. The paper cannot address the question what 'constructive' "really" means. To my knowledge, no precise answer is agreed upon by a majority of authors; see [6] and [38].

46. In spite of this "justification," I will from now on prefer 'by means of ZF only' to 'constructive'.

47. See note 44.

48. My proofs of this kind may be acceptable for one or the other "constructive" mathematician—I hope so, but I have not checked this.

49. In the corresponding formal way, there typically is a defining class term containing no free variable but x; see [24], p. 171.

50. Schuster (the author of [38] and in this respect an expert) thinks that this feature of "relative constructivity" of some proofs may be accepted by many, though not by all, exponents of constructive mathematics.

51. Other expositions might appear—at first glance—to use the principle of dependent choices ([3], pp. 176–77, e.g.; cf. [24], pp. 168–69, [38]). Like in Subsection 8.2, however, countability allows to well-order the codomain and so to "define" a map—cf. Russell's illustration by 'pairs of shoes vs. socks' ([18], p. 351). This is obvious for *all* the proofs I have seen, but the "recommended" proofs in [5] and in [30] are the only ones which make the definition explicit. The latter, however, might be supplemented by the hint that f^{-1} there is a *second* map which is constructed simultaneously with f and only in the end turns out to be the inverse map of the latter.

52. Consequences by means of ZF only, that is, as explained in Subsection 8.1.

53. In [37], Russell discusses $E = \exists_{-}$ again, as well $E = \exists_{+}$, which serves the same purpose, namely, giving an instant to each event to be "at"; moreover, $S \setminus SPS \neq \varnothing$ (SPS being defined analogously to PSP), which guarantees that there is any *one* instant at all (it is the negation of Thomason's ([40], Definition 3) condition corresponding to denseness of the *Walker* construction). Since he neither believes in AC nor in any of

his own conditions, he remains skeptical about existence of instants. He might have been happy to know that "instants" may be *constructed* in ways *other* than as maximal antichains—see end of Subsection 8.2. $E = \exists_-$, as well as $E = \exists_+$, even implies that $\langle E, P\rangle$ is 2-maximizing; see [46], p. 449. It is, however, too restrictive: in the example in note 34, for example, *no* event has a first instant. For an easier example take $\langle \mathbf{R} \cup \{\mathbf{R}\}, <\rangle$, where the event $\mathbf{R}$ has all the reals as instants, but none is a first one. $S \setminus SPS \neq \varnothing$ is too restrictive as is seen from all open real intervals (maybe of minimum length 1), for example. By contrast, Russell's $\langle E, P\rangle$ 2-maximizing is exactly the part of AC which is needed here for existence of instants.

54. For example, there is no well-ordering of the powerset of ω (equivalently, no well-ordering of the real numbers) by means of ZF only ([10], p. 138).

55. I am grateful to an anonymous referee for advising me to have a look at Walker's original [42], instead of just relying on Thomason's [40], Definition 2—who just relied on [45]. Like Thomason ([40], p. 85), I found it difficult to obtain a copy of [42]. I am very grateful to Leon Horsten for at last sending me one from Belgium. It turns out that Walker's [42] construction has been reported correctly by [40] and [45], at least essentially. One might object: (i) Thomason's [40] "event structures" bear three relations, while here and in Walker's article structures with just one relation are considered. Yet Thomason's additional relations are merely defined from the first one, which constitutes an interval order in the sense of (I). (ii) Walker [42] employs (W) of Subsection 8.3 instead of (I). Yet the same subsection explains that this does not make a difference. (iii) Unlike Thomason, Walker does not demand explicitly that one component of the instant triple shares no event with the other two, yet this follows immediately. The following observations may be more notable: (iv) Walker claims to prove that his instants form a complete linear order. Yet by 'complete' ('fermé') he understands that every strictly monotone bounded *sequence* ("indexed" by natural numbers—at least his use of dots hardly admits another interpretation) has a "limit." Thus Walker would consider $\omega_1 + \omega^*$, where ω_1 is the well-ordered set of countable ordinals and ω^* the inversed-ordered set of finite ordinals, bounded (assuming AC ([22], 10.21), so any strictly increasing sequence has bounds in the ω_1 part, so there is a least bound), although there is no least among the upper bounds of the *subset* (corresponding to) ω_1. (v) Moreover, Walker's proof of his completeness claim contains the same error that Thomason [40], p. 91, discovered in Whitrow's [45] rendering of Walker's [42]. Walker and Whitrow erroneously believed that each union of left-hand components of instant triples—bounded by another left-hand component of an instant triple—is a left-hand component of another instant triple.

56. Instead of time, Fishburn and other authors consider *utility* (theoretical economics) or other psychological dimensions (psychophysics, mathematical psychology). Often *weak* orders (not trichotomic) are constructed instead of linear orders, since this suffices for the representability theorems usually considered. Yet a weak order yields a linear order in the natural way (equivalence classes are turned into single points). Fishburn ([15], p. 41–45), "constructs" Russell's construction from his own. However, 'constructively' on [15], p. 45, must not be understood as if Fishburn had found a way to avoid AC in Russell's construction—some part of AC *is* needed for the existance of 'complex open gaps'. In particular, Fishburn does *not* show that Russellian instants exist at all in any interval order.

57. Hamblin starts from a very special kind of interval orders, almost like Burgess [7] (having a very "rich ontology"). Already this may raise doubts that Hamblin's construction is Walker's indeed. Yet there is a more important reason to doubt the claimed *identity*

of these constructions. Rather, they (or their results) are *isomorphic*. While a Walkerian instant is some *triple*, a Hamblinian instant seems to be a certain *equivalence class of ordered pairs* of "abutting" (cf. [41]) events. (So instants are "meeting places" of intervals, as [1], p. 10, put it.) However again, this only is a guess of mine in trying to understand Hamblin. Instead of introducing an equivalence relation among such pairs and forming the corresponding equivalence classes, he "defines identity" among such pairs—even nonidentical paris are *declared* "identical." I am suggesting to interpret his "identity" symbol as a symbol for a certain *equivalence* relation. The correspondence between Hamblin's and Walker's construction then might be this: When $\langle a, b\rangle$ represents an instant in Hamblin's sense, then $\langle H(A), U(A), E \setminus (H(A) \cup U(A))\rangle$, where $A = \{a\}$ is the corresponding instant in Walker's sense. And when $\langle A, B, C\rangle$ is a Walker instant, Hamblin's axioms seem to imply that there are $a \in A$ and $b \in B$ such that $\langle a, b\rangle$ represents a Hamblin instant—at least, about three persons apparently once believed this. For a counterexample, let $\langle E', P'\rangle$ be the natural interval order of open real intervals, and let $\langle E, P\rangle$ be the result of removing all the intervals from E' that have 0 as a (lower or upper) boundary. $\langle E, P\rangle$ satisfies Hamblin's axioms concerning the ordering of events. Yet the subset of E containing all the intervals with upper boundaries less than 0 is the left-hand component of a (unique, [40], Proposition 2) Walker instant $\langle A, B, C\rangle$ where A and B do *not* have elements a and b as thought above (a abutting b from the left). I owe these little discoveries to an anonymous referee who advised me to consider Hamblin's paper.

58. To see this, first derive reflexivity of S from irreflexivity of P, second, transitivity of P from (W) and reflexivity of S. Then, from $a\ P\ b, c\ P\ d$ but *not* $c\ P\ b$, similarly to the informal reasoning right after my introduction of (I) follows $b\ P\ c$ or $b\ S\ c$. $a\ P\ d$ follows by transitivity of P in the first case and directly by (W) in the second case. Thus, the universal closure of (I) follows from (W) and irreflexivity of P. Russell [37] starts with another equivalent set of conditions, namely, irreflexivity of P and transitivity of P and of SP. Unfortunately, he did not fully recognize the significance of the transitivity of SP—see note 61.

59. To see this, first derive reflexivity of S from irreflexivity of P, second, transitivity of P from (W) and reflexivity of S. Then, from $a\ P\ b, c\ P\ d$ but *not* $c\ P\ b$, similarly to the informal reasoning right after my introduction of (I) follows $b\ P\ c$ or $b\ S\ c$. $a\ P\ d$ follows by transitivity of P in the first case and directly by (W) in the second case. Thus, the universal closure of (I) follows from (W) and irreflexivity of P.

60. To see this, first derive reflexivity of S from irreflexivity of P, second, transitivity of P from (W) and reflexivity of S. Then, from $a\ P\ b, c\ P\ d$ but *not* $c\ P\ b$, similarly to the informal reasoning right after my introduction of (I) follows $b\ P\ c$ or $b\ S\ c$. $a\ P\ d$ follows by transitivity of P in the first case and directly by (W) in the second case. Thus, the universal closure of (I) follows from (W) and irreflexivity of P.

61. This was observed by Thomason ([40], p. 86) and overlooked by Anderson ([2], p. 263, note 6). Russell claimed transitivity of T derived from transitivity of P. The following counterexample refutes this claim. Let

$$E := \{ a_1, a_2, a_3, b_1, b_2, b_3 \} \text{ and}$$
$$P := \{ \langle a_1, a_2\rangle, \langle a_1, a_3\rangle, \langle a_2, a_3\rangle, \langle b_1, b_2\rangle, \langle b_1, b_3\rangle, \langle b_2, b_3\rangle \}.$$

P obviously is irreflexive and transitive. Now, among the elements of L there will be $t := \{a_1, b_2\}$ and $t' := \{a_2, b_1\}$, and we have $t\ T\ t'\ T\ t$ (witnesses a_1, a_2 and b_1, b_2), but not $t\ T\ t$; so T is not transitive. Russell should instead have pointed at (W) and derived it using essentially his assumption of transitivity of SP (cf. note 58). This would go as follows. Let $a\ \ PSP\ \ b$. Assume for reductio *not* $a\ \ P\ \ b$. Then $b\ \ P\ \ a$ or $a\ \ S\ \ b$. (I am now going to write compositions of relations as [15], p. 1, does.) In the first case, $a\ \ S\ \ a\ \ PSP\ \ b\ \ S\ \ b\ \ P\ \ a$ by reflexivity of S (which derives as for Lemma 8.3), so $a\ \ (SP)(SP)(SP)\ \ a$, and then $a\ \ SP\ \ a$ by transitivity of SP, contradicting irreflexivity of SP. (Note that irreflexivity of SP as proved for Lemma 8.3 does not depend on any assumption on P but is a consequence just of the definition of S from P.) In the second case, $b\ \ S\ \ a\ \ PSP\ \ b$, and this contradicts irreflexivity of SP similarly to the first case. *Equivalence* of the axiom set used in [37] (note 58) to each of the two two-element axiom sets suggested above now easily obtains, as transitivity of SP is a simple consequence of (W).

62. Questioning AC raises some problems with cardinalities; then the definition of [24], p. 83, 2.2, is meant to apply.

63. Note that neither $R!$ nor $R¡$ depends on X or Y.

64. Changing sides in dualizing now, of course, also applies to "lift marks" as defined at the beginning of Subsection 8.4: Ry becomes yR, and xR becomes Rx. By contrast, Λ and V are "self-dual."

65. But see note 29 for a reconsideration of the question of credits.

66. Using Lemma 8.6 for the proof so far, the proposition seems to rest on (*). However, a singleton version of Lemma 8.6 with $A = \{e\}$ may be inserted after Lemma 8.7 which only needs (*1) and suffices for the present proposition. When $e\ \ SP\ \ b \in t$, one then distinguishes whether $e \in \exists_-$ or not. In the second case, there is some c such that $e\ SP\ \ c\ \ PS\ e$ and $c\ \ P\ \ b$ (cf. [37]) which just needs (*1) to be a member of some $s \in Tt$.

67. By [15], p. 5, Theorem 2, weakly, as well as linearly, ordered sets are a special case of interval orders. In this special case, denseness and quasi-denseness of subsets coincide, because then *all* elements just have one—first and last—instant, so only the first condition from the definition of quasi-denseness is relevant, and this one reduces to the usual condition defining denseness of a subset.

68. Parentheses with PS and SP assure that it is the relation composites that are "lifted."

69. The case of finiteness is somewhat trivial: a finite subset can be neither dense nor quasi-dense.

70. With regard to my discussion of what can be obtained "by means of ZF only" and in view of [24], V.1.8, (comparability of cardinalities), 'at most' is too vague and should be understood to mean that there is a one-to-one map from such a subset of L or of E, respectively, into some set cardinality of which is κ.

71. Theorem 8.10 naturally generalizes to quasi-dense $D \subseteq E$ when κ is an *aleph* (see after the proof of Proposition 8.13) cardinal; so in this case (*) is entailed by the hypothesis.

72. I could have made better use of duality by showing $B = A\ !P$, but then making sufficiently clear how duality would work here might be too diffcult and expensive.

73. The case of upper bounds suffices by [34], Exercise 2.20; cf. the dual Theorem 9 in [20], p. 14.

74. Recall note 33.

75. (III) is just (f) as in the long footnote of [36], Lecture IV. At the passage (of Lecture V) to which I am referring here, Russell seems just to confuse it, intending to argue from (III) and not from the curious condition actually printed there. Moreover, Russell's discussion of conditions "required" for something sometimes gives the impression that he had confused sufficiency and necessity of the condition for something (density, e.g.). At the presently discussed passage, however, there is no actual need for such an unfavorable interpretation. To make clearer what has already been said: Russell searches for a *reason* to believe in continuity, and he does not know anything else than (III) which could yield such a reason. Therefore, he discusses whether what (III) requires may be accepted.

76. The latter proposition can be made precise thus: there are infinitely many events each of which has an instant between s and t.

77. For example, let $e\ S\ a\ P\ b\ S\ e$. If $a \notin \exists_+$, then $a\ P\ c\ S\ d\ P\ b$ for some c, d is not required for density by Corollary 7.4, while it would have been required by (III), and intrusion of an infinity of contemporaries of e seems to be blocked. $a \notin \exists_+$, however, implies that the events overlapping a and e are not simultaneous with each other, since otherwise they would comprise a last instant of a. Rather, there are contempories c_0 and d of a and e such that $c_0\ P\ d$. Now $a \notin \exists_+$ furthermore forces a "never ending" situation $c_0\ P\ c_1\ P\ c_2\ P \ldots c_n \ldots$ where the c_n are contemporaries of a and of e.

78. Kamp even indicates Russell could sometimes have pondered this solution but does not provide a reference.

79. Cf. the program according to Kamp [19], p. 381.

80. Extended (and differently arranged) versions of this article have been maintained as No. 72 of the preprint series "Forschungsberichte der DFG-Forschergruppe *Logik in der Philosophie*" at the University of Konstanz, Germany. You may find it on a World Wide Web search at some personal site of mine, or I will send you a PDF or DVI version by e-mail on your request.

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