The Aristotelian Continuum.
A Formal Characterization

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Abstract   While the classical account of the linear continuum takes it to be a totality of points, which are its ultimate parts, Aristotle conceives of it as continuous and infinitely divisible, without ultimate parts. A formal account of this conception can be given employing a theory of quantification for nonatomic domains and a theory of region-based topology.

1   Introduction

According to the standard view the structure of the straight line is the same as the structure of the so-called *Continuum of Real Numbers*.[1] The line is made up of points, the points are ordered in the same way as the real numbers are, and the topology of the points in the linear continuum is the familiar one, based on open and closed intervals.

The view that the continuum consists of points has of course been contentious for a long time. In Book VI of the *Physics* Aristotle takes up the discussion, addressing himself to already well-known views on the matter. He comes down on the side of those who deny that the continuum consists of points as its ultimate parts. According to Aristotle, a point is a limit of a line segment, but not part of the line. And the linear continuum is infinitely divisible in the sense that any part of the continuum itself has proper parts.

My aim in this paper is to defend this conception of the continuum. Aristotle's understanding of infinity as something merely potential, not actual, differs from that of the majority of mathematicians and logicians for whom actual infinite totalities are not problematic. In the case of infinite divisibility, however, Aristotle's view appears to be the right one. The process of dividing does not have an end, nor do the results of division approach a limit. It was the illegitimate assumption that it makes sense

to think of the division of the continuum as completable and to postulate points as the final result of completed division that leads to the paradoxes of Kant's Second Antinomy.

On the "point-free" conception, the typical parts of the continuum are intervals; the typical parts of n-dimensional spaces are n-dimensional regions. Although no space has points as constituent parts, points can be characterized (or identified) in terms of the genuine parts of spaces, for example, by certain sets of regions with convergence properties. The points of the linear continuum can also be identified by cuts in the continuous line. But the fact that points can be characterized in some such way does not mean that they are parts of spaces.[2]

Among Aristotle's reasons for rejecting pointlike parts of the line are the following:

(a) For a given point there is no next point on either side; two points never touch; there is a distance between any two points.

(b) Since points have no extension, they cannot generate an extended line. Points cannot fill the 1-dimensional space of a line. What composes the line must already have extension. Points, being extensionless, cannot be parts of the line.

Aristotle's view has considerable intuitive plausibility. So why did the point conception win out over the Aristotelian conception and form the basis of classical geometry?

One reason is metaphysical in character: the view that the parts of a whole are ontologically prior to the whole, combined with the view that an infinite regress of parts (and therefore infinite divisibility of the line), is impossible.

Another reason is that the logic invoked to describe structure, to account for structural relationships and functional dependence, is a logic based on (domains of) individuals. Only if the linear continuum consists of points can this logic be applied to describe its structure. An argument to this effect can be found in Russell,[3] and can be put thus:

1. The linear continuum is a structured totality; it has a particular structure that distinguishes it from, for example, a circle.
2. Structure has to be accounted for in terms of relations.
3. Relations require objects as relata.
∴ The continuum has to consist, ultimately, of simple, that is, unstructured, elements whose relationships determine the structure of the continuum.

It is this argument that is my concern in the present paper. I will disarm it by providing an account of the structure of the linear continuum that does not rely on relationships among points, but ultimately on relationships among its extended parts. In avoiding points in favor of extended parts the analysis is in the Aristotelian spirit. It is probably un-Aristotelian in that it implicitly appeals to classical quantification over the (extended, divisible) parts of the line and so treats them as forming a completed totality. Since the progressive division of the line is for Aristotle a process without an end point, he cannot conceive of the infinite totality of all parts; at any stage in the process there are only finitely many parts.

The account is meant to capture the intuitive or premathematical concept of the line. The classical analysis of the line as a collection of points, ordered in a certain

way, though inadequate because it postulates points as constituents of the line, nevertheless, does give a characterization of the structure of the line. It does that *indirectly* by describing the structure of the points (locations) associated with the line.

The two characterizations, the classical one and the one proposed here, are equivalent in the weaker sense that they can be systematically transformed one into the other, both telling us (indirectly or directly) what the structure of the line is. They are not equivalent in the stronger sense of providing the same account of the constitution of the continuous line.

The difference between the two accounts as regards the constitution of the line does affect predicates and functions defined over the line. The Aristotelian account allows for a symmetrical division of the line into a left and a right part. Not so if the line consists of points. Then the point where the line is divided, being a part of the line, must be counted either with the left or with the right part; the division is not symmetrical.

The account proposed here requires a theory of quantification for nonatomic domains, which will be described in the next section. It also requires a theory of topology which does not treat spaces as sets of points. Such a theory, formulated in terms of regions rather than points, will be introduced in Section 3. The proposed account of the Aristotelian Continuum will be presented in Section 4 and justified in Section 5.

2 Quantification Theory for Nonatomic Domains

Classical quantification theory (first-order and second-order) appears to be the only formal theoretical framework for the description of structure. A first-order theory characterizing a structured totality will have quantifiers ranging over the individuals making up the totality. A second-order theory will in addition have at least quantifiers ranging over sets that have these individuals as members. If only classical quantification theory is at our disposal, then it is indeed difficult to see how an account of the structure of the linear continuum can avoid quantifying over points.

But it is not correct that a relation must have objects as relata. There are sentences containing predicates and relational expressions which do not, or at least not obviously, apply to or involve individual objects. For example,

(i)   The surface is everywhere yellow

or, equivalently,

(i′)   All of the surface is yellow.

Again,

(ii)   Surface A is somewhere brighter than surface B is anywhere

or, equivalently,

(ii′)   Some of surface A is brighter than all of surface B.

The expressions 'yellow' and 'brighter than' as they occur in these examples are not correctly described as applying to *objects*, that is, points, but it is certainly these expressions which contribute the predicative or relational aspect of the two examples. Sentence (i) makes a claim about the whole surface in a distributive sense; it is universal without amounting to quantification over a domain of individuals. Sentence (ii) is doubly quantified; the ranges of the two quantifiers are different: neither of

those ranges is presented as a totality of individuals, yet 'is brighter than' is clearly a relational expression. Moreover, it is doubtful whether it makes sense to say of an extensionless point that it is yellow or of two points that one is brighter than the other.

I say of sentences like (i) and (ii) that they involve mass quantification or quantification over nonatomic domains. Sentences of this type are not frequently used and their inferential behavior is not intuitively obvious. There is, however, no intrinsic problem with developing the quantification theory (both first-order and second-order) of mass terms.[4] As one would expect, the Logic of Mass Terms is a generalization of the classical theory of quantification. Atomic statements are of the forms

$$(\forall p \in \alpha)Fp$$

and

$$(\forall p \in \alpha)(\forall q \in \beta)Rpq.$$

'All (of) α is F' with α a mass term will generally give the sense of '$(\forall p \in \alpha)Fp$'. If the range of quantification is a region, for example, a surface or a volume of space, then the form of words 'Region α is everywhere F' is a possible variant. '$(\forall p \in \alpha)(\forall q \in \beta)Rpq$' can be read as 'All of α is R to all of β' or 'Everywhere in α is R-related to everywhere in β'. The Greek letters $\alpha, \beta, \ldots$ indicate quantities, ranges of quantification that are parts or subquantities of some total domain ω.

A quantity α gives rise to a property $p \in \alpha$ in the way in which the quantity of all water is related to the property of being water, expressed by the predicate 'is water'. The statement 'all of the fluid in the container is water' is of the form

$$(\forall p \in \beta)p \in \alpha,$$

which asserts that β is a part or a subquantity of α and can be abbreviated to

$$\beta \subseteq \alpha$$

The distinguishing characteristic of the Logic of Mass Terms is the treatment of identity. Consider the appeal to identity made in reflexive statements $(\forall p \in \alpha)Rpp$ ('region α is everywhere R-related to there itself'). For this to be true it is necessary that there are regions β with $(\forall q \in \beta)(\forall r \in \beta)Rqr$ ('β is everywhere R-related to everywhere in β') and that these regions together add up to the total region α which is said to be everywhere R-related to there itself. This is how all predicates are understood that have been obtained by reflexivization. To indicate the identification with one another of argument places q and r under the label p in an arbitrary formula A, I write

$$p \left\{ \begin{array}{c} q \\ r \end{array} \right. A.$$

The rules governing this operation are specific to the Logic of Mass Terms. Restrictions on the identification of argument places also affect the formulation of the quantifier introduction and elimination rules.

The language QM of the Logic of Mass Terms and the characteristic first-order rules of inference for the logic are listed in Appendix A. ω is a constant, referring (distributively) to the total domain of quantification.

As noted above, Greek letters $\alpha, \beta, \ldots$ are interpreted as quantities that are parts of the domain ω. They not only indicate possible ranges of quantification for the first-order quantifiers, they are also used as second-order variables. They range over some or all of the parts or subquantities of ω. The set of subquantities of the domain

ω is called Ω; that is, $\Omega = \{\alpha \mid \alpha \subseteq \omega\}$. Members of Ω can be identified by properties A via the comprehension equivalence

$$\vdash A(p) \equiv p \in \{q \in \omega \mid A(q/p)\}$$

which mirrors the abstraction principle for sets. $\{q \in \omega \mid A(q/p)\}$ is the quantity of all that of ω of which A is true. It comes as no surprise then that Ω is a Boolean algebra under the ordering relation $\subseteq$ with ω the unit element. The comprehension equivalence allows the definition of the null-quantity 0 and of the familiar Boolean operations of complement $-$, meet $\cap$, join $\cup$, and infinite meet $\bigcap$ and join $\bigcup$.

Statements with a second-order quantifier are written formally

$$(\Pi\alpha \subseteq \beta)A.$$

The second-order quantifier here is restricted and ranges over all subquantities of β except 0. The unrestricted quantifier '$(\Pi\alpha \subseteq \omega)$' ranges over the elements of Ω other than 0.

Second-order logic also permits the definition of identity:

$$p = q \text{ for } (\Pi\alpha \subseteq \omega)(p \in \alpha \equiv q \in \alpha).$$

The second-order inference rules, including those governing the identity relation, are also listed in Appendix A.

An interesting result is that the statements of the first-order part of QM can be systematically translated into second-order statements (see Appendix A). This is useful, since due to the lack of familiarity with quantification over nonatomic domains, the second-order translations contribute to one's understanding of the first-order statements.

The formal characterization of the Aristotelian Continuum will employ the Logic of Mass Terms. Both first-order and second-order quantifiers will be required. Since the intended domain is a space, it is more appropriate to refer to the parts of the domain as *regions* rather than to use the more general expression 'quantities'.

3   Region-Based Topology

A satisfactory analytic description of the Aristotelian Continuum requires an account both of its order properties and of its topological properties. In order to obtain the latter it will be necessary to invoke a theory which describes the topological structure of spaces in terms of properties and relations of regions such as volumes, areas, and intervals. Regions are quantities in the sense of the logic of nonatomic domains; so the topological theory of regions, or *Region-Based Topology*, is a second-order theory in the language QM.[5]

The theory employs two primitive concepts, a 2-place relation of *connection*, symbolized by '∞', and the property of being *limited* (i.e., not infinite). There are three defined notions.

Definition 3.1 $\alpha \ll \beta$ (α is an inner part of β) for 'not: $\alpha\infty - \beta$'.

Definition 3.2 α is *coherent* for 'if β and γ are any nonnull regions such that $\beta \cup \gamma = \alpha$ then $\beta\infty\gamma$'.

Definition 3.3 α is *convex* for 'if β and γ are any nonnull regions such that $\beta \cup \gamma = \alpha$, then there is a region α' such that $\alpha' \ll \alpha$ and $(\alpha' \cap \beta)\infty(\alpha' \cap \gamma)$'.

Basic Region-Based Topology is characterized by the following 10 axioms. The domain of the theory, that is, the Boolean algebra Ω of regions with ω as its unit element, is assumed to be complete.

(A1)   If $\alpha \infty \beta$ then $\beta \infty \alpha$.

(A2)   If $\alpha \neq 0$ then $\alpha \infty \alpha$.

(A3)   not: $(0 \infty \alpha)$

(A4)   If $\alpha \infty \beta$ and $\beta \subseteq \gamma$ then $\alpha \infty \gamma$.

(A5)   If $\alpha \infty (\beta \cup \gamma)$ then $\alpha \infty \beta$ or $\alpha \infty \gamma$.

(A6)   0 is limited.

(A7)   If α is limited and $\beta \subseteq \alpha$ then β is limited.

(A8)   If α and β are both limited then $\alpha \cup \beta$ is limited.

(A9)   If $\alpha \infty \beta$ then there is a limited region β' with $\beta' \subseteq \beta$ and $\alpha \infty \beta'$.

(A10)   If α is limited, $\beta \neq 0$, and $\alpha \ll \beta$, then there is a region γ such that $\gamma \neq 0$, γ is limited, and $\alpha \ll \gamma \ll \beta$.

Region-based topologies can be systematically related to point-based topologies by characterizing points by means of the regions within which they are located and by giving appropriate definitions of the central notions of point-based topology, such as *open* and *closed* set. In this way a systematic correspondence between region-based and point-based topologies can be established. The result is that the region-based topologies that meet the conditions (A1) to (A10) correspond one-to-one to point-based topologies that are locally compact T_2-spaces.

   Not all locally compact T_2-spaces are of interest to us. The classical linear continuum **R** as well as the 2- and 3-dimensional spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ have more specific properties. They are known as *continua* and satisfy the further conditions of being connected, locally connected, and second countable. The corresponding region-based topologies satisfy, in addition to (A1) to (A10), three further conditions, (*Coherence*), (*Continuity*), and (*Convex Cover*).

(*Coherence*) ω is coherent.

(*Continuity*)   If β is limited and not : $\alpha \infty \beta$ then there is a region γ, which is the join $\gamma_1 \cup \cdots \cup \gamma_n$ of finitely many convex regions $\gamma_1, \ldots, \gamma_n$, such that $\beta \subseteq \gamma$ and not : $\alpha \infty \gamma$. (If β is limited and $\beta \ll \gamma$ then there is a finite join $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$ of convex regions $\gamma_1, \ldots, \gamma_n$ such that $\beta \subseteq \gamma$ and $\gamma \ll \alpha$.)

(*Convex Cover*) There is a countable set $\Delta \subseteq \Omega$ of limited regions such that for every limited region $\alpha \ll \beta$ there is a region γ in Δ with $\alpha \ll \gamma \subseteq \beta$.

The principles of Region-Based Topology implicitly fix the meaning of the relation 'connected with' and the predicate 'limited' to the extent that a formal theory can do that. A satisfactory characterization of the Aristotelian Continuum will describe it as a continuum in the sense of Region-Based Topology. It will identify extensions for the two basic notions, *connection* and *being limited*, so that the conditions (A1) to (A10), (*Coherence*), (*Continuity*), and (*Convex Cover*) are met.

4 The Axioms

The aim of this section is to give an axiomatic characterization of the Aristotelian Continuum using the Logic of Mass Terms. The axioms are of two kinds, namely, those describing the linear order which is characteristic of the continuum and those dealing with its topological features.

The linear order can be described by axioms that are not much different from the familiar total order axioms. For intuitive definiteness the 2-place relation '$<$' will be read as 'is to the left of'. The intended interpretation of the domain ω of first-order quantification is the Aristotelian Continuum. The second-order variables $\alpha, \beta, \ldots$ range over the subregions of ω; that is, the total domain of second-order quantification is the set $\Omega = \{\alpha \mid \alpha \subseteq \omega\}$ of subregions of ω.

Axiom 1 $(\forall p \in \omega)(\forall q \in \omega)(\forall r \in \omega)[(p < q \ \& \ q < r) \supset p < r]$.

Axiom 2 $(\forall p \in \omega)(\forall q \in \omega)(p < q \lor q < p)$.

Axiom 3 $(\forall p \in \omega) \sim p < p$.

The second-order translations of the three axioms can be simplified by defining

$$\alpha < \beta \text{ for} \qquad (\forall p \in \alpha)(\forall q \in \beta)p < q;$$

that is, 'region α is wholly to the left of region β'. Note that '$<$' signifies a first-order relation in the context '$p < q$' and a defined second-order relation in the context '$\alpha < \beta$'.

(**Axiom 1′**) $(\Pi\alpha \subseteq \omega)(\Pi\beta \subseteq \omega)(\Pi\gamma \in \omega)[(\alpha < \beta \ \& \ \beta < \gamma) \supset \alpha < \gamma]$.

(**Axiom 2′**) $(\Pi\alpha \subseteq \omega)(\Pi\beta \subseteq \omega)(\Sigma\alpha' \subseteq \alpha)(\Sigma \beta' \subseteq \beta)(\alpha' < \beta' \lor \beta' < \alpha')$.

(**Axiom 3′**) $(\Pi\alpha \subseteq \omega) \sim \alpha < \alpha$.

The first-order axioms appear to be inconsistent, Axiom 2 contradicting Axiom 3. However, the inconsistency is only apparent. The rules of the Logic of Mass Terms do not allow us to infer

$$(\forall p \in \omega)p < p$$

from Axiom 2, since an application of ($\forall$E) never results in the identification of argument places. On the other hand, Axioms 2 and 3 together imply

$$(\forall p \in \omega)(\forall q \in \omega) \sim p = q.$$

For[6]

(1)	$(\forall p \in \omega)(\forall q \in \omega)(p < q \vee q < p)$	Axiom 2
(2)	$(\forall p \in \omega) \sim p < p$	Axiom 3
(3)	$p < q \vee q < p$	1 ($\forall$E)
(4)	$\sim p < p$	2 ($\forall$E)
(5)	$p = q$	Assumption
(6)	$p < q$	Assumption
(7)	$p \left\{ \begin{matrix} p \\ q \end{matrix} \right. \quad p < q$	5, 6 (=E)
(8)	$p < p$	7 (Refl A)
(9)	$q < p$	Assumption
(10)	$p \left\{ \begin{matrix} p \\ q \end{matrix} \right. \quad q < p$	5, 9 (=E)
(11)	$p < p$	10 (Refl A)
(12)	$p < p$	3, 6–8, 9–11 ($\vee$E)
(13)	$\sim p = q$	5–12 ($\sim$I)
(14)	$(\forall p \in \omega)(\forall q \in \omega) \sim p = q$	13 ($\forall$I)

Far from being unwelcome, this result amounts to the infinite divisibility of the domain ω. For $(\forall p \in \omega)(\forall q \in \omega) \sim p = q$ is short for

$$(\Pi\alpha \subseteq \omega)(\Pi\beta \subseteq \omega)(\Sigma\gamma \subseteq \omega)(\Sigma\alpha' \subseteq \alpha)(\Sigma\beta' \subseteq \beta) \sim (\alpha' \subseteq \gamma \equiv \beta' \subseteq \gamma)$$

which implies

$$(\Pi\alpha \subseteq \omega)(\Sigma\gamma \subseteq \omega)(\Sigma\alpha' \subseteq \alpha)(\Sigma\alpha'' \subseteq \alpha)(\alpha' \subseteq \gamma \ \& \ \alpha'' \cap \gamma = 0).$$

The three axioms are, however, not sufficient to characterize the linear continuum completely. They suffice for the specification of its order properties but not for its topological properties. A model which satisfies Axioms 1 – 3 may fail to be a linear continuum for a number of reasons: the domain may fail to be 1-dimensional; the structure may fail to be infinite in both directions; the domain might not be continuous; it might not even be connected, that is, ω might consist of disconnected pieces.

In order to ensure that ω has the topological properties of being 1-dimensional, infinite and continuous more axioms are needed. These topological features of the continuum cannot be captured by first-order axioms; it is necessary to invoke second-order quantification. This is not surprising, since ordinary, set-theoretic topology is also a second-order theory.

Part of what the new axioms will be doing is to characterize the topological notions of *connectedness* and *limitedness* in terms of the order structure specified by the first three axioms. It will have to be shown that these stipulations are in line with the meaning of the primitive notions of Region-Based Topology as registered by the theory described in Section 3. It will have to be shown that, with these notions extensionally defined by further axioms, the principles of Region-Based Topology can be verified.

Before tackling the three concerns mentioned we note that Region-Based Topology presupposes that the regions form a *complete* Boolean algebra. The subquantities of the domain ω do of course constitute a Boolean algebra Ω as a matter of logic; the

completeness of Ω however needs to be stipulated: for any set Γ of subregions of ω the join $\bigcup \Gamma$ exists.

Axiom 4   *If $\Gamma \subseteq \Omega$, then $\bigcup \Gamma \in \Omega$.*

Central to the topology of the infinite line is the notion of an *interval*, a notion which can be defined within the first-order part of the language QM. An interval is a region without gaps; any parts of ω that are between parts belonging to an interval also belong to that interval. This is all we need for the definition, except that the empty region should not be counted as an interval.

Definition 4.1   *Interval(α) for '$(\exists p \in \omega)p \in \alpha$ & $(\forall p \in \alpha)(\forall r \in \alpha)(\forall q \in \omega)[(p < q$ & $q < r) \supset q \in \alpha]$'.*

Note that this definition does not tell us anything about the topological properties of intervals; in particular, it does not tell us that they are coherent regions.

The first concern mentioned, namely, the 1-dimensionality of the linear continuum, can be addressed by postulating that any region contains an interval.

Axiom 5 $(\Pi \alpha \subseteq \omega)(\Sigma \beta \subseteq \alpha)Interval(\beta)$.

In order to see that this is just the right condition, consider as a counterexample a 2-dimensional band which meets the three order axioms. By virtue of Definition 4.1 an interval is a stretch of the band that extends for its whole length to the full width of the band. But it is possible to mark out areas of the band that are everywhere clear of the edges of the band. Since such areas do not have parts that are intervals as defined, the condition of Axiom 5 is not met.

The second topological characteristic mentioned above is the infinity of the linear continuum. The order properties expressed by Axioms 1 to 3 do not distinguish between *limited* and *unlimited* structures. So it is necessary to stipulate, in conformity with our intuitive understanding, that a region is limited if and only if there are parts of ω to the right and to the left of the region.

Axiom 6   *Limited$(\alpha) \equiv [(\exists q \in \omega)(\forall p \in \alpha)p < q$ & $(\exists q \in \omega)(\forall p \in \alpha)q < p]$.*

Axiom 6 has as its second-order translation the following.

(**Axiom 6′**)   *Limited$(\alpha) \equiv [(\Sigma \beta \subseteq \omega)\alpha < \beta$ & $(\Sigma \beta \subseteq \omega)\beta < \alpha]$.*

This stipulation is not a definition in the proper sense, since the *concept* of a limited region is taken to be fixed by the theory of region-based topology described in Section 3. It will have to be confirmed that determining the extension of *limited* in this way accords with the meaning of 'limited' as registered by the theory of region-based topology.

Axiom 6 implies that ω is unlimited in both directions, and with Axiom 6 in place we can infer that every nonnull region contains a limited interval. (See Lemma 6 in Appendix B.) It then follows that every region in Ω is the join of limited intervals (Lemma 7).

The third concern was that a structure satisfying the order axioms might not be coherent. It might, for example, consist of two infinite half-lines which are not connected with one another. Recall that Region-Based Topology defines coherence in terms of the connectedness relation.

$$\alpha \text{ is } coherent \quad \text{iff} \quad (\Pi \beta \subseteq \omega)(\Pi \gamma \subseteq \omega)(\beta \cup \gamma = \alpha \supset \beta \infty \gamma).$$

Therefore the coherence of the structure has to be ensured by an appropriate determination of the extension of the connectedness relation ∞. But it is not only the infinite line as a whole that we expect to be coherent. Indeed, intuitively all intervals (and only intervals) are coherent parts of ω. Their coherence does not follow from Definition 4.1, which has no topological implications on its own; coherence must be secured by an account of connectedness in terms of the ordering structure of the domain.

Since we want intervals to be coherent it is necessary and sufficient that the characterization of the connectedness relation imply that if the join of two regions α and β is an interval, then α and β are connected.

$$\text{If } \alpha \cup \beta \text{ is an interval then } \alpha \infty \beta.$$

The converse, that regions α and β are connected only if their join is an interval, is of course not true. The extension of the connectedness relation is much wider. Regions α and β may each consist of disconnected parts and be connected with one another even if they do not together make up an interval. It is surely sufficient that there are respective parts α' and β' whose join is an interval.

$$\text{If, for some } \alpha' \subseteq \alpha \text{ and some } \beta' \subseteq \beta, \alpha' \cup \beta' \text{ is an interval, then } \alpha \infty \beta$$

Is the weaker condition, that there are nonnull parts of α and of β whose join is an interval, also necessary for α and β being connected? Can we stipulate that α is connected with β if there are nonnull parts of α and β, respectively, whose join is an interval; that is,

(#) $\alpha \infty \beta$ iff $(\Sigma \alpha' \subseteq \alpha)(\Sigma \beta' \subseteq \beta) Interval(\alpha' \cup \beta')$?

Adopting (#) as the criterion for the connectedness of the regions guarantees the coherence of the structure and of all intervals; so it takes care of one of our concerns. But there remains the question of continuity. Thanks to (#) the structure does not fall apart. When two intervals touch one another so that nothing separates them, they are connected with one another. A similar observation can be made about regions consisting of a finite number of intervals. Suppose α and β are finite joins of intervals, $\alpha = \alpha_1 \cup \cdots \cup \alpha_m$ and $\beta = \beta_1 \cup \cdots \cup \beta_n$. Then the stipulation (#) means that $\alpha \infty \beta$ if and only if there is at least one pair α_i, β_j so that $\alpha_i \cup \beta_j$ is an interval; that is, if and only if *there is at least one place where nothing separates α from β*. This condition, which is stronger than (#), can plausibly be taken as the criterion of continuity. If we do that and demand that always $\alpha \infty \beta$ if there is at least one place where no interval separates α from β, we find that this demand is not satisfied if we consider certain regions that are not finite joins of intervals. Take the region β to consist of the intervals $| 1/2^{2n+1}, 1/2^{2n} |$ for $i = 0, 1, 2, \ldots$; that is,

$$\beta = | 1/2, 1 | \cup | 1/8, 1/4 |, \cup | 1/32, 1/16 | \cup \cdots .^7$$

Let α be the left part of the line whose limit on the right is the point corresponding to 0. There are no parts α' and β' of α and β whose join is an interval. Hence α and β are not connected under criterion (#). However, there is a place where nothing separates α and β. So, (#) does not guarantee the desired continuity of the linear structure.

The crucial feature of the example is that the region β is made up of *infinitely* many disconnected intervals. Any way of covering β with *finitely* many limited intervals results in one of these intervals connecting with α. It is this observation

that leads to the formulation of the criterion of connectedness which guarantees the continuity of the domain.

Axiom 7 $\alpha \infty \beta$ if and only if *there are limited regions $\alpha' \subseteq \alpha$ and $\beta' \subseteq \beta$ such that if $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ and $\delta = \delta_1 \cup \cdots \cup \delta_n$ are any finite joins of intervals with $\alpha' \subseteq \gamma$ and $\beta' \subseteq \delta$ then, for at least one $i = 1, \ldots, m$ and one $j = 1, \ldots, n$, $\gamma_i \cup \delta_j$ is an interval.*

Given Axiom 7 the regions α and β in the example do count as connected and it is at least plausible that the axiom guarantees the continuity of the linear structure we want to characterize. At the same time it is worth noting that by adopting Axiom 7 the principle (*Continuity*) has effectively been assured.[8]

   The final axiom to be adopted states that the structure is sustained by a denumerable number of limited intervals. It mirrors the *Separation* property of the classical linear continuum.

Axiom 8   *There exists a countable set Θ of limited intervals such that whenever α is a limited interval and $\alpha \ll \beta$ there is an interval γ in Θ with $\alpha \ll \gamma \ll \beta$.*

5 Justification of the Axioms

5.1 Topology   The axioms that have been adopted ensure that the structure has the topological characteristics of a continuum. To see this one needs to verify the basic axioms (A1) to (A10) of Region-Based Topology and the additional axioms (*Coherence*), (*Continuity*), and (*Convex Cover*). All the proofs are given in Appendix B. It is also shown there that a region α is an interval, if and only if α is coherent, if and only if α is convex .

   The conclusion to be drawn at this point is that Axioms 1 to 8 succeed in characterizing the linear continuum in the sense that all models of the axioms exhibit a 1-dimensional linear order and have the topological structure of continua as defined in Region-Based Topology. That the axioms are categorical seems obvious. The proof will be given later in this section.

5.2 Points   The Aristotelian conception of the linear continuum does not deny the existence of points. But points, rather than being parts of the continuum, are locations in the continuum, limits of intervals, which can be identified in terms of its structure. It is obvious that the order structure of points is completely determined by the structure of the linear continuum. It is therefore reasonable to expect that the formal characterization of the continuum yields an account of the associated points and of their structure.[9]

   It was explained in Section 3 that points can be identified in region-based topologies by means of the regions within which they are located and that region-based topologies can be systematically related to point-based topologies. Since the structure characterized by Axioms 1 to 8 is a continuum (in the sense of Region-Based Topology) the corresponding point structure is a continuum in the sense that it is a connected, locally connected, locally compact T_2-space with a countable basis.

   There are alternative, but equivalent, ways of identifying points within region-based topologies. Given that our structure is a continuum, a particularly simple way draws only on limited convex regions, that is, in our case, on limited intervals: a point is identified by the set of limited intervals to which it is internal. Of course, in order to avoid circularity, these sets of intervals have to be described without reference to

points. This can be done as follows: A set Σ of limited intervals identifies a point if and only if

(i) if $\kappa_1 \in \Sigma$ and $\kappa_1 \subseteq \kappa_2$, then $\kappa_2 \in \Sigma$;
(ii) if $\kappa_1 \in \Sigma$ and $\kappa_2 \in \Sigma$, then $\kappa_1 \cap \kappa_2 \in \Sigma$;
(iii) if $\kappa_1 \in \Sigma$ then there exists $\kappa_2 \in \Sigma$ with $\kappa_2 \ll \kappa_1$;
(iv) Σ is maximal : if $\Sigma \subseteq \Sigma'$, where Σ' is a set of limited intervals satisfying (i) – (iii) then $\Sigma' = \Sigma$.

A set Σ satisfying these conditions is called a *contracting set of limited intervals*.

Still simpler is a familiar way of identifying points that is only available in a 1-dimensional space and is based on the idea that the points are located wherever the line can be divided or cut. The notion of a cut is also used in the characterization of the classical linear continuum. There a *Cut* is a partition [A, B] of the elements of the totality into two classes, the point identified (by the partition) is itself an element of the totality and must be a member of one or the other of the two classes. A cut in the present sense is an exhaustive division of the Aristotelian Continuum into a left infinite interval and a right infinite interval. The point identified by a cut is located where the two intervals abut. A cut in this sense is of course fully specified by just one of the infinite intervals, say the left one. So we use $[\alpha, -\alpha]$ to refer to a cut in the Aristotelian Continuum. A region α is a *left infinite interval* if it is an interval that is limited on the right but unlimited on the left.

Definition 5.1 α is a *left infinite interval* for

$$(\exists p \in \omega)p \in \alpha \ \& \ (\forall p \in \alpha)(\forall q \in \omega)(q < p \supset q \in \alpha) \ \& \ (\exists q \in \omega)(\forall p \in \alpha)p < q.$$

Definition 5.2 $[\alpha, -\alpha]$ is a *cut* if and only if α is a left infinite interval.

The two characterizations of points are equivalent. Given a cut $[\alpha, -\alpha]$, the set of limited intervals straddling the cut, that is, the set $\Sigma = \{\kappa | \kappa \text{ is a limited interval } \& \ \kappa \cap \alpha \neq 0 \ \& \ \kappa \cap -\alpha \neq 0\}$, is a contracting set of limited intervals and hence identifies a point. Conversely, if Σ is such a set and α is the infinite meet of the left infinite intervals in Σ, then $[\alpha, -\alpha]$ is a cut. And it is easy to see that if $[\alpha, -\alpha]$ is different from $[\beta, -\beta]$ then the sets of limited intervals straddling the two cuts, respectively, are also different.

The characterization of points in terms of cuts is preferable. For it is easier to work with, not least because of the straightforward definition of the ordering relation for points. The point associated with the cut $[\alpha, -\alpha]$ is to the left of the point associated with $[\beta, -\beta]$ just when α is a proper part of β.

$$[\alpha, -\alpha] < [\beta, -\beta] \quad \text{iff} \quad \alpha \subset \beta.$$

It is now possible to show that the points associated with the Aristotelian Continuum have the same order structure as the points constituting the classical linear continuum.

Although the classical account of the linear continuum is based on the assumption that the linear continuum consists of points, that it is an ordered totality of points, this ontological assumption is not part of the axiomatic theory itself. The theory, to be presented below, describes the structure of the totality of points, independently of what the ontological relationship between the points and the linear continuum might be.

It is therefore a criterion of the adequacy of the formal account of the linear continuum that I have given that it implies the classical axiomatic theory of the point structure of the linear continuum. The latter can be described by the following system R of axioms, where C is the totality of all points and $<$ the 2-place ordering relation. The axiom system characterizes the order type of the real numbers and is categorical.

Axiom System R

(R1.1)   If $x \neq y$, then $x < y$ or $y < x$.
(R1.2)   If $x < y$, then $x \neq y$.
(R1.3)   If $x < y$ and $y < z$, then $x < y$.
(R2)   If $[A, B]$ is a partition of C such that $(\forall x)(\forall y)[(x \in A \ \& \ y \in B) \supset x < y]$, then either A has a last element or B has a first element.
(R3)   There exists a nonempty, countable subset S of C such that, if $x \in$ C, $y \in$ C, and $x < y$, then there exists $z \in S$ such that $x < z < y$.

(R4)   C has no first element and no last element.

The characteristics (R1.1) to (R1.3) are mirrored by the relation $\subseteq$ on left infinite intervals and are therefore true of the ordering relation $<$ on cuts $[\alpha, -\alpha]$.

Suppose that A and B are sets of cuts partitioning the totality of cuts and such that if $[\alpha, -\alpha] \in A$ and $[\beta, -\beta] \in B$ then $\alpha \subset \beta$. By the completeness of Ω, $\gamma = \bigcup \{\alpha \mid [\alpha, -\alpha] \in A\}$ exists. $[\gamma, -\gamma]$ is the least upper bound of the cuts in A and the greatest lower bound of the cuts in B. $[\gamma, -\gamma]$ belongs to A or to B but not both, since $[A, B]$ is a partition of the totality of cuts. So (R2) is satisfied.

Let S consist of all the cuts $[\sigma, -\sigma]$ such that the left infinite interval σ is obtained by extending an interval in the set Θ of Axiom 8 to infinity on the left. Assume $[\alpha, -\alpha] < [\beta, -\beta]$. Then $-\alpha \cap \beta \neq 0$. By Lemma 6[10] there exists a limited interval γ with $\gamma \ll (-\alpha \cap \beta)$. By Axiom 8 there exists a limited interval $\delta \in \Theta$ such that $\gamma \ll \delta \ll (-\alpha \cap \beta)$ and by Lemma 5 there exists a cut $[\sigma, -\sigma] \in S$ such that $\alpha \subset \sigma \subset \beta$; that is, $[\alpha, -\alpha] < [\sigma, -\sigma] < [\beta, -\beta]$. Hence the separation property (R3) holds.

(R4) is true, since by Lemma 11 there exists for every cut $[\alpha, -\alpha]$ a cut $[\beta, -\beta]$ with $[\beta, -\beta] < [\alpha, -\alpha]$ and a cut $[\gamma, -\gamma]$ with $[\alpha, -\alpha] < [\gamma, -\gamma]$.

This result confirms that Axioms 1 to 8 correctly describe the order properties of the linear continuum. That the topological structure is implied by the axioms is implicit in the result that any model is a connected, locally connected, locally compact T_2-space with a countable basis. The result can also be had directly, using the definition of points as cuts. The sets of points $\{x \mid a < x < b\}$ are open and the sets $\{x \mid a \leq x \leq b\}$ are closed; that is, the theory correctly identifies the open and closed intervals of the classical linear continuum. It also implies, correctly, that the open intervals form a basis of the point-based topology. Consequently, Axioms 1 to 8 imply the topology of the straight line as classically conceived, as well as its order structure.

5.3 Categoricity   A final result to be obtained is the categoricity of the system consisting of Axioms 1 to 8. First, it needs to be clarified what the relevant models are. The axioms invoke four nonlogical primitives, the first-order 2-place predicate

$<$, the second-order predicates ∞ and *Limited* and the set Θ which figures in Axiom 8. The definition

$$\alpha < \beta \text{ for } \quad (\forall p \in \alpha)(\forall q \in \beta) p < q$$

allows us to replace the first-order relation by a second-order relation and to replace Axioms 1, 2, 3, and 6 by the equivalent second-order Axioms 1′, 2′, 3′, and 6′. Models for the resulting set of axioms (Axioms 1′, 2′, 3′, 4, 5, 6′, 7, and 8) are then interpretations of just the second-order part of the language QM.

Since limitedness and ∞ are explicitly defined by Axiom 6′ and Axiom 7, we can ignore them and the two axioms and take a model $\mathcal{M}$ to be a structure $\langle \Omega, <, \Theta \rangle$ consisting of a complete Boolean algebra Ω, a 2-place relation defined on Ω, which interprets the second-order relation symbol $<$, and a set Θ of regions, referred to in Axiom 8, a structure that satisfies Axioms 1′, 2′, 3′, 4, 5, and 8.

Assume then that there are two models $\mathcal{M}_1 = \langle \Omega_1, <_1, \Theta_1 \rangle$ and $\mathcal{M}_2 = \langle \Omega_2, <_2, \Theta_2 \rangle$ of these axioms. Consider, in each model, the associated set of points, that is, the set of cuts $[\alpha, -\alpha]$. The cuts of both models satisfy the axiom system R. But since this system is known to be categorical, there exists a 1-1 correlation between the cuts of $\mathcal{M}_1$ and those of $\mathcal{M}_2$ which preserves the ordering and maps S_1 onto S_2. (S_1 and S_2 are sets of cuts which play the role, in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, of the set S of (R3).) This isomorphism yields a 1-1 correlation between the left infinite intervals of the two models and between the right infinite intervals of the two models, preserving the inclusion relation $\subseteq$. Since every limited interval is the meet of a left infinite interval and a right infinite interval the 1-1 correlation extends to the limited intervals of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, preserving inclusion among all intervals. Since every region α in Ω is the join of the limited intervals that are subregions of α,

$$\alpha = \bigcup \{\kappa | Limited(\kappa) \ \& \ Interval(\kappa) \ \& \ \kappa \subseteq \alpha\}, \ ^{11}$$

the 1-1 correlation extends to all regions in $\mathcal{M}_1$ and $\mathcal{M}_2$, mapping the Boolean algebras Ω_1 and Ω_2 onto one another.

Since according to Lemma 10 $\alpha < \beta$ just when there is a cut $[\gamma, -\gamma]$ such that $\alpha \subseteq \gamma$ and $\beta \subseteq -\gamma$, the correlation also preserves the second-order relation $<$. And since the sets Θ_1 and Θ_2 can be recovered from S_1 and S_2, respectively (Lemma 12), the correlation, moreover, maps Θ_1 onto Θ_2. Hence if $\mathcal{M}_1 = \langle \Omega_1, <_1, \Theta_1 \rangle$ and $\mathcal{M}_2 = \langle \Omega_2, <_2, \Theta_2 \rangle$ are two models of Axioms 1 to 8, $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.

6 Conclusion

The structure of the Aristotelian Continuum can be described in the language QM, using the Logic of Mass Terms. The categorical system of Axioms 1 to 8 specifies both the order properties and the topological properties of the structure. The characterization is not only intuitively correct; its correctness has been proved by showing that

    (a) any model constitutes a region-based topology with the characteristics of a continuum;

    (b) the points associated with the structure have the same order properties as the points of the classical linear continuum.

Therefore, Russell's argument has been refuted, the Aristotelian Continuum is no less legitimate as a concept than the classical linear continuum. And it is not unreasonable

to argue that the Aristotelian Continuum is ontologically prior and that points are not parts of the continuum but are associated with it as limits, just as Aristotle claims.

Appendix A   The Language QM

A.1 Symbols

1. Denumerably many first-order variables: $p_1, p_2, \ldots$
2. Denumerably many second-order variables: $\alpha_1, \alpha_2, \ldots$
3. Countably many predicate letters
4. Countably many 2-place relation letters
5. $\sim$ & $\supset \vee \equiv \forall \, \Pi \in \subseteq | \{\,()\,\{\}$

A.2 Formation Rules

A.2.1 Formulas

(L1.i)   An expression consisting of a predicate letter followed by a first-order variable is a formula;

(L1.ii)   An expression consisting of a relation letter followed by two, not necessarily distinct, first-order variables is a formula;

(L1.iii)   If p is a first-order variable and μ a mass term, then $p\varepsilon\mu$ is a formula;

(L1.$\sim$) If A is a formula, then $\sim A$ is a formula;

(L1.&)   If A and B are formulas then $(A \& B)$ is a formula;

(L1.R)   If A is a formula and p is not free in A, unless p is identical with q or r, then $p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A$ is a formula;

(L1.$\forall$) If A is a formula, μ a mass term, and p a first-order variable, then $(\forall p\varepsilon\mu)A$ is a formula.

(L2.Π)   If A is a formula, μ a mass term, and α a second-order variable, then $(\Pi\alpha \subseteq \mu)A$ is a formula.

A.2.2 Mass Terms

(L1.iv)   Second-order variables are mass terms.

(L1.C)   If μ is a mass term and A a formula with no free first-order variable besides p, then $\{p \in \mu \mid A\}$ is a (complex) mass term.

The expressions $(A \vee B)$, $(A \supset B)$, and $(A \equiv B)$ are defined in the usual way; $(\exists p\varepsilon\mu)A$ is defined as $\sim(\forall p\varepsilon\mu) \sim A$. $(\Sigma\alpha \subseteq \mu)A$ is introduced as an abbreviation of $\sim(\Pi\alpha \subseteq \mu) \sim A$.

A.3 Rules of Inference of the Logic of Mass Terms

(PC)   The rules of truth-functional logic.

(Refl I)   If $\Gamma \vdash (\forall q \varepsilon \mu)(\forall r \varepsilon \mu)A$, then $\Gamma \vdash (\forall p \varepsilon \mu)p \left\{ \begin{array}{c} q \\ r \end{array} \right. A$, provided that

             if p is different from q and from r then p is not free in A.

(Refl E)   If $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash {\sim} (\forall q \varepsilon \kappa)(\forall r \varepsilon \kappa)A$, then $\Gamma \vdash (\forall p \varepsilon \mu)$

             ${\sim}p \left\{ \begin{array}{c} q \\ r \end{array} \right. A$, provided that κ does not occur in Γ, μ, or A.

(Refl A) $p \left\{ \begin{array}{c} q \\ r \end{array} \right. Rqr \dashv\vdash Rpp$

($\forall$I)   If $\Gamma, q \varepsilon \mu \vdash A(q/p)$, then $\Gamma \vdash (\forall p \varepsilon \mu)A$, provided that
             (a) q is free for p in A
             (b) $A(q/p)$ results from A by replacing every free occurrence of p by q
             (c) there are no free occurrences of q in $(\forall p \varepsilon \mu)A$
             (d) there are no free occurrences of q in Γ

($\forall$E)   If $\Gamma \vdash (\forall p \varepsilon \mu)A$, then $\Gamma, q \varepsilon \mu \vdash A(q/p)$, provided that
             (a) q is free for p in A
             (b) $A(q/p)$ results from A by replacing every free occurrence of p by q
             (c) there are no free occurrences of q in $(\forall p \varepsilon \mu)A$

(ΠI)   If $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash A(\kappa/\alpha)$, then $\Gamma \vdash (\Pi \alpha \subseteq \mu)A$, provided that
             (a) $A(\kappa/\alpha)$ results from A by replacing every free occurrence of α by κ
             (b) κ does not occur in $(\Pi \alpha \subseteq \mu)A$
             (c) κ does not occur in Γ

(ΠE)   If $\Gamma \vdash (\Pi \alpha \subseteq \mu)A$, then $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash A(\kappa/\alpha)$, provided that $A(\kappa/\alpha)$ results from A by replacing every free occurrence of α by κ.

A.4 Comprehension Principle

(*Compr*) $\vdash (\exists p \varepsilon \mu)A \supset (\Sigma \alpha \subseteq \mu)(\forall p \varepsilon \mu)(p \varepsilon \alpha \equiv A)$.

(*Compl* I)   If $\Gamma \vdash p \varepsilon \mu$ and $\Gamma \vdash A$, then $\Gamma \vdash p \varepsilon \{q \in \mu \mid A(q/p)\}$ provided that there is no free variable in A besides p, and q is free for p in A.

(*Compl* E)   If $\Gamma \vdash p \varepsilon \{q \in \mu \mid A(q/p)\}$, then $\Gamma \vdash p \varepsilon \mu$ and $\Gamma \vdash A$ provided that there is no free variable in A besides p, and q is free for p in A.

A.5 Identity

(Def=) $\vdash p = q \equiv (\Pi \alpha \subseteq \omega)(p \in \alpha \equiv q \in \alpha)$.

(=E)   If $\Gamma \vdash p = q$ and $\Gamma \vdash A$, then $\Gamma \vdash p \left\{ \begin{array}{c} p \\ q \end{array} \right. A$.

A.6 Translation into Second-Order Logic   The following equivalences serve to translate any statement of QM into a statement in which first-order quantifiers occur only in atomic statements.

(a) $(\forall p\varepsilon\mu) \sim B \dashv\vdash (\Pi\alpha \subseteq \mu) \sim(\forall p\varepsilon\alpha)B$.

(b) $(\forall p\varepsilon\mu)(B \mathbin{\&} C) \dashv\vdash (\forall p\varepsilon\mu)B \mathbin{\&} (\forall p\varepsilon\mu)C$.

(c) $(\forall p\varepsilon\mu)(B \supset C) \dashv\vdash (\Pi\alpha \subseteq \mu)((\forall p\varepsilon\alpha)B \supset (\forall p\varepsilon\alpha)C)$.

(d) $(\forall p\varepsilon\mu)p \left\lvert \begin{smallmatrix} q \\ r \end{smallmatrix} \right. B \dashv\vdash (\Pi\alpha \subseteq \mu)(\Sigma\beta \subseteq \alpha)(\forall q\varepsilon\beta)(\forall r\varepsilon\beta)B$.

(e) $(\forall p\varepsilon\mu)(\forall q\varepsilon\mu')B \dashv\vdash (\forall q\varepsilon\mu')(\forall p\varepsilon\mu)B$.

(f) $(\forall p\varepsilon\mu)(\Pi\alpha \subseteq \mu')B \dashv\vdash (\Pi\alpha \subseteq \mu')(\forall p\varepsilon\mu)B$, provided α is not free in μ.

Appendix B

B.1 Axioms for the Aristotelian Continuum

Axiom 1 $(\forall p \in \omega)(\forall q \in \omega)(\forall r \in \omega)[(p < q \mathbin{\&} q < r) \supset p < r]$.

(Axiom 1′) $(\Pi\alpha \subseteq \omega)(\Pi\beta \subseteq \omega)(\Pi\gamma \in \omega)[(\alpha < \beta \mathbin{\&} \beta < \gamma) \supset \alpha < \gamma]$.

Axiom 2 $(\forall p \in \omega)(\forall q \in \omega)(p < q \vee q < p)$.

(Axiom 2′) $(\Pi\alpha \subseteq \omega)(\Pi\beta \subseteq \omega)(\Sigma\alpha' \subseteq \alpha)(\Sigma\beta' \subseteq \beta)(\alpha' < \beta' \vee \beta' < \alpha')$.

Axiom 3 $(\forall p \in \omega) \sim p < p$.

(Axiom 3′) $(\Pi\alpha \subseteq \omega) \sim \alpha < \alpha$.

Axiom 4 *If* $\Gamma \subseteq \Omega$, *then* $\bigcup \Gamma \in \Omega$.

Axiom 5 $(\Pi\alpha \subseteq \omega)(\Sigma\beta \subseteq \alpha)Interval(\beta)$.

Axiom 6 $Limited(\alpha) \equiv [(\exists q \in \omega)(\forall p \in \alpha)p < q \mathbin{\&} (\exists q \in \omega)(\forall p \in \alpha)q < p]$.

(Axiom 6′) $Limited(\alpha) \equiv [(\Sigma\beta \subseteq \omega)\alpha < \beta \mathbin{\&} (\Sigma\beta \subseteq \omega)\beta < \alpha]$.

Axiom 7 $\alpha\infty\beta$ *if and only if there are limited regions* $\alpha' \subseteq \alpha$ *and* $\beta' \subseteq \beta$ *such that if* $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ *and* $\delta = \delta_1 \cup \cdots \cup \delta_n$ *are any finite joins of intervals with* $\alpha' \subseteq \gamma$ *and* $\beta' \subseteq \delta$ *then, for at least one* $i = 1, \ldots, m$ *and one* $j = 1, \ldots, n$, $\gamma_i \cup \delta_j$ *is an interval.*

Axiom 8 *There exists a countable set* Θ *of limited intervals such that whenever* α *is a limited interval and* $\alpha \ll \beta$ *there is an interval* γ *in* Θ *with* $\alpha \ll \gamma \ll \beta$.

B.2 Lemmas The proof that any structure described by Axioms 1 to 8 has the topological characteristics of a continuum draws on the following lemmas.

Lemma 1 $\alpha\infty\beta$ *if and only if there are limited regions* $\alpha' \subseteq \alpha$ *and* $\beta' \subseteq \beta$ *with* $\alpha'\infty\beta'$.

Proof By Axiom 7. □

Lemma 2 *(a) For any* α *there are limited intervals* $\alpha' \subseteq \alpha$ *and* $\alpha'' \subseteq \alpha$ *with* $\alpha' < \alpha''$. *(b) For any* α *there are limited intervals* $\alpha' \subseteq \alpha$, $\alpha'' \subseteq \alpha$, *and* $\alpha''' \subseteq \alpha$ *with* $\alpha' < \alpha'' < \alpha'''$.

Proof (a) By Axiom 2 there are regions $\alpha' \subseteq \alpha$ and $\alpha'' \subseteq \alpha$ with $\alpha' < \alpha''$. Because of Axioms 5 and 6 α' and α'' can be assumed to be limited intervals. (b) By (a). □

Lemma 3 *If* α *and* β *are limited intervals,* $\alpha < \beta$ *and* not: $\alpha\infty\beta$, *then there exists a limited interval* γ *between* α *and* β *(i.e.,* $\alpha < \gamma < \beta$).

Proof   By Axioms 7 and 5.   □

Lemma 4   *If α is a limited interval and* not: *$\alpha\infty\beta$, then there are limited intervals γ and δ with $\gamma < \alpha < \delta$ such that for every limited interval $\xi \subseteq \beta$ either $\xi < \gamma$ or $\delta < \xi$.*

Proof   By Lemma 3 and Axiom 7.   □

Lemma 5   *If α is a limited interval, then $\alpha \ll \beta$ if and only if there are limited intervals $\gamma \subseteq \beta$ and $\delta \subseteq \beta$ with $\gamma < \alpha < \delta$.*

Proof   By Lemma 4.   □

Lemma 6   *For any $\alpha \neq 0$ there exists a limited interval β with $\beta \ll \alpha$.*

Proof   By Lemma 2(b) there are limited intervals $\alpha' \subseteq \alpha$, $\beta \subseteq \alpha$, and $\alpha'' \subseteq \alpha$ with $\alpha' < \beta < \alpha''$. $\beta \ll \alpha$ by Lemma 5.   □

Lemma 7   *If α is any region in ω, then $\alpha = \bigcup\{\beta \mid Limited(\beta)$ & $Interval(\beta)$ & $\beta \subseteq \alpha\}$.*

Proof   By Axioms 5 and 6.   □

Lemma 8   *Let β, γ be intervals and $\beta < \gamma$. Then $\alpha = \{p \in \omega \mid (\exists q \in \beta)(\exists r \in \gamma) \; q < p < r\}$ is the smallest interval covering both β and γ, that is, such that $\beta \subseteq \alpha$ and $\gamma \subseteq \alpha$.*

Lemma 9   *Let α be a limited interval and $\alpha \ll \beta$. Then there exists a limited interval ξ with $\alpha \ll \xi \ll \beta$.*

Proof   By Lemma 5 there are limited intervals $\gamma \subseteq \beta$ and $\delta \subseteq \beta$ with $\gamma < \alpha < \delta$. By Lemma 6 there are limited intervals γ' and δ' with $\gamma' \ll \gamma$ and $\delta' \ll \delta$. By Lemma 8, $\xi = \{p \in \omega \mid (\exists q \in \gamma')(\exists r \in \delta')q < p < r\}$ is the smallest interval such that $\alpha \subseteq \xi$, $\gamma' \subseteq \xi$, and $\delta' \subseteq \xi$. $\alpha \ll \xi \ll \beta$ by Lemma 5.   □

Lemma 10   *$\alpha < \beta$ if and only if there exists a left infinite interval γ with $\alpha \subseteq \gamma$ and $\beta \subseteq -\gamma$.*

Proof   Put $\gamma = \{p \in \omega \mid (\forall q \in \beta)p < q\}$.   □

Lemma 11   *If α is a left infinite interval, then there exist left infinite intervals β and γ with $\beta \subset \alpha \subset \gamma$.*

Proof   By Lemma 2 there are limited intervals $\beta' \subseteq \alpha$, $\beta'' \subseteq \alpha$, $\gamma' \subseteq -\alpha$, and $\gamma'' \subseteq -\alpha$ with $\beta' < \beta''$ and $\gamma' < \gamma''$. $\beta = \{p \in \omega \mid (\exists q \in \beta')p < q\}$ and $\gamma = \{p \in \omega \mid (\exists q \in \gamma')p < q\}$ are left infinite intervals and $\beta \subset \alpha \subset \gamma$.   □

Lemma 12   *If α is a limited interval then there exist a left infinite interval β and a right infinite interval γ such that $\alpha = \beta \cap \gamma$.*

Proof   Take $\beta = \{p \in \omega \mid (\exists q \in \alpha)p < q\}$ and $\gamma = \{p \in \omega \mid (\exists q \in \alpha)q < p\}$.
                                                                            □

B.3 Verification of the Topological Axioms   (A1), (A2), and (A3) are immediate consequences of Axiom 7.

(A4)   If $\alpha\infty\beta$ and $\beta \subseteq \gamma$ then $\alpha\infty\gamma$. (If $\alpha \ll \beta$ and $\beta \subseteq \gamma$ then $\alpha \ll \gamma$.)

Proof   Suppose $\alpha\infty\beta$ and $\beta \subseteq \gamma$. Then there are limited regions $\alpha' \subseteq \alpha$ and $\beta' \subseteq \beta$ with $\alpha'\infty\beta'$ by Lemma 1. But $\beta' \subseteq \gamma$. So $\alpha\infty\gamma$ by Lemma 1.   □

(A5)   If $\alpha\infty(\beta \cup \gamma)$ then $\alpha\infty\beta$ or $\alpha\infty\gamma$. (If $\alpha \ll \beta$ and $\alpha \ll \gamma$ then $\alpha \ll (\beta \cap \gamma)$.)

Proof   Because of Lemma 1 we may assume that α and $\beta \cup \gamma$, and hence β and γ, are limited. If neither $\alpha\infty\beta$ nor $\alpha\infty\gamma$ then there exist finite joins of limited intervals δ_1, δ_2, ξ, and θ with $\alpha \subseteq \delta_1$, $\alpha \subseteq \delta_2$, $\beta \subseteq \xi$, and $\gamma \subseteq \theta$ such that neither $\delta_1\infty\xi$ nor $\delta_2\infty\theta$ (Axiom 7). $\delta = \delta_1 \cap \delta_2$ is a finite join of intervals disconnected from ξ and θ by A4 and hence disconnected from the finite join of all those intervals that make up ξ and θ. $\alpha \subseteq \delta$; $\beta \cup \gamma \subseteq \xi \cup \theta$. Hence not: $\alpha\infty\beta \cup \gamma$.   □

(A6)   0 is limited.

Proof   By the definition of limited and the truth $(\forall p \in 0)A(p)$.   □

(A7)   If α is limited and $\beta \subseteq \alpha$ then β is limited.

Proof   By the definition of *limited*.   □

(A8)   If α and β are limited then $\alpha \cup \beta$ is limited.

Proof   By Axiom 6.   □

(A9)   If $\alpha\infty\beta$ then there is a limited region β' with $\beta' \subseteq \beta$ and $\alpha\infty\beta'$.

Proof   By Lemma 1.   □

(A10)   If α is limited, $\beta \neq 0$, and $\alpha \ll \beta$, then there is a region γ such that $\gamma \neq 0$, γ is limited, and $\alpha \ll \gamma \ll \beta$.

Proof

Case 1 $\alpha = 0$. Then there exists a limited interval γ with $\gamma \ll \beta$ by Lemma 6. $0 \ll \gamma$ by (A3).

Case 2 $\alpha \neq 0$. Suppose $\alpha \ll \beta$. By Axiom 7 there is a finite join $\xi = \xi_1 \cup \cdots \cup \xi_n$ of limited intervals such that $\alpha \subseteq \xi$ and $\xi \ll \beta$. Hence, for each $i = 1, \ldots, n$, (1) $\xi_i \ll \beta$ and (2) by Lemma 9 there is a limited interval γ_i such that $\xi_i \ll \gamma_i \ll \beta$. Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$. γ is limited, $\alpha \ll \gamma$, and $\gamma \ll \beta$ by A4 and A5.   □

B.4 Coherence and Convexity   A region α is coherent if and only if, for any non-null regions β and γ, $\beta \cup \gamma = \alpha$ implies $\beta\infty\gamma$. That α is coherent if α is an interval follows from Axiom 7. And if α is not an interval, then there is an interval β, $\alpha \cap \beta = 0$ so that there are parts of α to the left and parts to the right of β. These are not connected. Hence α is not coherent.

A region α is *convex* if and only if when β and γ are nonnull regions such that $\beta \cup \gamma = \alpha$, then there is a region α' such that $\alpha' \ll \alpha$ and $(\alpha' \cap \beta)\infty(\alpha' \cap \gamma)$. So let α be an interval, $\beta \neq 0 \neq \gamma$, and $\beta \cup \gamma = \alpha$. By Lemma 6 there are limited intervals $\delta \ll \beta$ and $\theta \ll \gamma$. Let α' be the smallest interval with $\delta \subseteq \alpha'$ and $\theta \subseteq \alpha'$ (Lemma 8). $\alpha' \ll \alpha$ by (A4). Let $\beta' = \beta \cap \alpha'$, $\gamma' = \gamma \cap \alpha'$. Then $\beta' \neq 0 \neq \gamma'$ and

$\beta' \cup \gamma' = \alpha'$. Hence α' is coherent and $\beta' \infty \gamma'$. Hence α is convex. So every interval is convex. And, as we have seen, if α is not an interval, α is not coherent, and hence not convex.

B.5 Coherence, Continuity and Countable Cover

(*Coherence*) ω is coherent. For ω is an interval.

(*Continuity*)   If β is limited and not: $\alpha \infty \beta$ then there is a region γ, which is the join $\gamma_1 \cup \cdots \cup \gamma_n$ of finitely many convex regions $\gamma_1, \ldots, \gamma_n$, such that $\beta \subseteq \gamma$ and not: $\alpha \infty \gamma$. This is proved by Axiom 7.

(*Convex Cover*)   There is a countable set Δ of limited regions such that whenever α is limited and $\alpha \ll \beta$ there is a region γ in Δ with $\alpha \ll \gamma \subseteq \beta$. For the proof we need to identify the set Δ: Let Δ be the set of all finite joins of intervals in the set Θ of Axiom 8. Assume $\alpha \ll \beta$. There is a finite join $\delta = \delta_1 \cup \cdots \cup \delta_n$ of limited intervals, $\alpha \subseteq \delta \ll \beta$ by Axiom 7. For each $i = 1, \ldots, n$ there exists $\theta_i \in \Theta$ with $\delta_i \ll \theta_i \ll \beta$ by Axiom 8. $\gamma = \theta_1 \cup \cdots \cup \theta_n \in \Delta$ by the definition of Δ. $\delta \ll \gamma$ and $\gamma \ll \beta$ by A4 and A5. Hence $\alpha \subseteq \delta \ll \gamma \ll \beta$. So, $\alpha \ll \gamma$ and $\gamma \subseteq \beta$.

Notes

1. David Hilbert's *Grundlagen der Geometrie* [2] is a classical text.

2. In a space of more than one dimension the boundaries between regions are not themselves regions in that space. For example, the 2-dimensional boundary between a spherical region in 3-dimensional space and the rest of that space is not part of that space even though it is a space in its own right.

3. Russell [7], pp. 189–90.

4. See Roeper [4] and [6] for a detailed account of the logic of mass terms.

5. For a full treatment of Region-Based Topology see Roeper [5]. Similar accounts have also been given by Mormann [3] and Forrest [1]. Region-based topology is not the only type of pointless topology in existence. More widely known is the theory developed by Sambin under the name "formal topology" (see [8]). Formal topology is a *generalization* of point-based topology. Starting with a family of sets which is an open basis for a topology, the conditions for being such a family are generalized so that they can be met by sets of elements that are not themselves sets of points. In this sense formal topology is a point-free theory. It is more abstract in that it describes structures that can be realized in all kinds of domains that may have nothing to do with spaces.

    Region-based topology, on the other hand, is, if anything, less abstract than point-based topology. It is meant to deal with, for example, volumes of 3-dimensional space, items that are more concrete than points are. The aim of region-based topology is to characterize the same structures as point-based topology does, using concepts that are more accessible than *point* and *open set of points*.

    While corresponding region-based and point-based topologies give accounts of the same structures, they differ in their ontology and therefore there is no complete correspondence between their respective properties and functions. Take the linear continuum and consider the domain of points corresponding to the real numbers. The three functions, one of them partial,

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \qquad f_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \qquad f_3(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ 1 & \text{if } x > 0 \end{cases}$$

have as counterpart just one function in the point-free domain, a function whose value is 0 on the left half-line, 1 on the right half-line. The two half-lines together make up the whole line. So the function has been fully described and is total.

A further point of difference between formal topology and region-based topology is that formal topology is intended as a constructivist topology and uses intuitionistic logic. The considerations motivating the present analysis are not epistemic but concern the constitution and structure of the line. There is therefore no reason not to use classical logic.

6. For the rules of inference of the Logic of Mass Terms see Appendix A.

7. $|a, b|$ is meant to be the interval with the points corresponding to the real numbers a and b as left and right limits. So, $|1/8, 1/4|$ is the interval limited by the points corresponding to $1/8$ and $1/4$.

8. Cf. Section 3.

9. See Roeper [5].

10. See Appendix B.

11. See Section 4.

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