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Universality for Orders and Graphs Which Omit Large Substructures

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Abstract This paper will examine universality spectra for relational theories which cannot be described in first-order logic. We will give a method using functors to show that two types of structures have the same universality spectrum. A combination of methods will be used to show universality results for certain ordered structures and graphs. In some cases, a universal spectrum under GCH will be obtained. Since the theories are not first-order, the classic model theory result under GCH does not hold.

1 Introduction

This paper gives universality results for many different kinds of structures. These structures are all relational structures which omit substructures of a certain size. Because of these omitted substructures, the structures cannot be defined from a first-order theory. Thus, the classic general universality results from model theory do not apply in these cases. This paper examines a number of methods for determining universality for these types of structures. Universality results for related first-order structures (such as linear orders) will also be mentioned for contrast.

To clarify our use of universality, we will present the basic definitions. In general, the embeddings are injective "structure-preserving" functions where structure-preserving can be interpreted in different ways. In this paper, we make a distinction between weak embeddings, which preserve some structure, and strong embeddings, which preserve all structure. Embeddings for ordered sets are normally injective order-preserving maps and strong embeddings will also preserve incomparability. For graphs (directed and undirected), an embedding is an injective function which preserves edges (and directions in directed graphs) and strong embeddings also preserve "nonedges" or the property that two nodes do not have an edge between them.

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Given a set of structures $\mathcal{A}_\lambda$ each of size λ, a *(strong) universal model* for $\mathcal{A}_\lambda$ is one which (strongly) embeds all other structures in $\mathcal{A}_\lambda$. If there does not exist a universal model for $\mathcal{A}_\lambda$, then we consider its *complexity*, or the smallest size of a family of structures of $\mathcal{A}_\lambda$ which embeds the rest. This family of structures is called a *universal family*. The *universal spectrum* for a class of structures $\mathcal{A}$ is the family of cardinals for which $\mathcal{A}$ has a universal model (given a universe of set theory and cardinal arithmetic assumptions). All of these notions have weak and strong counterparts depending on the type of embedding used.

Another question one could ask when there does not exist a universal model is what is the smallest size of a family of structures which do not embed into a single element? This can be considered as the dual notion to complexity, which we call the *simplicity number*. This notion was first considered in the context of cardinal invariant; see, for instance, Mekler and Väänänen [6]. For consistency, we may set the simplicity number of a family which has a universal element to ∞.

Throughout the paper, let κ and λ be infinite cardinals. For convenience, we will use the following abbreviations for the structures considered in this paper. We assume that these sets of structures are representatives under isomorphism.

1. $\mathcal{C}(\lambda, \kappa)$ = posets of size λ which omit chains of size κ.
2. $\mathrm{wfpo}(\lambda, \kappa)$ = well-founded posets of size λ which omit chains of size κ.
3. $\mathrm{tree}(\lambda, \kappa)$ = trees of size λ which omit branches of height κ.
4. $\mathrm{OG}_p(\lambda, \kappa)$ = oriented graphs of size λ which omit paths of length κ.
5. $\mathrm{OG}_i(\lambda, \kappa)$ = oriented graphs of size λ which omit independent sets of size κ.
6. $G_i(\lambda, \kappa)$ = undirected graphs of size λ which omit independent sets of size κ.
7. $G_c(\lambda, \kappa)$ = undirected graphs of size λ which omit cliques of size κ.
8. $\mathrm{LO}(\lambda, \kappa)$ = linear orders of size λ which omit well-ordered suborderings of size κ.
9. $\mathrm{LO}_*(\lambda, \kappa)$ = linear orders of size λ which omit suborderings which are the inverse of well-orders of size κ.

An *oriented graph* in this context will be a directed graph which omits all cycles and multi-edges.

Definition 1.1 A κ*-path* in an oriented graph is a κ-sequence $\langle n_\alpha : \alpha < \kappa \rangle$ such that the transitive closure of $\{(\alpha, \beta) : R(n_\alpha, n_\beta)\}$ is $\{(\alpha, \beta) : \alpha < \beta < \kappa\}$ where $R(n_\alpha, n_\beta)$ indicates that there is a directed edge from n_α to n_β. For convenience, for any element a of the oriented graph, let (a, a) be a path of length 0.

We say path when we mean a κ-path for some κ. This notion of an infinite directed graph path was considered in Brochet and Pouzet [1]. Note that for $\kappa \leq \omega$, these paths are exactly the traditionally defined graph paths. Also, any finite subpath of a κ-path is a path and behaves as expected.

The main results in this paper are as follows. In Section 2 we show that if there exist functors with certain strong properties between two categories, then the two categories have similar "model-counting" properties. For the next theorem, for structures X, Y, let $X \hookrightarrow Y$ denote that X embeds into Y using a definition of embedding appropriate for these structures.

Theorem 1.2 *Suppose C_1 and C_2 are categories each given as a type of object with the embeddings as their morphisms. Further suppose there exist functors*

$F : C_1 \to C_2$ and $G : C_2 \to C_1$ which preserve the respective embeddings and the size of the objects and have the following property. For $X \in C_1$, $Y \in C_2$ we have that $Y \hookrightarrow F(G(Y))$ and $X \hookrightarrow G(F(X))$. Then,

1. *the classes of objects have universal models in exactly the same cardinals; moreover, they have the same complexity in each cardinal;*
2. *the classes of objects have the same simplicity number in each cardinal;*
3. *the classes of objects have prime models in exactly the same cardinals;*
4. *if the functors F and G are both injective maps, the two classes of objects have the same number of pairwise nonisomorphic models in each cardinal.*

The notion of a prime model is well known in a first-order model theoretic sense for elementary embeddings. However, it can be generalized to include any set of objects of a certain size and any type of embedding. Namely, we can say that a *prime model* for a set of objects A is a structure in A which embeds into every other structure in A.

As examples of the type of structures one can use in this context, we obtain the following in Section 3.

Theorem 1.3

1. *Oriented graphs and posets have the same universality spectrum.*
2. $\mathrm{OG}_p(\lambda, \kappa)$ *has the same universality spectrum as* $\mathcal{C}(\lambda, \kappa)$ *for any cardinals* λ *and* κ.
3. $\mathrm{LO}(\lambda, \kappa)$ *has the same universality spectrum as* $\mathrm{LO}_*(\lambda, \kappa)$.

The first part of the theorem was already known. Both posets and oriented graphs have the strict order property, so they have the same weak universality spectrum as linear orders by Kojman and Shelah [3].

In the Brochet-Pouzet paper [1], directed graphs with various properties, such as well-founded and scattered, are defined. For instance, a "well-founded directed graph" is one whose transitive closure is well-founded. By the arguments above, structures such as well-founded oriented graphs and well-founded posets have the same universal spectra. Thus, the results in Thompson [10] (see also Džamonja and Thompson [2]) giving the universal spectrum for well-founded posets as the same for ordinals also apply to well-founded oriented graphs. These results are summarized in Table 2.

In Section 4, the σ-functor is used to prove universality results. It was used in Todorčević and Väänänen [12] to prove that there are no weak universal models for certain types of ordered structures and graphs. We extend their results to show the following.

Theorem 1.4 *For $\lambda^{<\kappa} = \lambda$ there is no universal model in* $\mathcal{C}(\lambda, \kappa)$, $\mathrm{wfpo}(\lambda, \kappa)$, *and* $\mathrm{tree}(\lambda, \kappa)$. *By the results in Section 3, there is also no universal model in* $\mathrm{OG}_p(\lambda, \kappa)$ *under the above conditions.*

Komjáth and Shelah [4] proved a universality result analogous to the one below for graphs of size λ omitting cliques of size κ. In Section 5 we use similar methods to prove the statements below.

Theorem 1.5 *Assuming* GCH, *let λ be a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and* $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$. *Then there exists a strong universal in* $\mathrm{OG}_i(\lambda, \kappa)$, $G_i(\lambda, \kappa)$, $\mathrm{tree}(\lambda, \kappa)$, *and* $\mathrm{wfpo}(\lambda, \kappa)$.

The commonality of the methods in Sections 3 and 4 is that in each case a tree is constructed which gives a skeletal structure in order to build the rest of the structure. In Section 6 we combine the results above to get the following universality spectra.

Theorem 1.6 *Assuming* GCH*, the universality spectrum for structures* $\mathrm{tree}(\lambda, \kappa)$ *and* $\mathrm{wfpo}(\lambda, \kappa)$ *is as follows: there is a strongly (and weakly) universal poset in* $\mathrm{tree}(\lambda, \kappa)$ *or* $\mathrm{wfpo}(\lambda, \kappa)$ *if and only if* λ *is a (strong) limit cardinal with* $\lambda > \kappa \geq \aleph_0$ *and* $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$.

In order to show the context of the above results, there are below a number of known results for related first-order structures.

Linear orders	$\exists$ saturated model at λ iff $\lambda = \lambda^{<\lambda}$ $\nexists$ a universal at $\lambda \in (\aleph_1, 2^{\aleph_0})$ (see [3]) Con($\exists$ universal at $\aleph_1 + \neg$CH) (see Shelah [7]) Con($\nexists$ universal at $\aleph_1 + \neg$CH) (see [7])
posets	same as linear orders
oriented graphs	same as linear orders
graphs	$\exists$ saturated model at λ iff $\lambda = \lambda^{<\lambda}$ Con($\exists$ universal at $\aleph_1 + \neg$CH) (see Shelah [8]) Con($\nexists$ universal at $\aleph_1 + \neg$CH) (see [7])

Table 1 Universality results for first-order relational structures

One can see from the above table that the first-order relational structures have quite similar universality spectra. They all have a saturated model whenever $\lambda = \lambda^{<\lambda}$ by a well-known model theory result. The results for graphs (or any universal relational theory with the 3-amalgamation property) at $\aleph_1$ extend to include any power λ such that $\lambda = \lambda^{<\lambda}$ and $2^\lambda > \lambda^+$. In contrast to the results above, Table 2 shows known and new universality results for relational non-first-order theories.

$\mathcal{C}(\lambda, \kappa)$	$\nexists$ a universal if $\lambda = \lambda^{<\kappa}$
$\mathrm{OG}_p(\lambda, \kappa)$	same as $\mathcal{C}(\lambda, \kappa)$
$\mathrm{LO}(\lambda, \kappa)$	$\nexists$ a universal if $\lambda = \lambda^{<\kappa}$
$\mathrm{LO}_*(\lambda, \kappa)$	same as $\mathrm{LO}(\lambda, \kappa)$
ordinals	$\nexists$ a universal at any cardinal > 1 (see [2])
well-founded posets	same as ordinals (see [2])
trees	same as ordinals
$\mathrm{wfpo}(\lambda, \kappa)$	(GCH) $\exists$ universal iff $\lambda > \kappa$ limit and $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$
$\mathrm{tree}(\lambda, \kappa)$	same as $\mathrm{wfpo}(\lambda, \kappa)$
$G_c(\lambda, \kappa)$	(GCH) $\exists$ universal iff $\lambda > \kappa$ limit and $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$ (see [4])
$\mathrm{OG}_i(\lambda, \kappa)$	$\exists$ universal if $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$
$G_i(\lambda, \kappa)$	$\exists$ universal if $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$

Table 2 Universality results for non-first-order relational structures

One can see that the universality spectrum for relational structures which omit large substructures is quite similar. There are no contradictory results, only gaps in the classification. Under the assumption of GCH, it is unknown whether or not there is a universal model if $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$ for $\mathcal{C}(\lambda, \kappa)$, $\mathrm{OG}_p(\lambda, \kappa)$, $\mathrm{LO}(\lambda, \kappa)$, and $\mathrm{LO}_*(\lambda, \kappa)$. Note that these are exactly the theories considered which omit chains. On the other hand, we do not know if there is a universal when $\mathrm{cf}(\kappa) \leq \mathrm{cf}(\lambda)$ for $\mathrm{OG}_i(\lambda, \kappa)$ and $G_i(\lambda, \kappa)$. Both of these theories omit independent sets.

2 Embedding Preserving Functors

In this section, functors will be used to show that one can translate one type of structure into another in order to preserve certain embedding related "model-counting" properties. These properties include universality, the existence of prime models, and the number of pairwise nonisomorphic models.

Definition 2.1

1. A *category* C is a class $\mathrm{Ob}(C)$ of objects each of which is a set, together with a class of functions known as morphisms, denoted $\mathrm{Mor}(C)$. The following properties for the morphisms must hold:
   (a) $\mathrm{Mor}(C)$ must contain the identity function,
   (b) $\mathrm{Mor}(C)$ must be closed under the composition of functions.
2. A *functor* is a mapping between categories which preserves identities and the composition of maps. That is, for two categories $C_i = (\mathrm{Ob}(C_i), \mathrm{Mor}(C_i))$ for $i = 1, 2$ a functor $F : C_1 \to C_2$ maps $\mathrm{Ob}(C_1)$ to $\mathrm{Ob}(C_2)$ and $\mathrm{Mor}(C_1)$ to $\mathrm{Mor}(C_2)$.

In this context, the morphisms will be a subclass of the class of all functions between the objects. A category with this type of morphism is called a *concrete category* as in Krishnan [5]. All of the categories in this chapter will be concrete and henceforth, we shall omit this adjective. Also, functors as defined above are covariant functors, but we shall omit this adjective as well as is standard in [5].

We can define a type of structure as a category with the appropriate embedding as its morphism. Then the existence of size-preserving functors between two categories shows that two types of structures have similar properties in the same cardinals as we will demonstrate here.

Theorem 2.2 *Suppose C_1 and C_2 are categories each given as a type of object with the embeddings as their morphisms. Further suppose there exist functors $F : C_1 \to C_2$ and $G : C_2 \to C_1$ which preserve the respective embeddings and the size of the objects and have the following property. For $X \in C_1$, $Y \in C_2$ we have that $Y \hookrightarrow F(G(Y))$ and $X \hookrightarrow G(F(X))$. Then,*

1. *the classes of objects have universal models in exactly the same cardinals; moreover, they have the same complexity in each cardinal;*
2. *the classes of objects have the same simplicity number in each cardinal;*
3. *the classes of objects have prime models in exactly the same cardinals;*
4. *if the functors F and G are both injective maps, the two classes of objects have the same number of pairwise nonisomorphic models in each cardinal.*

Proof Fix a cardinal λ and assume that all structures defined in this proof have size λ.

1 For simplicity, we will prove this for the case of one universal model; however, the proof is the same for any size of a universal family.

Assume that there is a universal model U for C_2 in λ. We will show that $G(U)$ is universal for C_1. For any object $X \in C_1$, we have that $F(X)$ is a C_2-object so $F(X) \hookrightarrow U$. Now we have

$$X \hookrightarrow G(F(X)) \hookrightarrow G(U)$$

by the embedding preservation of G. The composition of two embeddings is itself an embedding; thus $G(U)$ is universal for C_1. The other direction proceeds in a similar way.

2 Assume without loss of generality that the simplicity number of C_2 is smaller than that of C_1. Let $\mathcal{A}_2 \subset C_2$ witness the simplicity number of C_2; that is, $\mathcal{A}_2$ has minimal cardinality such that there does not exist $Y \in C_2$ such that for all $A \in \mathcal{A}_2$, A embeds into Y. Consider $G(A)$ for all $A \in \mathcal{A}_2$. By the fact that the simplicity number of C_1 is greater than C_2 there must exist X such that X embeds $G(A)$ for all $A \in \mathcal{A}_2$. Note here that $|\{G(A) : A \in \mathcal{A}_2\}| \leq |\mathcal{A}_2|$. However, by the embedding preservation of the functors, $F(X)$ must embed all elements of $\mathcal{A}_2$, which is a contradiction.

3 Assume that there exists a prime model P in C_2 of size λ. We will show that $G(P)$ is a prime model for C_1. Suppose it is not, so there exists $X \in C_1$ such that X does not embed $G(P)$. However, $F(X)$ embeds P and the functors preserve embeddings. This leads to a contradiction as above.

4 Trivial. □

3 Examples of Theories with Embedding Preserving Functors

We showed that if two categories have embedding preserving functors with strong properties between them, then these categories have the same universal spectra. We will demonstrate the existence of such functors for certain types of posets, oriented graphs, and linear orders.

First we will define the specific categories that we will use. Let POS be the category of all posets whose universes are sets together with all injective order-preserving functions as morphisms. Let OG be the category of all oriented graphs whose universes are sets together with all injective directed edge-preserving functions as morphisms. Hence, the embeddings considered here are weak embeddings.

It is easily seen that these are indeed categories using the definition above, as embeddings are closed under composition of functions.

Lemma 3.1 *There exists a size-preserving functor* F : POS $\to$ OG.

Proof We will define F by specifying its value $(F_p(P), F_R(\leq))$ on each $(P, \leq_P)$ in POS. Let F_p be the pointwise injective identity map from POS-objects to OG-objects. That is, if P is the universe of a POS-object then $F_p(P)$ is the universe of an OG-object such that each $a \in P$ maps to the same a in $F_p(P)$. Thus, the size of the object is preserved.

Define $F_R(\leq)$ by letting $(F_p(a), F_p(b))$ be a directed edge if and only if $a <_P b$. For $f : P_1 \to P_2$ an embedding, the following square commutes.

$$\begin{array}{ccc} (P_1, \leq_1) & \xrightarrow{\quad f \quad} & (P_2, \leq_2) \\ \Big\downarrow F & & \Big\downarrow F \\ (F_p(P_1), F_R(\leq_1)) & \xrightarrow[\quad f \quad]{} & (F_p(P_2), F_R(\leq_2)) \end{array}$$

We can see that if (a, b) is an edge in $F(P_1)$ then $(f(a), f(b))$ is an edge in $F(P_2)$. Also, if (a, b) is an edge in $F(P_1)$, then $a <_{P_1} b$. Because f is an embedding, we have $f(a) <_{P_2} f(b)$ and thus, $(f(a), f(b))$ is an edge in $F(P_2)$. □

Lemma 3.2 *There exists a size-preserving functor $G : \mathrm{OG} \to \mathrm{POS}$.*

Proof We shall define G by specifying its value $(G_p(D), G_R(e))$ on each $(D, e) \in \mathrm{OG}$. Let G_p be the pointwise injective identity map from OG-objects to POS-objects as in the proof of Lemma 3.1. This preserves size as before.

To define G_R we need to take the transitive closure of the relations e. So let $G_R(e) = \{(a, b) : a, b \in D$ and there is a finite path from a to $b\}$.

Thus, for $g : D_1 \to D_2$ an embedding, the following square commutes.

$$\begin{array}{ccc} (D_1, e_1) & \xrightarrow{\quad g \quad} & (D_2, e_2) \\ \Big\downarrow G & & \Big\downarrow G \\ (G_p(D_1), G_R(e_1)) & \xrightarrow[\quad g \quad]{} & (G_p(D_2), G_R(e_2)) \end{array}$$

We can see that if (a, b) is a relation in $G(D_1)$ then $(g(a), g(b))$ is a relation in $G(D_2)$. Also, if (a, b) is a relation in $G(D_1)$, then either $(a, b) \in e(D_1)$ or there is a path in D_1 from a to b. Because g is an embedding, we have $(g(a), g(b)) \in e(D_2)$ or there is a path in D_2 from $g(a)$ to $g(b)$ and thus by the transitive closure of the relations, $(g(a), g(b))$ is a relation in $G(D_2)$. □

Lemma 3.3 *Posets and oriented graphs have the same universal spectra under weak embeddings.*

Proof Given Lemmas 3.1 and 3.2, we must show that any POS or OG objects embed into their image under the appropriate composition of the functors. First, given $X \in \mathrm{POS}$, we will show that $X \hookrightarrow G(F(X))$. So $F(X)$ is an OG object which is constructed by mapping all poset relations into oriented graph relations. Thus, $F(X)$ is a transitively closed oriented graph. Applying G will not add any new relations to $F(X)$ as it is already transitively closed. Thus, $G(F(X))$ is isomorphic to X.

Now given $Y \in \mathrm{OG}$, we will show that $Y \hookrightarrow F(G(Y))$. By applying G to Y, we are taking the transitive closure of the relations of Y. Thus, relations are added, but none are removed. Then F will simply map all poset relations into oriented graph relations, which neither add nor subtract any relations. So as the embeddings are weak, the addition of relations will not affect any of the embeddings. Thus by Theorem 2.2 posets and oriented graphs have the same universal spectra. □

We will demonstrate below that by a different argument, linear orders have weak universal models in exactly the cardinals that posets have them. Another proof of this can also be found in [3] using the fact that both linear orders and posets have the strict order property.

The method that we will employ relies on Szpilrajn's Theorem (in [9]) which states that every partial order can be extended to a linear order with the same universe. Also recall that partial order embeddings and linear order embeddings are both one-to-one functions which preserve order. These facts are crucial to the following theorem.

Theorem 3.4 *For any cardinal λ, the set of all posets of size λ has a universal model if and only if the set of all linear orders of size λ has one.*

Observation 3.5 This result is true for any complexity; that is, posets have the same complexity in any given cardinal as linear orders. For simplicity, we will just prove it for the case of one universal model. For complexity > 1, replace the word "universal" with "universal family" in the proof below.

Proof First assume there exists a universal partial order P in cardinality λ. Extend P to a linear order P' using Szpilrajn's Theorem. The set of all partial orders of size λ includes all linear orders of size λ, so P' is universal for linear orders of size λ. Since the embeddings we are using are weak, and thus preserve order but not incomparability, the extra relations on P' will not affect any of the embeddings of the set of partial (or linear) orders. In other words, for $f : L \to P$ an embedding from a linear order L into P, it is also the case that f when viewed as a function from L to P' is an embedding.

By the argument given above, a universal linear order is automatically a universal partial order, since a linear order is just a specific type of partial order. □

Thus by Lemma 3.3, we have that linear orders, posets, and oriented graphs all have universals in exactly the same cardinals.

These results (including the Shelah-Kojman results in [3] for linear orders) are stated for weak universal models, that is, universals under weak embeddings. We cannot get a strongly universal linear order for posets, for instance, because there is no incomparability in a linear order. Also, because we are taking the transitive closure of the oriented graphs, the functors do not preserve incomparability. Therefore, this method will not suffice to show that oriented graphs and posets have the same strong universal spectra.

However, we can extend these weak universality results to more restrictive types of graphs and posets.

Fix a cardinal κ and a cardinal $\lambda \geq \kappa$. Let $\mathrm{POS}_{\kappa cc}$ be the category of all posets with universe a set of ordinals of size λ which omit chains of size κ, together with all injective order-preserving functions. Let $\mathrm{OG}_{\kappa cc}$ be the category of all oriented graphs with universe a set of ordinals of size λ for any λ, which omit paths of size κ, together with all injective directed edge-preserving functions. These are indeed categories for similar reasons as for POS and OG.

Theorem 3.6 $\mathrm{OG}(\lambda, \kappa)$ *has the same complexity as* $\mathcal{C}(\lambda, \kappa)$ *under weak embeddings.*

Proof Let F_p and F_R be as in Theorem 3.1. We know that these are functors by that same theorem, so we only need to show that the chain condition is preserved.

If $(P, \leq)$ is a $\mathrm{POS}_{\kappa\mathrm{cc}}$-object and (a, b) is a directed edge in $F_R(\leq)$, then $a <_P b$. This means that any path in $F(P)$ has a one-to-one correspondence to a chain in P. So since there are no chains of forbidden size in P, there are no paths of forbidden size in $F(P)$.

Let G_p and G_R be as in Theorem 3.2. Again, we know these are functors so we only need to show that the chain condition holds.

Suppose that $\langle x_\alpha : \alpha < \kappa \rangle$ is a chain in $G(D)$. In defining a poset $G(D)$ from an oriented graph D, we take the transitive closure of the graph relations. This can only add relations to $G(D)$ where a path is already present in D. So for any $\alpha, \beta < \kappa$ there is a path in D connecting x_α and x_β. Thus, $\langle x_\alpha : \alpha < \kappa \rangle$ is a path in D. Therefore, no new chains are added when the transitive closure is taken.

Thus, by Theorem 2.2 for any cardinal λ there exists a universal poset of size λ which omits κ-chains if and only if there exists a universal oriented graph of size λ which omits κ-paths. The complexities of these categories are also the same. □

The functor F given above is not injective, so this method says nothing about whether the number of nonisomorphic models for these structures are the same. Below is an example of functors which are injective.

Theorem 3.7 *There exist size-preserving functors between* $\mathrm{LO}(\lambda, \kappa)$ *and* $\mathrm{LO}_*(\lambda, \kappa)$.

Proof For both functors, one must simply reverse the order of the structure. □

4 The σ Functor

In this section, the functor considered will be used to map one object in a category to another object in the same category.

The σ functor was first introduced by Kurepa to prove that a Suslin line exists if and only if a Suslin tree exists. This functor can also be used to produce a counterexample to any assumed universal. Todorčević [11] gives this definition of the σ functor for general sets with a single binary relation.

Definition 4.1 For any structure (A, R) where A is a set with one binary relation R, let $(\sigma A, \subseteq)$ be the set of all injective functions s from some ordinal δ into A such that $\alpha < \beta < \delta$ implies $s(\alpha) R s(\beta)$, ordered by end-extension (which in this context is equivalent to $\subseteq$).

Note that σA is a structure partially ordered by the subset relation. Based on this definition, Todorčević gives the following general result. The proof is included here as it is short and instructive.

Theorem 4.2 *For any structure* (A, R) *where* A *is a set with one binary relation* R*,* σA *does not embed into* A.

Proof Assume there is an embedding $f : \sigma A \to A$. Define a function s recursively on all ordinals by $s(\alpha) = f(s \restriction \alpha)$. This function is well-defined for all ordinals as f preserves the relations and thus $s \restriction \alpha \in \sigma A$. However, $\mathrm{ran}(s)$ forms a proper class, which contradicts A being a set. □

Using this theorem, Todorčević and Väänänen in [12] prove that there are no weak universal models for the following structures: $\mathrm{LO}(\kappa, \aleph_1)$, $\mathrm{LO}_*(\kappa, \aleph_1)$, posets of size κ omitting increasing $\aleph_1$-chains, and graphs of size κ omitting $\aleph_1$-cliques, where $\kappa \leq 2^{\aleph_0}$. These results can easily be extended to include structures of the same type of size λ omitting substructures of size κ where $\lambda = \lambda^{<\kappa}$ (see similar proof below).

Using the same functor, we get similar results for $\mathrm{tree}(\lambda, \kappa)$ and $\mathcal{C}(\lambda, \kappa)$ where $\lambda = \lambda^{<\kappa}$. The proof is given below for $\mathcal{C}(\lambda, \kappa)$, but since the functor creates a tree omitting κ branches and $\mathrm{tree}(\lambda, \kappa) \subseteq \mathcal{C}(\lambda, \kappa)$, the same proof holds for $\mathrm{tree}(\lambda, \kappa)$.

Theorem 4.3 *Assume $\lambda = \lambda^{<\kappa}$. Then there is no weak universal model for $\mathcal{C}(\lambda, \kappa)$.*

Proof Suppose λ and κ are cardinals as in the statement of the theorem.

Assume $P \in \mathcal{C}(\lambda, \kappa)$ is universal and we will construct a counterexample to its universality. Let σP be the tree whose elements are of the form $p = \langle p_i : i < i^* \rangle$ for some $i^* < \kappa$ where each $p_i \in P$ and for $i < j < i^*$ we have $p_i <_P p_j$, and ordered by $\subset$. That is, $\bar{p} <_{\sigma P} \bar{p}'$ if and only if $\bar{p} \subset \bar{p}'$. (Note that as we identify sequences of elements of P with functions from an ordinal into P, the subset relation $\bar{p} \subset \bar{p}'$ actually means that $\bar{p}$ is an initial segment of $\bar{p}'$.) Thus, σP is as defined in Definition 4.1. We will show that $\sigma P \in \mathcal{C}(\lambda, \kappa)$ and that P does not embed σP.

Lemma 4.4 *σP is a tree of size λ which omits chains of size κ.*

Proof Note first that σP is partially ordered by $\subset$. To see that σP is a tree, we must show that the predecessors of any element form a linear order and that the structure is well-founded. To show the former, let $\bar{p} \in \sigma P$ and consider $\bar{q_1}, \bar{q_2} \in \sigma P$ such that $\bar{q_1}, \bar{q_2} <_{\sigma P} \bar{p}$. Both are initial segments of $\bar{p}$ so either $\mathrm{dom}(\bar{q_1}) \geq \mathrm{dom}(\bar{q_1})$, in which case $\bar{q_1} \geq_{\sigma P} \bar{q_2}$, or $\mathrm{dom}(\bar{q_1}) < \mathrm{dom}(\bar{q_1})$, indicating that $\bar{q_1} \leq_{\sigma P} \bar{q_2}$. So $\{\bar{q} \in \sigma P : \bar{q} <_{\sigma P} \bar{p}\}$ is totally ordered.

Now for contradiction suppose $\langle \bar{p_n} : n < \omega \rangle$ is a strictly decreasing chain in σP. For each $\bar{p_n}$ let l_n be the length of that sequence. Then $\langle l_n : n < \omega \rangle$ would form a strictly decreasing sequence of ordinals, a contradiction.

The size of σP is given as follows:

$$|\sigma P| \leq |{}^{<\kappa} P| = \lambda^{<\kappa} = \lambda.$$

As for all $p \in P$ we have $\langle p \rangle \in \sigma P$, we also conclude that $|\sigma P| \geq \lambda$, so $|\sigma P| = \lambda$.

Since σP is a tree, a chain can only occur along a branch. So suppose that σP has a branch B of size κ. Each element of this branch is an increasing sequence in P. If we choose a distinct element of P from each node in B, then this would form a κ-chain in P. This can be done as follows. Let B have elements $\{\bar{p}_i : i < \kappa\}$ such that $\bar{p}_i <_{\sigma P} \bar{p}_j$ for all $i < j < \kappa$. For each $j < \kappa$ there exists $a_j \in \mathrm{ran}(\bar{p}_{j+1}) \setminus \mathrm{ran}(\bar{p}_j)$ by the definition of σ_P. Then $\{a_j : j < \kappa\}$ forms a κ-chain in P. □

Lemma 4.5 *There does not exist an embedding $f : \sigma P \to P$.*

Proof In fact, this is proved by Theorem 4.2, but we do not need the full strength of the proof; therefore, we shall include the weaker version here. Suppose such an f exists and consider the empty sequence $\langle\rangle \in \sigma P$. If we set $x_0 = f(\langle\rangle) \in P$ then we have $\langle\rangle <_{\sigma P} \langle x_0 \rangle$. Now let $x_1 = f(\langle x_0 \rangle)$ and thus, $x_0 <_P x_1$ as f preserves order. This also means that $\langle x_0 \rangle <_{\sigma P} \langle x_0, x_1 \rangle$.

We can continue in this way, taking the union at limit ordinals. When we reach stage κ, we have constructed $\{x_\alpha : \alpha < \kappa\}$ which is a κ-chain in P. □

Therefore, for any assumed universal P there exists σP, an element of $\mathcal{C}(\lambda, \kappa)$ which does not embed into P. □

Corollary 4.6 *Assume $\lambda = \lambda^{<\kappa}$ and κ is infinite. There are no universal oriented graphs in* $\mathrm{OG}_p(\lambda, \kappa)$.

Proof By the functors in Section 3, $\mathrm{OG}_p(\lambda, \kappa)$ has universal models in exactly the same cardinals as $\mathcal{C}(\lambda, \kappa)$. By Theorem 4.3, $\mathcal{C}(\lambda, \kappa)$ has no universal model with the hypotheses above. □

Corollary 4.7 *Assume $\lambda = \lambda^{<\kappa}$ and κ is infinite. Then there is no weak universal model for* $\mathrm{wfpo}(\lambda, \kappa)$ *or* $\mathrm{tree}(\lambda, \kappa)$.

Proof The poset σP as constructed in the proof of Theorem 4.3 is a well-founded poset and a tree. We also have that $\mathrm{tree}(\lambda, \kappa) \subseteq \mathrm{wfpo}(\lambda, \kappa) \subseteq \mathcal{C}(\lambda, \kappa)$. So, it follows that the same poset may be used as a counterexample to any universal in $\mathrm{tree}(\lambda, \kappa)$ or $\mathrm{wfpo}(\lambda, \kappa)$. □

5 Strong Universals

The results in this section are considered "positive"; that is, it is proved that universal models do exist in some circumstances.

Theorem 5.1 *Assuming* GCH, *let λ be a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda) > \omega$. Then there exists a strongly universal oriented graph in* $\mathrm{OG}_i(\lambda, \kappa)$.

Proof Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be a continuous sequence of regular cardinals increasing to λ such that $2^{\lambda_\alpha} \leq \lambda_{\alpha+1}$ with $\lambda_0 = 0$ and $\lambda_1 > \mathrm{cf}(\lambda)$. This is possible by the definition of a strong limit cardinal. Let T be a rooted tree of height $\mathrm{cf}(\lambda)$ such that each level, α of T for $1 \leq \alpha < \mathrm{cf}(\lambda)$, has $\lambda_{\alpha+2}$ nodes while every branch of $T \restriction \alpha$ has $\lambda_{\alpha+2}$ extensions on the αth level.

One may construct such a tree by constructing $T \restriction \alpha$ by induction on α. Level 0 of T is its root r. Having constructed $T \restriction \alpha$ for $1 \leq \alpha < \mathrm{cf}(\lambda)$, we may make the following calculations. The number of nodes of the tree $T \restriction \alpha$ is at most $\sup_{\beta<\alpha} \lambda_{\beta+2}$. If α is a limit then the number of nodes of $T \restriction \alpha$ is thus λ_α. If $\alpha = \gamma + 1$ for some γ, then the number of nodes of $T \restriction \alpha$ is $\lambda_{\gamma+2} = \lambda_{\alpha+1}$. The number of branches of $T \restriction \alpha$ is $|{}^\alpha(T \restriction \alpha)|$. If α is a limit then the number of branches of $T \restriction \alpha$ is λ_α^α which is λ_α by GCH and the fact that for $\alpha \geq 1$, we set $\lambda_\alpha > \mathrm{cf}(\lambda)$ and $\alpha < \mathrm{cf}(\lambda)$. If $\alpha = \gamma + 1$, then $\lambda_{\alpha+1}^\alpha$ is $\lambda_{\alpha+1}$ by GCH. In conclusion, there are at most $\lambda_{\alpha+1}$ branches in $T \restriction \alpha$ for every $\alpha < \mathrm{cf}(\lambda)$. Thus, if each branch of $T \restriction \alpha$ has $\lambda_{\alpha+2}$ extensions on the αth level, then there are $\lambda_{\alpha+2} \cdot \lambda_{\alpha+1} = \lambda_{\alpha+2}$ nodes on the αth level.

The idea of this proof is to use the tree T to give a "skeletal structure" to the universal model in the following way. In order to construct the embeddings of the other members of the set, we will define partial embeddings which increase at each successive level of the tree T. Each level α will code all the relevant possibilities for the oriented graphs of size $\lambda_{\alpha+1}$. To construct the full embedding, one can then follow the appropriate T-branch through the oriented graph.

We will now define $U \in \mathrm{OG}_i(\lambda, \kappa)$ such that it is universal in this set. Let $A(r)$ be the empty graph (recall that r is the root of the tree T). On each level α of T for $\alpha \geq 1$, the nodes $t \in T$ will have attached to them oriented graphs $A(t)$

whose universe is a subset of $\lambda_{\alpha+2}$ of size $\lambda_{\alpha+1}$ and which we will define to omit independent sets of size κ. We will choose the $A(t)$ by induction on $\alpha = \text{height}(t)$. The induction hypothesis is that if $t, t' \in T \restriction \alpha$ and $t <_T t'$, then $A(t)$ is a subgraph of $A(t')$ and $A(t')$ omits independent sets of size κ. The universe of U will be $\bigcup\{A(t) : t \in T\}$ after its construction as $U = \bigcup_{\alpha < \text{cf}(\lambda)} S_\alpha$ where $\langle S_\alpha : \alpha < \text{cf}(\lambda)\rangle$ are an increasing sequence of oriented graphs defined by induction on α as follows.

Let S_0 be the empty graph. Coming to level α, by the induction hypothesis we have constructed S_α as an oriented graph omitting independent sets of size κ whose universe is $\bigcup\{A(t) : t \in T \restriction \alpha\}$ where for each $t \in T \restriction \alpha$, if the height of t in T is $\beta \in [1, \alpha)$ then the universe of $A(t)$ is a subset of $\lambda_{\beta+2}$ of size $\lambda_{\beta+1}$. Note that this implies that the universe of S_α is a subset of $\lambda_{\alpha+2}$. As calculated above, the number of nodes of $T \restriction \alpha$ is at most $\lambda_{\alpha+1}$ and for each such node t we have $|A(t)| \leq \lambda_{\alpha+1}$. So $|S_\alpha| \leq \lambda_{\alpha+1} \cdot \lambda_{\alpha+1} = \lambda_{\alpha+1}$. Let $\langle B_i^\alpha : i < i^*(\alpha)\rangle$ for some $i^*(\alpha) \leq \lambda_{\alpha+1}$ be an enumeration of all branches of $T \restriction \alpha$ of height α. Let $\{H_i^\alpha : i < i^*(\alpha)\}$ be a disjoint partition of $\lambda_{\alpha+2} \setminus S_\alpha$ into pieces of size $\lambda_{\alpha+2}$, which is possible since $|S_\alpha| \leq \lambda_{\alpha+1}$. These sets will be the universes of various $A(t) \setminus S_\alpha$ for $t \in T$ of height α.

We now consider all branches of $T \restriction \alpha$ whose height is α. Let $B = B_i^\alpha$ for some $i < i^*(\alpha)$ be such a branch. For $\beta < \alpha$ let t_β be the node of B of height β. By the induction hypothesis, $\langle t_\beta : \beta < \alpha\rangle$ gives a sequence $\langle A(t_\beta) : \beta < \alpha\rangle$ of oriented graphs such that $\beta < \gamma < \alpha$ implies $A(t_\beta)$ is a subgraph of $A(t_\gamma)$. Hence, $A(B) := \bigcup\{A(t_\beta) : \beta < \alpha\}$ is an oriented graph whose universe is a subset of $\lambda_{\alpha+1}$ of size λ_α.

Note that $A(B)$ has no independent sets of size κ. Namely, assume that it does, so let K be an independent set of size κ. Since $\alpha < \text{cf}(\kappa)$ there must be $\beta < \alpha$ such that $K \cap A(t_\beta)$ has size κ which is a contradiction with the induction hypothesis.

We will construct representatives under isomorphism of all possible oriented graphs of size $\lambda_{\alpha+1}$ that contain $A(B)$ and that have no independent sets of size κ and as their universes we shall take subsets of $A(B) \cup H_i^\alpha$. This type of oriented graph exists; for example, one could make all new elements into a path and set them all greater than all elements of $A(B)$. Between every pair of distinct elements a, b in this extension, there are three possibilities; $a < b$, $a > b$, or $a >< b$. Because of the fact that the formation of independent sets of size κ is forbidden, not all choices can be made. However, even if all choices were possible, we have $\leq \lambda_{\alpha+2}$ possibilities as by the definition of the sequence, $2^{\lambda_{\alpha+1}} \leq \lambda_{\alpha+2}$.

We shall, in fact, for each relevant isomorphism type, choose some $t \in T$ with $\text{height}(t) = \alpha$ and $B <_T t$ and define an oriented graph $A(t)$ of that type so that the universes $\{A(t) \setminus A(B) : B <_T t \text{ and } \text{height}(t) = \alpha\}$ form a disjoint family $\langle H_{i,j}^\alpha : j < j^*(i)\rangle$ of subsets of H_i^α for some $j^*(i) \leq \lambda_{\alpha+2}$ whose exact value is determined by the number of isomorphism types needed. Thus, for $t \in T$ with $\text{height}(t) = \alpha$ that satisfy $B <_T t$, we will let $A(t) = H_{i,j}^\alpha \cup A(B)$ for some j and let the ordering extend $A(B)$ and form no independent sets of size κ according to the isomorphism type in question.

Before finishing this step of the induction, we shall totally order level α of T by a relation $<'_\alpha$. Now we define $S_{\alpha+1}$ to extend S_α, have the universe $S_\alpha \cup \bigcup\{A(t) : t \in T \text{ and } \text{height}(t) = \alpha\}$ and satisfy $a < b$ in $S_{\alpha+1}$ if and only if one of the following holds:

1. $a, b \in S_\alpha$ and $a < b$ holds in S_α,
2. for some $t \in T$ of height α we have $a, b \in A(t)$ and $a < b$ holds in $A(t)$,

3. for some $t, t' \in T$ both of height α we have $t <'_\alpha t'$ and $a \in A(t) \setminus S_\alpha$ while $b \in A(t') \setminus S_\alpha$.

Note that $S_{\alpha+1}$ is a well-defined oriented graph. Consequently, the oriented graph $U = \bigcup_{\alpha < \mathrm{cf}(\lambda)} S_\alpha$ defined at the end of the induction is a well-defined oriented graph.

Claim 5.2 *The oriented graph U defined above is a universal oriented graph of size λ omitting independent sets of size κ.*

Proof The cardinality of U is $\sup_{\alpha < \mathrm{cf}(\lambda)} |S_\alpha| \leq \sup_{\alpha < \mathrm{cf}(\lambda)} \lambda_{\alpha+2} = \lambda$. On the other hand, it will follow from the universality of U for $\mathrm{OG}_i(\lambda, \kappa)$ that the size of U is $\geq \lambda$. To see that there are no independent sets of size κ, assume there is one, call it A. Because $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$, there must be α such that $|A \cap (S_{\alpha+1} \setminus S_\alpha)| = \kappa$. In other words, $|A \cap \bigcup_{\mathrm{height}(t)=\alpha} A(t) \setminus S_\alpha| = \kappa$. Since for $t \neq t'$ of height α in T we have that all elements of $A(t) \setminus S_\alpha$ are connected to all elements of $A(t') \setminus S_\alpha$ and A is an independent set, there is at most one t of height α such that $A(t) \setminus S_\alpha \neq \varnothing$. However, A cannot be contained in a single $A(t) \setminus S_\alpha$ as this set was constructed to avoid independent sets of size κ.

Now we will show that given any $G \in \mathrm{OG}_i(\lambda, \kappa)$, we can find a strong embedding from G to U. The idea is to follow the appropriate branch of T to get the embedding. Let $G = \bigcup_{\alpha < \mathrm{cf}(\lambda)} G_\alpha$ be a decomposition of G into an increasing union of subgraphs $G_\alpha \subseteq G$ with $|G_\alpha| \leq \lambda_{\alpha+1}$. We will find the appropriate branch $B = \{t_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ of T by choosing t_α by induction on α, matching G_α with $A(t_\alpha)$. Here t_α is the node of B of height α. Assume we have matched everything up to α; that is, there exists a strong embedding $\bigcup_{\beta<\alpha} G_\beta \hookrightarrow \bigcup_{\beta<\alpha} A(t_\beta)$. Now we want to extend the embedding to G_α while preserving the previous embedding. Because $\{A(t) : \mathrm{height}(t) = \alpha\}$ contains isomorphism types of all the possible extensions of $\bigcup_{\beta<\alpha} A(t_\beta)$ to oriented graphs of size $\lambda_{\alpha+1}$ omitting independent sets of size κ, there exists $t_\alpha > t_\beta$ for all $\beta < \alpha$ such that $A(t_\alpha)$ strongly embeds G_α. □

The same proof works for undirected graphs and so there exists a strong universal in $G_i(\lambda, \kappa)$ whenever $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$.

Creating a similar tree structure to the one above, we can get an analogous result for $\mathrm{tree}(\lambda, \kappa)$ and $\mathrm{wfpo}(\lambda, \kappa)$.

Theorem 5.3 *Assuming* GCH, *let λ be a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and* $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$. *Then there exists a strong universal in* $\mathrm{tree}(\lambda, \kappa)$ *and* $\mathrm{wfpo}(\lambda, \kappa)$.

Proof We will concentrate on well-founded posets, but the proof for trees will be similar. Define the tree T as in Theorem 5.1 with $\varnothing$ as the root. The construction of the universal in $\mathrm{wfpo}(\lambda, \kappa)$ using the skeletal structure of T will be almost exactly that of the proof of Theorem 5.1 except that we want to ensure that as we build the poset through increasing levels of T, the rank of the well-founded poset will also increase. So, in connecting the elements of $A(t_\alpha)$ to those of $A(t_{\alpha+1})$ where $\mathrm{rk}_T(t_\alpha) = \alpha$ and $\mathrm{rk}_T(t_{\alpha+1}) = \alpha + 1$, all the relations in the poset will be increasing. This will prevent chains from forming along each level of T. Also, the T-nodes will not be ordered in the end.

As before, we will construct a poset $U = \bigcup_{\alpha < \mathrm{cf}\lambda} S_\alpha$ by induction on α. Assume that we have constructed S_α, a well-founded poset omitting chains of size κ whose universe is $\bigcup\{A(t) : t \in T \restriction \alpha\}$. By the same calculations as before, $|S_\alpha| \leq \lambda_{\alpha+1}$. Let $\langle B_i^\alpha : i < i^*(\alpha)\rangle$ and $\{H_i^\alpha : i < i^*(\alpha)\}$ be as before.

Choose an arbitrary branch $B = B_i^\alpha$ of $T \upharpoonright \alpha$. For $\beta < \alpha$ let t_β be the node of B of height β. By the induction hypothesis, assume that $\langle A(t_\beta) : \beta < \alpha \rangle$ is a sequence of well-founded posets such that $\beta < \gamma < \alpha$ implies that

1. $A(t_\beta)$ is a subposet of $A(t_\gamma)$,
2. if $a \in A(t_\beta)$ and $b \in A(t_\gamma)$ then $\mathrm{rk}_{S_\alpha}(a) \leq \mathrm{rk}_{S_\alpha}(b)$.

Note the $A(B) = \bigcup\{A(t_\beta) : \beta < \alpha\}$ omits κ-chains as before.

We will construct representatives under isomorphism of all possible well-founded posets of size $\lambda_{\alpha+1}$ which extend $A(B)$, have no κ-chains and such that for $t_\alpha \in T$ of height α the elements of $A(t_\alpha) \setminus A(B)$ have rank greater or equal to the rank of the elements of $A(B)$.

Define $S_{\alpha+1}$ to extend S_α with universe $S_\alpha \cup \bigcup\{A(t) : t \in T \ \mathrm{height}_T(t) = \alpha\}$ and if $a, b \in S_{\alpha+1}$ then $a <_{S_{\alpha+1}} b$ if and only if either $a <_{S_\alpha} b$ or there is $t \in T$ of height α such that $a <_{A(t)} b$. Note that $S_{\alpha+1}$ is a well-founded poset as S_α is assumed to be well-founded and for all $t \in T$ $A(t)$ is a well-founded poset.

If $U = \bigcup_{\alpha < \mathrm{cf}\lambda} S_\alpha$ then the calculations for the size of U are the same as before; namely, $|U| = \lambda$. To show that U is well-founded, assume not, so there exists $\langle p_i : i < \omega \rangle$ such that $p_i > p_j$ for all $i < j < \omega$. The sequence could not exist in a single S_α as these were defined to be well-founded. Since $\mathrm{cf}(\lambda) > \omega$, there must exist an α such that $\langle p_i : i < \omega \rangle \subseteq S_\alpha$, which is a contradiction.

We need to show that U omits κ-chains. Assume there is one; call it A. Since $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$, there exists α such that $|A \cap (S_{\alpha+1} \setminus S_\alpha)| = \kappa$. We will show that for $t, t' \in T$ both of height α, if t is incomparable to t' in T then any element $a \in A(t) \setminus A(B)$ is incomparable in $S_{\alpha+1}$ to any element $b \in A(t') \setminus A(B')$ where B, B' are the branches of T of which t, t' are top members, respectively. It can only happen that $a < b$ if there exists $c \in A(t_\beta)$ for some $\beta < \alpha$ such that $a < c < b$. However, if $a < c$ this implies $\mathrm{rk}(a) \leq \mathrm{rk}(c)$ which cannot happen by the construction of $S_{\alpha+1}$. So all elements of $S_{\alpha+1} \setminus S_\alpha$ are incomparable. There also cannot be a κ-chain in a single $A(t) \setminus A(B)$ by construction of $A(t)$. So A cannot exist.

Now we will see that U embeds any $P \in \mathrm{wfpo}(\lambda, \kappa)$. Let $P = \bigcup_{\alpha < \mathrm{cf}(\lambda)} P_\alpha$ be such that $|P_\alpha| = \lambda_\alpha$ and for all $\beta < \alpha < \mathrm{cf}(\lambda)$ we have $P_\beta \subseteq P_\alpha$. Furthermore, for $a \in P_\beta, b \in P_\alpha$ with $\beta < \alpha$ require $\mathrm{rk}_P(a) \leq \mathrm{rk}_P(b)$. Then we may find the appropriate branch through T as in the proof of Theorem 5.1. Assume that there exists a strong embedding $\bigcup_{\beta<\alpha} P_\beta \hookrightarrow \bigcup_{\beta<\alpha} A(t_\beta)$. We will extend the embedding to P_α while preserving the rest of the embedding. We have defined the decomposition of P such that for $a \in P_\beta$ and $b \in P_\alpha$ it is the case that $\mathrm{rk}_P(a) \leq \mathrm{rk}_P(b)$ This is also the case for well-founded partial orders extending $\bigcup A(t_\beta)$. So there exists a $t \in T$ with $\mathrm{height}_T(t) = \alpha$ and $t > t_\beta$ for $\beta < \alpha$ such that $P_\alpha \hookrightarrow A(t)$. □

6 Conclusions and Open Questions

The result in this section combines the results in the previous two sections to give a picture of what what these methods have accomplished.

Theorem 6.1 *Assuming* GCH*, there is a strongly (and weakly) universal poset in* $\mathrm{wfpo}(\lambda, \kappa)$ *and* $\mathrm{tree}(\lambda, \kappa)$ *if and only if λ is a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and* $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$.

Proof All limits are strong limits under GCH and $\lambda^{<\kappa} = \lambda$ implies that $\mathrm{cf}(\kappa) \leq \mathrm{cf}(\lambda)$. All the the cases where $\mathrm{cf}(\kappa) \leq \mathrm{cf}(\lambda)$ are covered by Corollary 4.7

which says there are no weak universal models. The cases where $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$ are covered by Theorem 5.3 where strong universals are constructed. □

It is unknown what happens in the case when $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$ for $\mathcal{C}(\lambda, \kappa)$, $\mathrm{LO}(\lambda, \kappa)$, and $\mathrm{LO}_*(\lambda, \kappa)$. The proofs in Section 5 do not work as the construction would produce a forbidden chain along any one level of the tree T.

Also, the σ functor in Section 4 does not give a means for proving the nonexistence of universals for structures such as $\mathrm{OG}_i(\lambda, \kappa)$ or $G_i(\lambda, \kappa)$. In particular, one cannot use this method to inductively build a graph without large independent sets by taking the independent sets from another graph as elements.

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