Notre Dame Journal of Formal Logic
Volume 47, Number 1, 2006

Variations on a Theme of Curry

Lloyd Humberstone

Abstract After an introduction to set the stage (§1), we consider some variations on the reasoning behind Curry's Paradox arising against the background of classical propositional logic (§2) and of *BCI* logic and one of its extensions (§3), in the latter case treating the "paradoxicality" as a matter of nonconservative extension rather than outright inconsistency. A question about the relation of this extension and a differently described (though possibly identical) logic intermediate between *BCI* and *BCK* is raised in a final section (§4), which closes with a handful of questions left unanswered by our discussion.

1 Introduction

We present some familiar considerations in this section in order to fix notation and terminology. The Naïve Comprehension axiom—(NCA) below—is the axiom schema providing for every formula $\psi(x)$, with at most the variable x free, a term $\{x|\psi(x)\}$ with, for any term t,

$$t \in \{x|\psi(x)\} \leftrightarrow \psi(t), \qquad \text{(NCA)}$$

where $\psi(t)$ is the result of replacing any free occurrences of x in $\psi(x)$ with the term t. (Naïve Comprehension together with a Principle of Extensionality are taken to provide a basis for Naïve Set Theory; we do not consider Extensionality here.)

For this section and the next, we envisage for any given background logic (whose language is assumed to have—as primitive or defined—the binary connective $\leftrightarrow$ employed here, so that (NCA) is well formed) a set theory with the instances of the schema (NCA) as nonlogical axioms, and theorems derivable from them in accordance with the given background logic.[1] In Sections 3 and 4 we turn our attention to purely implicational logics, for which case (NCA) should be taken to be the pair

Received December 7, 2004; accepted July 12, 2005; printed March 22, 2006
2000 Mathematics Subject Classification: Primary, 03B47; Secondary, 03B05
Keywords: Curry's Paradox, truth-functions, fixed points, conservative extension, substructural logics, BCI logic

of schemata

$$t \in \{x|\psi(x)\} \to \psi(t) \qquad \text{and} \qquad \psi(t) \to t \in \{x|\psi(x)\}.$$

Although it is natural given the content of (NCA) to use a quantified logic in this capacity, for our purposes it is the propositional aspects of the logic that will be central.

Let $\mathsf{E} = \mathsf{E}(p, q_1, \ldots, q_n)$ be a formula constructed from exactly the propositional variables (assumed distinct) $p, q_1, \ldots, q_n$. Then for any n formulas $\varphi_1, \ldots, \varphi_n$, we have a formula $\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n)$ for which an instance of the schema (NCA) can be represented in the following form,

$$\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n) \leftrightarrow \mathsf{E}(\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n), \varphi_1, \ldots, \varphi_n)), \tag{1}$$

where, on the right, $\mathsf{E}(\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n), \varphi_1, \ldots, \varphi_n))$ is the result of substituting $\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n)$ for p and φ_i for $q_i (i = 1, \ldots, n)$ in $\mathsf{E}(p, q_1, \ldots, q_n)$. To obtain (1) from (NCA), for a given sequence $\varphi_1, \ldots, \varphi_n$ of formulas, let 'a' abbreviate the term

$$\{x|\mathsf{E}(x \in x, \varphi_1, \ldots, \varphi_n)\}$$

and observe that (2) is then an instance of (NCA):

$$a \in a \leftrightarrow \mathsf{E}(a \in a, \varphi_1, \ldots, \varphi_n). \tag{2}$$

Thus the left-hand side of (2), in which recall that the term a tacitly depends on $\varphi_1, \ldots, \varphi_n$ (as well as on the way E is constructed), may be taken as the promised $\mathsf{F}_\mathsf{E}(\varphi_1, \ldots, \varphi_n)$. This is all well known,[2] as also is the fact that the same result follows from some principles of "naïve semantics" in place of (NCA).[3] For our purposes here it doesn't matter where the various cases of (1) come from (set theory, semantics, or elsewhere) so much as what they lead to, and we may as well think of F_E (for any given E) itself as simply an n-ary sentential connective given to us as governed by (1) as an axiom-schema, whose consequences can be investigated at the level of propositional logic. (The increase in clarity from taking this point of view is analogous to that afforded by modal provability logic—cf. Boolos [7]—in which the main outlines of various metamathematical arguments become easily visible, where before they were buried in a mound of notation for Gödel numbering, provability predicates, and so on.) To emphasize this perspective, we rewrite (1), dropping the parentheses and commas of the general functional notation which suggests that its left-hand side is just some formula whose identity is fixed by those of $\varphi_1, \ldots, \varphi_n$, in the following more familiarly "connectival" style:[4]

$$\mathsf{F}_\mathsf{E}\varphi_1 \ldots \varphi_n \leftrightarrow \mathsf{E}(\mathsf{F}_\mathsf{E}\varphi_1 \ldots \varphi_n, \varphi_1, \ldots, \varphi_n). \tag{1}$$

When $n = 0$, so E is a formula $\mathsf{E}(p)$, F_E becomes a nullary connective (a sentential constant) and (1) becomes $\mathsf{F}_\mathsf{E} \leftrightarrow \mathsf{E}(\mathsf{F}_\mathsf{E})$, so that F_E can be regarded as a fixed point of the function represented by E—which it literally would be if we were attending not to the formulas themselves but to the elements of the corresponding Lindenbaum algebra in the case of any underlying logic conferring appropriate properties on the connective $\leftrightarrow$.[5] The best known subcase of this $n = 0$ case arises when $\mathsf{E}(p)$ is the formula $\neg p$, so the fixed point equivalence is $\mathsf{F}_\mathsf{E} \leftrightarrow \neg\mathsf{F}_\mathsf{E}$, making for inconsistency in the theory (in case F_E is regarded as an abbreviation) or the logic (if, as just suggested, we think of the underlying logic supplemented by (1) for each—or even for just this—E as itself a propositional logic), given certain suppositions as to how $\leftrightarrow$ and $\neg$ behave according to what we have been calling the underlying logic.

(Inconsistency here may be taken as a matter of providing for the provability of some formula and its negation, or alternatively of every formula in the language. In what follows we generally have the latter interpretation in mind.) This fixed point arises from the set-theoretical background sketched above—or any of several variants, including the untyped λ-calculus, cf. van Benthem [55], p. 51—as Russell's Paradox, and in the semantical case as the Liar Paradox. In the general case of (1), one could say that what we have is a fixed point equivalence "with parameters," though we will not explicitly be including this qualification. Our notation is intended mnemonically thus: the 'F' (or more explicitly, '$\mathsf{F_E}$') is a fixed point operator, while the 'E' is intended to suggest that the right-hand side provides an "elaboration" of its left-hand side.[6]

Let us now recall a particularly dramatic instance of (1), capable of wreaking havoc over a wide variety of background logics and requiring only the presence of an implication connective $\to$ with a few relatively uncontroversial properties. As mentioned above apropos of (NCA) for such a restricted language, (1) should be thought of as abbreviating the pair of schemata consisting of its left-to-right implication and its right-to-left implication. We have in mind Curry's Paradox, for which $n = 1$ and $\mathsf{E}(p, q_1)$, or as we might as well say, $\mathsf{E}(p, q)$, is the formula $p \to q$. (1) then becomes (3), in which, since no confusion will arise, we omit the subscript "E" on the fixed point operator:

$$\mathsf{F}\varphi \leftrightarrow (\mathsf{F}\varphi \to \varphi). \tag{3}$$

For an arbitrary formula taken as φ, this threatens to render φ derivable, since if $\to$ "contracts" according to the underlying logic from the left-to-right direction of (3) we derive the implication $\mathsf{F}\varphi \to \varphi$ in which case if a modus ponens rule holds in that logic for $\to$ we may infer $\mathsf{F}\varphi$ from that implication and the right-to-left implication of (3), whence we obtain φ by a second modus ponens from that same implication and the newly inferred $\mathsf{F}\varphi$. Thus one popular response[7] —from those wishing to save some version of naïve set theory or semantics—to Curry's Paradox (and the label has been used for the trouble we have just seen with (3), from either of these sources) has been to remove the principle of contraction (and any of its close relatives).[8] We are concerned with assessing the sources and extent of the problem here, not with suggesting or comparing solutions. In particular, to keep the discussion close to that supplied by Curry's example, we mainly concentrate on the case in which the n in (1) is 1 (except for briefly returning to the $n = 0$ case at one point in the following section, with results from Proposition 2.2 on there addressing the "arbitrary n" case.). In Section 2 we look at the possibility of "trouble" arising from sources other than that of $\mathsf{E}(p, q) = p \to q$, driving Curry's Paradox itself, while keeping the background logic as classical propositional logic. "Trouble" here can be interpreted as inconsistency or any kind of nonconservative extension, where this means the provability of some $\mathsf{F_E}$-free formula on the basis of (1), since the Post completeness of that logic means that these two coincide. (We assume that all logics are closed under Uniform Substitution. More than this is needed for the reference to Post completeness to make sense: we also need closure under a consequence relation, for which purposes that isolated in note 12 below will do—or equivalently closure under modus ponens. For further details see notes 9 and 10 of Humberstone [19] and the text to which they are appended.) In Section 3 we will weaken the logic greatly and be on the lookout for any signs of nonconservative extension coming from (1), whether or not they amount to inconsistency/triviality. (Section 4 pursues an issue raised by a logic figuring in Section 3.)

2 Classical Propositional Logic

The great advantage of classical propositional logic is that we can investigate its n-variable formulas by just studying the n-ary truth-functions, and thus in particular we can survey the effects of the $n = 1$ case of (1) in the previous section by looking at the possible truth-functional interpretations available for $\mathsf{E}(p, q)$. Since there are 16 binary truth-functions, there are 16 candidates to survey, for which purpose we employ the following notation: $\wedge, \vee, \rightarrow$, and $\leftrightarrow$ for the Boolean connectives commonly so written, with $+$ for exclusive disjunction, $\leftarrow$ for the converse of $\rightarrow$, with the composite labels $\neg\wedge$, $\wedge\neg$, for $\mathsf{E}(p, q)$ as $\neg p \wedge q$ and $p \wedge \neg q$, respectively, and $\wedge'$, $\vee'$ for negated conjunction (nand) and disjunction (nor). Of the binary truth-functions that leaves the six which are "not essentially" binary, which we notate by means of $\top$ and $\bot$ for the constant true and constant false functions, ① and ② for the first and second projections, and ①$'$ and ②$'$—for their respective negations. However, in writing formulas as opposed to labels for truth-functions, we will use only $\wedge, \vee, \rightarrow$, and $\leftrightarrow$, supplemented by the 1-ary $\neg$ (negation), translating the remaining notations into these for that purpose. (Exception: we use the above labels for the not essentially binary connectives in the first column of entries (4.11) – (4.16) below, in the interests of perspicuity.)

In Table 1 we give a list of the fixed point equivalences for each choice of $\mathsf{E}(p, q)$. In (4.1) we have the case of $p \wedge q$, for example, indicated by writing F_{E} (from (1)) as '$\mathsf{F}_{\wedge}$'. By truth-functional reasoning this can be seen to be equivalent to the entry in the second column, headed "simplified," which exhibits the formula as a truth-functional compound of $\mathsf{F}_{\mathsf{E}}\varphi$ and φ—though we omit the subscript on the 'F' to reduce clutter (since it can be recovered from the first column). In the third—"derived truth-function"—column we indicate the mode of truth-functional composition (of those arguments taken in that order) by bracketing our label for that function. The idea is that if we start with the truth-function represented in the subscript on the 'F'—that used to construct $\mathsf{E}(p, q)$ from p and q—we end up with a particular (derived) truth-function which, applied to arbitrary arguments $\mathsf{F}_{\mathsf{E}}\varphi$ and φ, in that order, yields as value the fixed point equivalence $\mathsf{F}_{\mathsf{E}}\varphi \leftrightarrow \mathsf{E}(\mathsf{F}_{\mathsf{E}}\varphi, \varphi)$. The significance of the asterisks on the numbers of some of the entries will be explained below.

The case of Curry's Paradox proper, with $\mathsf{E}(p, q) = p \rightarrow q$, appears as (4.3). We see that the derived truth-function is conjunction and recall a remark from [55], p. 50, in which 'A', 'B' are used as schematic letters for formulas: ". . . the sober-minded reader must have realized already that $(A \leftrightarrow (A \rightarrow B)) \leftrightarrow (A \wedge B)$ is a propositional tautology." The reference to sober-mindedness is van Benthem's reaction to descriptions in the literature (cited by van Benthem) of the arguments in the style of Curry and Löb as apparently "magical." Similarly (4.1) could be put in van Benthem's terms by saying that $(A \leftrightarrow (A \wedge B)) \leftrightarrow (A \rightarrow B)$ "is a propositional tautology," by which he presumably means a truth-functional tautology (a classical propositional tautology, or substitution instance thereof). Two points deserve notice concerning this way of speaking. First, while it is perfectly correct in the context of a gloss on our table, which addresses specifically classical logic, it is somewhat misleading to associate Curry's Paradox with that logic—for example, the two biconditionals just cited are equally provable in, for example, intuitionistic propositional

f.p. equivalence	simplified	derived truth-function	
$\mathsf{F}_\wedge\varphi \leftrightarrow (\mathsf{F}_\wedge\varphi \wedge \varphi)$	$\mathsf{F}\varphi \to \varphi$	$[\to]$	(4.1)
$\mathsf{F}_\vee\varphi \leftrightarrow (\mathsf{F}_\vee\varphi \vee \varphi)$	$\varphi \to \mathsf{F}\varphi$	$[\leftarrow]$	(4.2)
$\mathsf{F}_\to\varphi \leftrightarrow (\mathsf{F}_\to\varphi \to \varphi)$	$\mathsf{F}\varphi \wedge \varphi$	$[\wedge]$	(4.3)$*$
$\mathsf{F}_\leftarrow\varphi \leftrightarrow (\varphi \to \mathsf{F}_\leftarrow\varphi)$	$\mathsf{F}\varphi \vee \varphi$	$[\vee]$	(4.4)
$\mathsf{F}_\leftrightarrow\varphi \leftrightarrow (\mathsf{F}_\leftrightarrow\varphi \leftrightarrow \varphi)$	φ	$[②]$	(4.5)$*$
$\mathsf{F}_{\neg\wedge}\varphi \leftrightarrow (\neg\mathsf{F}_{\neg\wedge}\varphi \wedge \varphi)$	$\neg(\mathsf{F}\varphi \vee \varphi)$	$[\vee']$	(4.6)$*$
$\mathsf{F}_{\wedge\neg}\varphi \leftrightarrow (\mathsf{F}_{\wedge\neg}\varphi \wedge \neg\varphi)$	$\neg(\mathsf{F}\varphi \wedge \varphi)$	$[\wedge']$	(4.7)
$\mathsf{F}_{\vee'}\varphi \leftrightarrow \neg(\mathsf{F}_{\vee'}\varphi \vee \varphi)$	$\neg\mathsf{F}\varphi \wedge \varphi$	$[\neg\wedge]$	(4.8)$*$
$\mathsf{F}_{\wedge'}\varphi \leftrightarrow \neg(\mathsf{F}_{\wedge'}\varphi \wedge \varphi)$	$\mathsf{F}\varphi \wedge \neg\varphi$	$[\wedge\neg]$	(4.9)$*$
$\mathsf{F}_+\varphi \leftrightarrow (\mathsf{F}_+\varphi \leftrightarrow \neg\varphi)$	$\neg\varphi$	$[②']$	(4.10)$*$
$\mathsf{F}_{①}\varphi \leftrightarrow (\mathsf{F}_{①}\varphi ① \varphi)$	$\mathsf{F}\varphi \leftrightarrow \mathsf{F}\varphi$	$[\top]$	(4.11)
$\mathsf{F}_{②}\varphi \leftrightarrow (\mathsf{F}_{②}\varphi ② \varphi)$	$\mathsf{F}\varphi \leftrightarrow \varphi$	$[\leftrightarrow]$	(4.12)
$\mathsf{F}_{①'}\varphi \leftrightarrow (\mathsf{F}_{①'}\varphi ①' \varphi)$	$\mathsf{F}\varphi \leftrightarrow \neg\mathsf{F}\varphi$	$[\bot]$	(4.13)$*$
$\mathsf{F}_{②'}\varphi \leftrightarrow (\mathsf{F}_{②'}\varphi ②' \varphi)$	$\mathsf{F}\varphi \leftrightarrow \neg\varphi$	$[+]$	(4.14)
$\mathsf{F}_\top\varphi \leftrightarrow (\mathsf{F}_\top\varphi \top \varphi)$	$\mathsf{F}\varphi$	$[①]$	(4.15)
$\mathsf{F}_\bot\varphi \leftrightarrow (\mathsf{F}_\bot\varphi \bot \varphi)$	$\neg\mathsf{F}\varphi$	$[①']$	(4.16)

Table 1

logic.[9] (Similarly, although one could draw attention to Peirce's Law in obtaining Curry's Paradox, saying that from the $\leftarrow$ direction of (3) in Section 1 and an instance of Peirce's Law, we derive $\mathsf{F}\varphi$ by modus ponens, whence, proceeding as in the original discussion after (3), we obtain φ; this would give the misleading impression that deriving φ required an appeal to Peirce—whereas the original, entirely standard, derivation showed that only Contraction and not the intuitionistically unacceptable Peirce's Law was needed.)[10] Secondly, in the two tautologies just cited, we note that the connectives '$\to$' and '$\wedge$' have simply changed places. In terms of our table, if we start with (4.1), we find that the derived connective when we start with conjunction is implication, whereas when we start with implication (4.3), we derive conjunction. This situation is entirely representative, the derivative of the derivative being the original.

Proposition 2.1 *Where e is the equivalential truth-function (that is, $e(x, y) = T$ if and only if $x = y$) and f, g are arbitrary binary truth-functions, we have, for all $x, y \in \{\mathrm{T}, \mathrm{F}\}$,*

$$\textit{If } e(x, f(x, y)) = g(x, y) \textit{ then } e(x, g(x, y)) = f(x, y).$$

Proof Suppose (i) $e(x, f(x, y)) = g(x, y)$. Then, "e-multiplying" both sides by x, we get (ii): $e(x, e(x, f(x, y))) = e(x, g(x, y))$, and simplifying the left-hand side of (ii), since for any z, $e(x, e(x, z)) = z$, we get $f(x, y) = e(x, g(x, y))$ as required. □

It would accordingly not be out of place to describe truth-functions f and g related as in the above proposition as *dual*: given either as starting point, the other emerges as the derived function (so we are dealing with an involution of period 2). Below, we will speak of this as *Curry duality*. Note that this point in no way depends on the fact that, in the interests of sticking within the prototype provided by Curry, we restrict attention to *binary* f and g here. Proposition 2.1 holds more generally, the same proof establishing the implication (and thus also its converse) for n-ary f, g:

$$e(x, f(x, y_1, \dots, y_{n-1})) = g(x, y_1, \dots, y_{n-1}) \\ \Rightarrow e(x, g(x, y_1, \dots, y_{n-1})) = f(x, y_1, \dots, y_{n-1}).$$

Indeed, the point holds more generally still, in the form

$$e(z, f(x, y_1, \dots, y_{n-1})) = g(x, y_1, \dots, y_{n-1}) \\ \Leftrightarrow e(z, g(x, y_1, \dots, y_{n-1})) = f(x, y_1, \dots, y_{n-1}).$$

Returning to the binary case, we can put Proposition 2.1 in more linguistic dress by saying that for any definable binary connectives # and ✡, the formulas

$$(p \leftrightarrow (p\#q)) \leftrightarrow (p✡q) \qquad \text{and} \qquad (p \leftrightarrow (p✡q)) \leftrightarrow (p\#q)$$

are logically equivalent (in the sense that the biconditional linking them is a tautology); as in the observation just made the left-most occurrences of 'p' in these two formulas could be replaced by occurrences of some other variable without jeopardizing the claim of equivalence. In this incarnation the fact appealed to in the proof above to the effect that $e(x, e(x, z)) = z$, for any choice of x, z, emerges as the logical equivalence between $\varphi \leftrightarrow (\varphi \leftrightarrow \psi)$ and ψ (in the present case, going from the left formula to the right formula, φ is taken as ψ as $p\#q$)—an equivalence which notoriously fails in intuitionistic logic, revealing the availability of both (4.1) and (4.3) for that logic, noted above, to be something of a lucky accident.[11] (Indeed of course the "simplification" steps leading to column two in our table, while they hold in the two cases just mentioned, in general fail intutionistically, as in (4.4) and (4.5), the latter involving precisely the intuitionistically invalid equivalence just remarked on—to say nothing of the fact that since there are infinitely many nonequivalent formulas constructed out of p, q in intuitionistic logic, we couldn't even given a corresponding finite table in that case.)

As to the lines to which asterisks have been appended, these represent the cases in which the fixed point equivalence on that line leads to trouble, in the sense of having a consequence (according to the consequence relation of classical logic)[12] which does not contain the fixed point operator (the F of that line) and is not tautologous (not a consequence of the empty set according to the consequence relation just mentioned). (4.1) and (4.2) obviously cause no such trouble, since the fixed point operator in them can be interpreted as expressing, respectively, the constant false or the constant true 1-ary truth-function. (The identity truth-function would also serve here, in both cases.) A constant-true interpretation (of F) similarly suffices for the tautologousness of the fixed point equivalence in (4.4) and (4.15), while the constant-false interpretation serves in the case of (4.7) and (4.16). (For (4.4) we could also interpret F as negation.) The remaining unasterisked cases, (4.11), (4.12), and (4.14) are handled by interpreting F as, respectively, any truth-function whatever, the identity truth-function, and the negation function. The cases marked by an asterisk as problematic all have nontautologous F-free formulas among their classical consequences (in the sense of note 11). For (4.3), Curry's prototype, this is what is schematically represented by 'φ', which can be instantiated as a propositional variable (or by the conjunction of a propositional variable and its negation), as happens even more directly in case (4.5), as well as in (4.8), while for (4.6), (4.9), and (4.10) we have $\neg\varphi$ as a consequence, and the final case (4.13) has—as is evident from the simplified form—a tautologous negation. Thus aside from the source—(4.3)—of Curry's Paradox, in respect of its appearance in naïve set theory (or semantics) based on classical logic, we encounter six further binary truth-functional variations in the remaining asterisked cases. That trouble is threatened by more than (as we would put it) just the fixed point equivalence for $\mathsf{F}_{\to}\varphi$, the Curry prototype, has been noted before: for example, Rogerson and Butchart [41] note that the corresponding equivalence in the case of $\mathsf{F}_{\leftrightarrow}\varphi$ (line (4.5) in Table 1) is equally problematic. But it seems worthwhile setting out the precise extent of the problem insofar as the binary truth-functions are involved, as we have done here.

The relation of Curry duality interacts somewhat unpredictably with status as "problematic," in that we find sometimes neither a truth-function nor its Curry dual is problematic, sometimes both are, and sometimes one is while the other is not. The cases are surveyed here, using the bracketed labels for the truth-functions (as in our third column), marking the problematic cases with an asterisk (as in the table, but borrowed from the line number); we indicate that f and g are Curry duals by writing $f \sim g$:

$[\wedge] \sim [\to]^*$	$[\vee] \sim [\leftarrow]$	$[\leftrightarrow]^* \sim [②]$	$[\neg\wedge]^* \sim [\vee']^*$
$[\wedge\neg] \sim [\wedge']^*$	$[+]^* \sim [②']$	$[①] \sim [\top]$	$[①']^* \sim [\bot]$

Tabulating the Curry Dualities (among Binary Truth-Functions)

It is not clear whether the notion of Curry duality will turn out to be of any theoretical interest in the end. (It certainly proved its worth on a practical level, enabling the author to detect a mistake in the information tabulated as (4) above, for which an earlier draft of the present paper had the same entry in the "simplified truth-function" column (namely, $\mathsf{F}\varphi$) for both (4.15) and (4.16): the duality implies that the "derivative of" function is an injection, so that could not be right.)

One advantage of studying the effects of (NCA) by propositional logic of the fixed point operators F_{E} is that instead of taking them all on board at once, as would be the effect of (NCA), we can look at them one at a time. For any fixed background logic, we can ask what the effects are of adding the fixed point equivalence for any given (E and) F_{E}, whether this F_{E} is already definable, or can be added conservatively, and so on. For the background logic of the present section, the case of classical propositional logic, in which we have been writing '$\mathsf{F}_{\wedge}$', '$\mathsf{F}_{\to}$',. . ., for F_{E} when $\mathsf{E} = p\wedge q$, $p \to q$, and so on, what we have seen is the 16 fixed point operators (for E in two variables) fall into two disjoint classes: those whose fixed point equivalences nonconservatively extend—which in the present case amounts to "inconsistently extend"—the background logic, and those whose fixed point equivalences are already in the background logic upon a suitable truth-functional reinterpretation of the fixed point operators (automatically demonstrating conservativity). A third possibility turns out not to be realized—the possibility of an E and F_{E} for which we can add (1), of the preceding section, conservatively to classical propositional logic, and yet F_{E} is susceptible of no truth-functional interpretation.[13] Putting this in the terminology of Humberstone [19], we never obtain a consistent *contraclassical* logic—such as is provided by various (e.g., modal) examples there.[14] (Proposition 2.1 of that paper shows that the behavior of necessity, or perhaps better put, of the box operator, renders S6 a contraclassical logic. A nicer example will be recalled at the end of the present section, in which contraclassicality is consistently exhibited in a congruential logic, unlike S6.) It is even easier to see that this is also the case for E of the simpler form $\mathsf{E}(p)$ and F_{E} a nullary connective, since we have only 4 instead of the earlier 16 cases to consider. Their status as problematic or not, as well as their Curry dualities, we indicate using the above notation supplemented by labels for the singulary truth-functions, naming the identity, constant true, constant false, and negation truth-functions [ ], $[\top_1]$, $[\bot_1]$, and $[\neg]$ for this purpose: $[\,] \sim [\top_1], [\neg]^* \sim [\bot_1]$. The starred case is that of Russell/Epimenides, and in the remaining cases we have a truth-functional interpretation of F_{E}—a truth-value, that is, since F_{E} is nullary—validating the fixed point equivalences as follows: for [ ], either truth-value; for $[\top_1]$, the value T; and for $[\bot_1]$, the value F.

Our final aim for this section is to show that what we have just seen for the 1- and 2-ary cases is quite general. The result is stated as Corollary 2.3 below, and we lead up to it by working through two representative cases in which ψ is a formula in three variables of the more general observation that any formula in three variables (indeed in any number of variables, as we show in Proposition 2.2), $\psi(p, q_1, q_2)$, not just one of the special form $p \leftrightarrow \mathsf{E}(p, q_1, q_2)$, which is such that for a new binary connective #, $\psi(\#q_1q_2, q_1, q_2)$ can consistently be added as an axiom to classical propositional logic, # can be assigned a truth-functional interpretation (verifying the axiom). This then gives the result we are particularly interested in when the relevant $\psi(\#q_1q_2, q_1, q_2)$ takes the form $\#q_1q_2 \leftrightarrow \mathsf{E}(\#q_1q_2, q_1, q_2)$, for which case we have been writing $\#q_1q_2$ as $\mathsf{F}q_1q_2$ (or more fully, $\mathsf{F}_{\mathsf{E}}q_1q_2$).

Consider first $\psi(p, q_1, q_2) = (q_1 \to p) \to (p \to q_2)$, so that the axiom we'd be adding would be $(q_1 \to \#q_1q_2) \to (\#q_1q_2 \to q_2)$. We want to interpret $\#q_1q_2$ as a truth-function in such a way to make this tautologous. One starts with a truth-table to be completed:

$(q_1$	$\to$	$\#q_1q_2)$	$\to$	$(\#q_1q_2$	$\to$	$q_2)$
T			T		T	T
T			T			F
F			T		T	T
F			T			F

Here we have filled in not only the possible combinations of truth-values for the propositional variables, but also their immediate effects given the truth-function associated with $\to$, in lines 1 and 3, as well as the column of Ts under the main connective, since we are trying to arrange matters so that the whole formula comes out true in every case. Since this is already secured by the Ts we have entered in lines 1 and 3, our formula will come out true whatever truth-function is associated with #, the truth-values of $\#q_1q_2$ on these lines having no effect on the truth-value of the whole formula. In line 2, we ask if we can consistently assign the value T, say, to $\#q_1q_2$. This would make the antecedent of the formula true, since it would be an implication with a true consequent, but the consequent of the formula false since it would itself be an implication with a true antecedent and a false consequent, making the whole formula false. So we must try instead the possibility of assigning the value F to $\#q_1q_2$; now the antecedent of the whole formula would be a false implication and the formula itself would be true. So the desired function, f, call it, must satisfy $f(\mathrm{T}, \mathrm{F}) = \mathrm{F}$. Finally in line 4, the same considerations show that $\#q_1q_2$ cannot have the value T, but can consistently with verifying the whole formula, be assigned the value F; accordingly $f(\mathrm{F},\mathrm{F}) = \mathrm{F}$. So any of the four binary truth functions satisfying these two conditions can be used as a successful interpretation of #.

For a second example, consider the simpler formula $\psi(p, q_1, q_2) = (q_1 \to p) \to q_2$, so our candidate axiom would be $(q_1 \to \#q_1q_2) \to q_2$. Again one starts by constructing and then trying to complete a partial truth table, which we will not draw here, noting only that trouble strikes in the bottom line (both q_i false, that is). If we were to have $\#q_1q_2$ true here the whole formula would have a true antecedent and a false consequent, and likewise if we tried to declare $\#q_1q_2$ false. So no truth-functional interpretation is possible. But of course our would-be axiom leads to inconsistency for this very reason: its antecedent is provably implied by $\neg q_1$, which would therefore in the envisaged extension, nonconservatively and hence inconsistently, lead to $\neg q_1$'s provably implying q_2. We now verify that this is always the case. The phrase "Boolean valuation"[15] in the proof means an assignment of truth-values to all formulas of the language which associates with any Boolean connectives (all those except #, that is) a stipulated pre-assigned truth-function.

Proposition 2.2 *Let $\psi(p, q_1, \ldots, q_n)$ be any formula (in the variables exhibited) of the language of classical propositional logic in any functionally complete set of connectives and # be a new n-ary connective. Then if $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$ can consistently be added as a new axiom to classical propositional logic, there is an n-ary truth-function which when assigned as the interpretation of # renders the formula $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$ valid in the sense that $v(\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)) = \mathrm{T}$ for every Boolean valuation v associating that truth-function with #.*

Proof We check that for each of the 2^n assignments of truth-values $\{\mathrm{T}, \mathrm{F}\}$ to $q_1, \ldots, q_n$, there is at least one possible assignment to $\#q_1 \ldots q_n$, so that picking one in each case gives a truth-functional interpretation of # which validates $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$. Suppose, on the contrary, that for some assignment of truth-values to $q_1, \ldots, q_n$, there is no way of assigning a truth-value to $\#q_1 \ldots q_n$ which will make $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$ come out true. It follows that where $\sigma_i q_i$ is q_i for the cases in which the assignment in question verifies q_i and is $\neg q_i$ when the assignment falsifies q_i that

$$\sigma_1 q_1, \ldots, \sigma_n q_n \vdash_{\mathrm{CL}} \neg\psi(p, q_1, \ldots, q_n),$$

since otherwise, $\psi(p, q_1, \ldots, q_n)$ is verified by some Boolean valuation v which extends the assignment in question, in which case $v(p)$ could have been used as the value of $\#q_1 \ldots q_n$ for that case. By uniform substitution, then

$$\sigma_1 q_1, \ldots, \sigma_n q_n \vdash_{\mathrm{CL}} \neg\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n),$$

and so, contraposing, our logic with $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$ as new axiom will have the nontautologous $\neg(\sigma_1 q_1 \wedge \cdots \wedge \sigma_n q_n)$ as a theorem. □

Corollary 2.3 *If the extension of classical propositional logic by any fixed point equivalence is contraclassical, then it is inconsistent.*

Proof The envisaged extension may be taken to be the extension with an axiom involving the new symbol F, of the form $\mathsf{F}q_1 \ldots q_n \leftrightarrow \mathsf{E}(\mathsf{F}q_1 \ldots q_n, q_1, \ldots, q_n)$, where the right-hand side results from the $(n+1)$-variable formula $\mathsf{E}(p, q_1, \ldots, q_n)$

by substituting $\mathsf{F}q_1 \ldots q_n$ for p. The claimed corollary follows by taking the formula $p \leftrightarrow \mathsf{E}(p, q_1, \ldots, q_n)$ as the $\psi(p, q_1, \ldots, q_n)$ of Proposition 2.2 (with F as #). □

The potential axioms $\psi(\#q_1 \ldots q_n, q_1, \ldots, q_n)$ treated in Proposition 2.2, and not just the special cases addressed in the above corollary, have an important feature that distinguishes them from the examples in Humberstone [19] giving rise to consistent contraclassical extensions of classical logic; namely, that although they have substitution instances in which there are occurrences of # within the scope of other occurrences of #, no such embedding arises in the axioms themselves. Allowing such embedding easily gives rise to the contraclassical possibility alluded to above, for example, adding to the language a 1-ary # and to classical logic the axiom $\#\#q \leftrightarrow \neg q$ gives a consistent logic which, furthermore, remains consistent if we impose the further requirement that # is congruential in the logic (in the sense of note 5) and yet admits of no truth-functional interpretation for the new connective. Full details may be found in Section 3 of Humberstone [19].

Corollary 2.4 *Where Γ is a set of fixed point equivalences, each of which gives a consistent extension of classical propositional logic, then the extension of classical propositional logic obtained by simultaneously adding all of Γ is consistent.*

Proof Just use the truth-functional interpretations of the various fixed point operators involved, as provided by Proposition 2.2. □

It would be a mistake to look at the two unasterisked lines (4.1) and (4.16) above, for example, and say that since the former fixed point equivalence simplifies to the schema $\mathsf{F}\varphi \rightarrow \varphi$ and the latter to $\mathsf{F}\varphi$, by using them simultaneously we can infer (arbitrary) φ. The two fixed point operators here are different and the modus ponens just performed was a fallacy of equivocation. The subscripts distinguishing them were simply omitted from (4.1) – (4.16) to reduce clutter.

One aspect of the interest of Corollary 2.4 is that in general it is not the case that if a logic is conservatively extended by principles (axioms, rules) governing one new connective, and also conservatively extended by principles governing a second new connective, then the logic is conservatively extended by the simultaneous addition of both sets of principles. An example illustrating this point may be found at p. 429–30 of Humberstone [20]. As we see, however, in the present case there is no problem about such "joint conservativity."

3 Substructural Variations

As intimated in Section 1, it has been a popular pastime for logicians to conjecture or prove that this or that logic can be used as the underlying logic for a consistent (or nontrivial) version of Naïve Set Theory (or semantics)—or at least that (NCA) does not lead to the proof of every formula on the basis of the logic in question. Examples of such work include Skolem ([44], [45], [46]), Brady ([8], [9], [10]), and White ([56], [57], [58]) as well as Slaney [47] (a venture into second-order propositional logic, with a very liberal comprehension principle). To keep our bibliography within bounds we direct the reader to Petersen [34] for further references on work in this vein (itself an example of such work), especially by Akama, Chang, Grišin, and Komori, and also refer the reader to Terui [52], and references therein, for more recent work in the area inspired by Girard's Linear Logic. As was also mentioned in Section 1, much of this work concentrates on removing the axiom of contraction (W below) from a suitable axiomatization—or a corresponding rule from a natural deduction or sequent-calculus systematization—of the underlying logic. For introductory purposes we work in the axiomatic tradition (a sequent-calculus presentation appears below)[16] but concentrating on the pure implicational fragments of the logics concerned, whose axioms—given here as axiom-schemata so that the only rule to be used is modus ponens ($\varphi, \varphi \to \psi / \psi$)—are drawn from the following list:

$$
\begin{array}{ll}
B & (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi)) \\
C & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
I & \varphi \to \varphi \\
K & \varphi \to (\psi \to \varphi) \\
W & (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi).
\end{array}
$$

BCK logic, or *BCK* for short, has as axioms B, C, and K, and figures prominently in White [57] and derivatively in White [56] since the infinite-valued Łukasiewicz logic there considered is a contraction-free extension of *BCK*. This logic properly extends the similarly defined system *BCI*, and since in neither logic is W provable, the axiom corresponding in a natural way to the structural rule of contraction, both are termed substructural logics. *BCI* is substructural also in lacking K, which corresponds to the structural rule of thinning or "weakening" (on the left).[17] *BCI* is the implicational fragment of linear logic (intuitionistic or classical linear logic) while *BCIW* is the implicational fragment of the relevant logic **R**, which is known to satisfy Belnap's relevance criterion: no implicational formula whose antecedent and consequent do not share a propositional variable is provable. Thus a fortiori *BCI* satisfies this variable-sharing condition, a fact we shall make some use of below. (On the other hand *BCKW* is the implicational fragment of intuitionistic logic, and like any other extension of *BCK*, does not satisfy the condition.)

Now the fact that *BCK*, say, permits a consistent set-theory with (NCA) (or "does not trivialize naïve comprehension," as it is sometimes put) means that the system which we may regard as a propositional logic got by adding the fixed point operator F to the language and any one of the fixed point equivalences given as (1) in its second occurrence in Section 1 above (or indeed several such operators with several such equivalences) is a *consistent* extension of *BCK*. But it does not guarantee that the extension is *conservative*. There might, after all, be *some* F-free formulas provable via (1) that were not *BCK*-provable, even if not *every* formula becomes provable. That question will not be settled here, though we shall be able to supply an answer in the case of a certain implicational logic intermediate between *BCI* and *BCK*, whence the "substructural" in our section title. (Of course since *BCIW* is similarly substructural and runs foul of Curry's Paradox, much to the chagrin of those who had hoped the relevant logic **R** would double as an all-purpose paraconsistent logic—cf. Meyer et al. [29]—this aspect of the present discussion is not new. The novelty lies in searching elsewhere, among the contractionless logics[18] in particular, and in looking for nonconservativity rather than outright inconsistency.) Some would be disinclined to pursue such a question with ever weaker logics, and just their implicational fragments at that, perhaps because it would not show the consistency with (NCA) of any logic suited to play a full foundational role. But our interest here is simply in the disruptive effects the fixed point property can have. Inconsistency is only one special case, after all, of nonconservativity—so why not, then, study the general phenomenon in the simplest possible setting?

For the moment, we attend to *BCI*, for which there is a simple and elegant sequent calculus with structural rules,

$$\text{(Identity)} \quad \varphi \succ \varphi \qquad\qquad \text{(Cut)} \quad \frac{\Gamma \succ \varphi \qquad \Delta, \varphi \succ \psi}{\Gamma, \Delta \succ \psi},$$

in which we use '$\succ$' as our sequent-separator (following [5]), thinking of what lies to its left as a finite multiset of formulas (and to its right, a single formula), supplemented by operational rules for implication:

$$(\to \text{Left}) \quad \frac{\Gamma \succ \varphi \qquad \Delta, \psi \succ \chi}{\Gamma, \Delta, \varphi \to \psi \succ \chi} \qquad (\to \text{Right}) \quad \frac{\Gamma, \varphi \succ \psi}{\Gamma \succ \varphi \to \psi}.$$

The sense in which this—which is just a part of a standard sequent calculus for (intuitionistic) linear logic—is a sequent calculus version of *BCI* is given by the fact that for any multiset of formulas $\varphi_1, \ldots, \varphi_n, \psi$, the sequent $\varphi_1, \ldots, \varphi_n \succ \psi$ is provable from these rules just in case the formula $\varphi_1 \to (\varphi_2 \to \cdots \to (\varphi_n \to \psi)\cdots)$ is provable in the axiomatic system *BCI*, described above (i.e., is deducible by modus ponens from some instances of the schemata B, C, and I.)

Now a fixed point extension of axiomatic *BCI* with new axiom-schema

$$\mathsf{F}_{\mathsf{E}}\varphi \leftrightarrow \mathsf{E}(\varphi, \mathsf{F}_{\mathsf{E}}\varphi)$$

for some formula $\mathsf{E}(p, q)$—and, as remarked in Section 1, what we mean really is, "with axioms $\mathsf{F}_{\mathsf{E}}\varphi \to \mathsf{E}(\varphi, \mathsf{F}_{\mathsf{E}}\varphi)$, $\mathsf{E}(\varphi, \mathsf{F}_{\mathsf{E}}\varphi) \to \mathsf{F}_{\mathsf{E}}\varphi$"—can be matched by supplementing the above *BCI* sequent calculus with suitable rules for the new F (we drop the subscript 'E', taken as fixed for any given context): one replacing a fixed point formula $\mathsf{F}\varphi$ on the left with its elaboration $\mathsf{E}(\varphi, \mathsf{F}\varphi)$ and another making a similar replacement on the right. We call these rules (F Left) and (F Right), though note that unlike such labels as ($\to$ Right), ($\to$ Left), the rules do not insert a formula with F as main connective: they simply "deal with" F on the left or the right, respectively (what they insert being rather $\mathsf{E}(\varphi, \mathsf{F}\varphi)$), and moreover, deal with it in conformity with the usual rationale for such rules: every formula occurring in a premise-sequent for an application of one of the rules appears as a subformula of some formula occurring in the conclusion-sequent. Since the only rule lacking this property is the structural rule (Cut), the usual subformula property—that is, any provable sequent has a proof in which the only formulas to occur are subformulas of formulas in the sequent in question—is established by a demonstration that this last rule is redundant.[19] While, however, such redundancy obtains in the case of the sequent calculus for *BCI*, it is apt to fail for the fixed point extensions we are now envisaging.[20] But first, the promised rules for such extensions:

$$(\mathsf{F}\ \text{Left}) \quad \frac{\Gamma, \mathsf{F}\varphi \succ \psi}{\Gamma, \mathsf{E}(\varphi, \mathsf{F}\varphi) \succ \psi} \qquad (\mathsf{F}\ \text{Right}) \quad \frac{\Gamma \succ \mathsf{F}\varphi}{\Gamma \succ \mathsf{E}(\varphi, \mathsf{F}\varphi)}.$$

As in the case of the sequent calculus for *BCI*, this gives a corresponding sequent calculus for the extension of *BCI* by the fixed point equivalence (we are loosely writing as) $\mathsf{F}\varphi \leftrightarrow \mathsf{E}(\mathsf{F}\varphi, \varphi)$, in the sense that in the sequent calculus with (F Left) and (F Right), a sequent $\varphi_1, \ldots, \varphi_n \succ \psi$ is provable just in case the formula $\varphi_1 \to (\varphi_2 \to \cdots \to (\varphi_n \to \psi)\cdots)$ is deducible by applications of modus ponens from instances of B, C, I and implications of the form $\mathsf{F}\varphi \to \mathsf{E}(\mathsf{F}\varphi, \varphi)$ and $\mathsf{E}(\mathsf{F}\varphi, \varphi) \to \mathsf{F}\varphi$.

If for a certain E and $\mathsf{F}(= \mathsf{F}_\mathsf{E})$, a cut-elimination theorem can be proved for the extension of the *BCI* sequent calculus with the above rules, that shows that the extension is conservative, since by the subformula property any provable sequent not involving F (and so, with formulas constructed using only the connective $\to$) can be proved without appeal to (F Left), (F Right). However, as already intimated, in general no such cut-elimination property obtains.

Proposition 3.1 *Cut elimination fails for the fixed point extension of the BCI sequent calculus with the Curry fixed point operator* F_E *with* $\mathsf{E}(p, q) = p \to q$.

Proof Dropping the subscript 'E', the fixed point equivalence we are concerned with is $\mathsf{F}\varphi \leftrightarrow (\mathsf{F}\varphi \to \varphi)$, and the rules ($\mathsf{F}$ Left) and (F Right) encode its $\leftarrow$ and $\to$ directions, respectively. Consider the following proof of the sequent $\mathsf{F}p, \mathsf{F}p \succ p$, in which unlabeled steps are instances of (Identity).

$$\dfrac{\dfrac{\mathsf{F}p \succ \mathsf{F}p}{\mathsf{F}p \succ \mathsf{F}p \to p}\,(\mathsf{F}\text{ Right}) \qquad \dfrac{\mathsf{F}p \succ \mathsf{F}p \qquad p \succ p}{\mathsf{F}p, \mathsf{F}p \to p \succ p}\,(\to\text{ Left})}{\mathsf{F}p, \mathsf{F}p \succ p}\,(\text{Cut})$$

There is no cut-free proof of the end-sequent, since it is not an instance of (Identity) or a potential conclusion of an application of any of the remaining rules: (F Left), (F Right), ($\to$ Left), or ($\to$ Right). □

Although we do not, then, have the guarantee that cut elimination would give of the conservativity of the fixed point extension in this case or in other cases, the author has not been able to find a single case in which an F-free sequent not already *BCI*-provable can be proved with the aid of (F Left) and (F Right): it turns out to be remarkably difficult to "get rid of" all the Fs from a sequent involving them. We emphasize that it is a matter of getting rid of the fixed point operator as one passes down a proof and not of the corresponding left-to-right disappearance. There is no difficulty in finding examples of F for which we can prove $\varphi_1, \ldots, \varphi_n \succ \psi$ where this sequent is not a substitution instance of a *BCI*-provable sequent and ψ is F-free, one of which we give here so as to use it to make another point also.

Example 3.2 Take the fixed point equivalence $\mathsf{F}\varphi \leftrightarrow ((\varphi \to \mathsf{F}\varphi) \to \mathsf{F}\varphi)$. ($\mathsf{F}$ Left) then gives $(\varphi \to \mathsf{F}\varphi) \to \mathsf{F}\varphi \succ \mathsf{F}\varphi$ (from $\mathsf{F}\varphi \succ \mathsf{F}\varphi$), and we also have the easily proven and well-known *BCI* principle $\varphi \succ (\varphi \to \mathsf{F}\varphi) \to \mathsf{F}\varphi$. Thus we have $\varphi \succ \mathsf{F}\varphi$, from which again by familiar moves (essentially Suffixing, as in note 10) we obtain

$$\mathsf{F}\varphi \to \psi \succ \varphi \to \psi.$$

Instantiating the schematic letters as propositional variables, say in the form

$$\mathsf{F}p \to q \succ p \to q,$$

we obtain a sequent delivering an F-free formula on the right from a formula on the left, where this sequent is not a substitution instance of any sequent provable in pure implicational *BCI*. But as already emphasized, getting rid of 'F' horizontally, as here, is not getting rid of it vertically, the sequent proved being anything but F-free.

Our attempts to find a nonconservative fixed point extension of *BCI* have not been particularly systematic, and have not, for example, made any play with occurrences of the fixed point operator within its own scope. We sound one note of caution here on that score, withdrawing for this purpose from the specifically sequent calculus presentation of these extensions. Take F again as the Curry fixed point operator of Proposition 3.1. We have the equivalence (speaking abbreviatively) $\mathsf{F}\varphi \leftrightarrow (\mathsf{F}\varphi \to \varphi)$ (= (3) above) for arbitrary φ and hence in particular when φ is, say, $\mathsf{F}p$,

$$\mathsf{FF}p \leftrightarrow (\mathsf{FF}p \to \mathsf{F}p),$$

and thus, since this replacement is made within the scope only of $\to$, by the case of (3) in which φ is just plain p,

$$\mathsf{FF}p \leftrightarrow (\mathsf{FF}p \to (\mathsf{F}p \to p)).$$

What would not have been justified—so here comes the cautionary note—would have been the transition from the case of (3) with φ as p, to

$$\mathsf{FF}p \leftrightarrow \mathsf{F}(\mathsf{F}p \to p),$$

by "applying F to both sides." Such a would-be justification presumes that the fixed point operators of the extensions of *BCI* they inhabit are congruential in those logics (in the sense of note 5), which we have done nothing to secure. In fact reflection on the motivating source of the introduction of these F in Section 1, via Naïve Set Theory and (NCA), will show that the issue of their congruentiality is bound up with the Axiom of Extensionality and the relation of identity, which we have deemed outside of the scope of the present discussion. (Putting matters very informally, if we have φ and ψ provably equivalent in a *BCI*-based set theory, then the sets $\{x|x \in x \to \varphi\}$ and $\{x|x \in x \to \psi\}$—to stick with the Curry example—will be provably coextensive. Call these sets a_φ and a_ψ, respectively. For congruentiality, we want $\mathsf{F}\varphi$ and $\mathsf{F}\psi$ to be equivalent, which, before we took them as the results of applying a new sentential operator, started life (in Section 1) as the claims that $a_\varphi \in a_\varphi$ and $a_\psi \in a_\psi$, respectively, so it is the equivalence of these two that we are after. Since a_φ and a_ψ are coextensive, we do have that $a_\varphi \in a_\varphi$ and $a_\varphi \in a_\psi$ are equivalent, as are $a_\psi \in a_\psi$ and $a_\psi \in a_\varphi$; but there is nothing here to tell us that $a_\varphi \in a_\varphi$ and $a_\psi \in a_\psi$ are equivalent, for which we would need instead precisely what we do not have, that $a_\varphi = a_\psi$.) We return to the simpler kinds of cases we have been considering.

As mentioned, there was an ulterior motive behind giving Example 3.2, namely, to remark apropos of it that this particular choice of F would be a nonstarter in the search for a nonconservative extension of *BCI*, since there is already in *BCI* itself an "endogenous" candidate for $\mathsf{F}\varphi$ equivalent to $(\varphi \to \mathsf{F}\varphi) \to \mathsf{F}\varphi$—namely, take $\mathsf{F}\varphi$ as φ itself. If, by way of a contrast, we discard the consequent, and consider instead $\mathsf{F}\varphi$ equivalent to $\varphi \to \mathsf{F}\varphi$, then we obtain an equivalence which holds in neither direction in *BCI* for any implicational reconstrual of '$\mathsf{F}\varphi$', since, taking φ as p, for instance, there is no formula ψ for which either $(p \to \psi) \to \psi$ or $\psi \to (p \to \psi)$ is *BCI*-provable. The easiest way to see this is to note that *BCI* is a sublogic of classical equivalential logic—that is, interpreting the '$\to$' as the material biconditional all the theorems of *BCI* (or all the provable sequents, if one prefers that formulation) are tautologous, which of course cannot be so for either of $(p \to \psi) \to \psi$, $\psi \to (p \to \psi)$, since whatever ψ is, these formulas will have an odd number of occurrences of the variable p. The same conclusion—no endogenous candidate—applies for the same reasons in the case of the Curry fixed point operator of Proposition 3.1, whose elaboration is the converse of that just considered: nothing of the form $(\psi \to p) \to \psi$ or $\psi \to (\psi \to p)$ is *BCI*-provable. (The issue of endogenous candidates is an abstraction to the case of arbitrary logics—though here applied to *BCI*—from the issue of the possible truth-functional interpretation of the fixed point operators in the preceding section, which amounts to endogenousness in a functionally complete classical logic. The issue of Curry duality does not arise here, in view of the points made in Section 2 about the case of intuitionistic logic—and we are now even further down from classical logic than that.)

To recapitulate, then, with regard to the claim that every fixed point extension of *BCI* is conservative, we are in possession neither of a proof of the claim, nor of a counterexample to it. Accordingly, we pass to a stronger logic, between *BCI* and *BCK*, for which we can give a counterexample to the corresponding claim for that logic. As it happens, this logic raises a further question somewhat tangential to our pursuit of variations on a theme of Curry, but of some interest in its own right, and accordingly addressed in a section of its own below. For the extension we have in mind, we revert to the axiomatic presentation of *BCI* given earlier, and replace I with

$$(\varphi \to \varphi) \to (\psi \to \psi). \qquad (I^*)$$

I can be derived from I^* (with modus ponens)—for instance, put $\psi \to \psi$ for φ and we have an instance of I^* whose antecedent is also an instance of I^* and whose consequent is I (relettered). BCI^* is one of several extensions of *BCI* considered in Bunder [12]. (See further note 22 below.) It is a proper extension of *BCI*, because instantiating the schematic letters to distinct propositional variables yields a provable implication whose antecedent and consequent do not share a variable—a violation of Belnap's relevance criterion which places it outside of *BCI* (or even *BCIW*, alias $\mathbf{R}_{\to}$, the setting in which that criterion was originally wielded). Using one of the fixed point equivalences mentioned in the previous paragraph, namely,

$$\mathsf{F}\varphi \leftrightarrow (\varphi \to \mathsf{F}\varphi),$$

we can easily show that this gives a nonconservative extension of BCI^*, thereby illustrating the generalized phenomenon of Curry-paradoxicality we were after and had failed to turn up with *BCI* itself.

Proposition 3.3 *The extension of BCI^* by the fixed point equivalence above is nonconservative.*

Proof Note that, like *BCI* itself, BCI^* is a sublogic of the equivalential fragment of classical logic (reading '$\to$' as the biconditional), in which therefore, the formula $q \to (p \to p)$ is not provable. But the $\to$ direction of the equivalence above, once we permute (as we may because of C) its antecedents, is $\varphi \to (\mathsf{F}\varphi \to \mathsf{F}\varphi)$. In particular, then, the formula $q \to (\mathsf{F}q \to \mathsf{F}q)$ is provable. But I^* gives us as one of its instances

$$(\mathsf{F}q \to \mathsf{F}q) \to (p \to p),$$

so by Transitivity in the terminology of note 10 (appeal to B and modus ponens twice, that is), we have a proof of $q \to (p \to p)$ showing the nonconservativity of the fixed point extension in this case. $\square$

We remark that the formula whose provability was used to show that the extension treated in Proposition 3.3 was nonconservative is nothing but a "permuted antecedents" version of a representative instance, $p \to (q \to p)$, of the schema K,[21] so the fixed point equivalence featured there—and note that only one half of the equivalence was actually appealed to—takes us from BCI^* all the way up to BCK, at least.

4 Monothetic *BCI* and Other Open Problems

Our main concern here is with the identity of the logic BCI^* that figured in Proposition 3.3. Let us call a logic, conceived of (for definiteness, merely as a set of formulas) *monothetic* if any two formulas provable in the logic are interreplaceable in longer formulas salva provabilitate. Clearly BCK is monothetic and BCI is not—witness the BCI-unprovability I^*. By *monothetic BCI* we mean the smallest extension of BCI (identified with its set of theorems) which is monothetic. This logic, which for brevity we will call simply μBCI, can be axiomatized by means of B, C, I and, alongside modus ponens, the following further rule expressly securing that any theorems be interreplaceable (since the provability of an implication and its converse guarantees such interreplaceability for its antecedent and consequent): from premises φ, ψ, to conclusion $\varphi \to \psi$.[22] Of course I^* is provable in μBCI, since its antecedent and consequent are instances of the schema I and thus, since they are therefore provable, so is the implication connecting them, via the new rule. Thus $BCI^* \subseteq \mu BCI$. But what about the converse inclusion? That will be our concern in this section. Is the logic BCI^* none other than monothetic BCI? If not, one may ask whether μBCI can be presented as an axiomatic extension of BCI—that is, with additional axioms but no new proper rules.[23] A preliminary observation in that direction, Proposition 4.1, reducing the two-premise rule to a one-premise rule is all we shall offer here. It will set off in a promising direction for settling the question just raised as to whether $BCI^* = \mu BCI$.

Proposition 4.1 *In the axiomatization of μBCI the two-premise rule $\varphi, \psi / \varphi \to \psi$ can be replaced by the one-premise rule $\varphi / \varphi \to (\psi \to \psi)$.*

Proof In view of the provability of $\psi \to \psi$, the one-premise rule is derivable from the two-premise rule. For the other direction, suppose that φ and ψ are provable. In view of the provable BCI-equivalence of ψ with $(\psi \to \psi) \to \psi$, and the conclusion of an application of the one-premise rule, $\varphi \to (\psi \to \psi)$, we obtain the conclusion $\varphi \to \psi$ of the two-premise rule by Transitivity. □

The above proof takes us half way toward showing—if it can be shown—that $\mu BCI \subseteq BCI^*$. The BCI-equivalence mentioned, of an arbitrary formula ψ with the formula $(\psi \to \psi) \to \psi$ means that every formula is provably implied by a self-implication (an instance of the schema I, that is).[24] All we would really need to derive the inclusion in question is that every *theorem* is provably implied by a self-implication—if we also knew that every theorem provably implied a self-implication. If we had both of these results, then we could derive the rule $\varphi, \psi / \varphi \to \psi$ as follows. Suppose φ is provable and that ψ is. Then since every theorem provably implies a self-implication, $\varphi \to (\varphi_0 \to \varphi_0)$ is provable for some formula φ_0, and since every theorem is provably implied by a self-implication we have for some ψ_0, that $(\psi_0 \to \psi_0) \to \psi$.[25] By I^* we have that $\varphi_0 \to \varphi_0$ provably implies $\psi_0 \to \psi_0$, so by Transitivity we conclude that $\varphi \to \psi$ is provable, as desired.

The trouble with the above argument is of course that we *don't* know that every theorem provably implies (in *BCI*) a self-implication. The remainder of our discussion will address (without managing to fill) this gap in the argument. We begin with some extensions of *BCI* which evidently have the required property. Principally, we have in mind extensions in the same language and mention an extension not of this kind briefly first (and then again at the end of our discussion). The constant T with the axiom schema $\varphi \to T$ commonly encountered in relevant logic (and, differently notated, in linear logic) provides—or rather *BCI* in the language with it as a new nullary connective and the above formulas $\varphi \to T$ as new axioms provides—a logic in which every formula (not just every theorem) provably implies a self-implication because for any ψ we can always prove $\psi \to (T \to T)$. (Indeed, we can prove any instance of $\psi \to (\chi \to T)$. If fusion or "multiplicative conjunction," $\circ$, were present with its usual logical powers, then we could obtain this by taking the φ of the T schema as $\psi \circ \chi$. We leave it as a pleasant exercise for the reader to provide a derivation not appealing to $\circ$.) Let us return to the purely implicational language and to possible extensions of *BCI* therein.

In *BCIW*, as with the above T extension of *BCI*, not just every theorem, but every *formula* provably implies a self-implication, because we can substitute any formula for p in the *BCIW* theorem,

$$p \to [(p \to (q \to q)) \to (p \to (q \to q))], \tag{5}$$

which is not hard to prove, as is mentioned at p. 18 of Thistlewaite et al. ([53], q.v. for additional references). (In fact the authors are discussing the **R**-provability of a version of (5) with the antecedents p and $p \to (q \to q)$ permuted, and are not particularly concerned, as we are, with the provability of implications whose consequents are instances of I. Indeed they go so far as to say that its provability "is of no special interest in relevant logic," though they think of the speed with which it can be proved as an indicator of the efficiency of automated theorem-provers for nonclassical logics.) The closest the present author has been able to get to (5) in *BCI* itself, however, is (6), with of course no means of contracting the repeated antecedent:

$$p \to [(p \to (q \to q)) \to [(p \to (q \to q)) \to (p \to (q \to q))]]. \tag{6}$$

An incidental addendum is called for by the above remark that "not just every theorem but every formula" provably implies a self-implication, since in fact whenever one knows that in *BCIW* every theorem implies something of a prescribed form, one can always infer that every formula does so also, since in this logic every formula provably implies a theorem—as (5) itself shows.[26] By the considerations about odd and even occurrences of variables aired two paragraphs after Example 3.2, we know that matters stand very differently in *BCI*: nothing of the form $p \to (\psi_0 \to \psi_0)$ is provable, since such a formula would have an odd number of occurrences of 'p'. (Indeed these considerations show that more generally, in *BCI* a propositional variable can never provably imply any provable formula. Note that the T extension of *BCI* mentioned above destroys these properties.) It is for this reason that we pose the question as one of whether every *theorem*—rather than every *formula*—provably implies a self-implication in *BCI*.

Another well-known extension of *BCI*, in which, as with *BCIW*, every formula provably implies a self-implication, is the extension with the additional axiom-schema (Mingle), namely, $\varphi \to (\varphi \to \varphi)$, which secures this property by fiat in the simplest possible way. Like the logic treated in Proposition 3.3, BCI^*, *BCI* + (Mingle) is intermediate between *BCI* and *BCK*. Let us call a formula φ a *mingler*, relative to a certain logic, if that logic proves $\varphi \to (\varphi \to \varphi)$. Thus in *BCI* + (Mingle) every formula is a mingler. What about *BCI*? If, in *BCI*, all the theorems were minglers, then every theorem would imply a self-implication, solving our problem. Every instance of I is a mingler (in *BCI*—this relativization will be left implicit for the most part from now on), because we have for $\varphi \to \varphi$ the following instance of B:

$$(\varphi \to \varphi) \to ((\varphi \to \varphi) \to (\varphi \to \varphi)). \tag{7}$$

Thus all self-implications are minglers, and it is not hard to see that anything *BCI*-equivalent to a self-implication is a mingler, which includes all instances of the schema C, in view of the *BCI*-provability of (8), by which of course we mean that each of the $\to$ and $\leftarrow$ directions is provable:

$$[(p \to (q \to r)) \to (q \to (p \to r))] \leftrightarrow [(p \to (q \to r)) \to (p \to (q \to r))]. \tag{8}$$

The fact that any instance of I or C—we will get to B presently—is a mingler is due not so much to the fact that that instance is provable, being (equivalent to) a self-implication, as to the fact that it has a converse which is provable—which in this special case happens to coincide with the given formula. That is, our observations concerning I and C can be subsumed under the following generalization.

Proposition 4.2 *Any formula which is the converse of a BCI-theorem is a mingler.*

Proof One needs only to check the *BCI*-provability of formulas of the form $(\varphi \to \psi) \to [(\psi \to \varphi) \to ((\psi \to \varphi) \to (\psi \to \varphi))]$, from which the claimed result follows by appeal to modus ponens. □

Since having a provable converse is sufficient for being a mingler in *BCI*, one might also wonder whether it is necessary. A negative answer is suggested by the fact that provably equivalent formulas in general have nonequivalent converses. (9), for instance, a representative instance of a schema which is sometimes used to replace C in axiomatizations of *BCI* and *BCK* and their extensions (and is referred to as *Assertion* in the "relevant logic" tradition):

$$p \to ((p \to q) \to q), \tag{9}$$

has a *BCI*-unprovable converse but is equivalent (permuting antecedents) to the self-implication with $p \to q$ as antecedent and consequent. Thus (9), too, is a mingler, and we state the appropriate generalization as a corollary to Proposition 4.2.

Corollary 4.3 *Any formula which is BCI-equivalent to a formula with a BCI-provable converse is a mingler.*

We further illustrate this point—which would no doubt be more useful if we had a "characterization of converses" for *BCI* (as it is put in Humberstone [21], where such characterizations are supplied, inter alia, for various proper extensions of *BCI*)—with a formula related to (9), and provable from it by using $p \to p$ in place of p and then applying modus ponens, namely, (10):

$$((p \to p) \to q) \to q. \tag{10}$$

Example 4.4 We can show that (10) is a mingler, because it is equivalent to the converse of a *BCI*-theorem—in fact again to a self-implication, namely, the formula

$$((p \to p) \to q) \to ((p \to p) \to q).$$

It is clear that this formula provably implies (10), since we can permute the second '$p \to p$' to the front and then detach. That (10) in turn provably implies this formula is less obvious and to demonstrate it, we give a natural deduction proof illustrating the Lemmon-style system alluded to in note 16.

1	(1)	$((p \to p) \to q) \to q$	Assumption
2	(2)	$(p \to p) \to q$	Assumption
3	(3)	$p \to p$	Assumption
4	(4)	$p \to p$	Assumption
5	(5)	p	Assumption
3, 5	(6)	p	3, 5($\to$E)
3, 4, 5	(7)	p	4, 6($\to$E)
3, 4	(8)	$p \to p$	5–7($\to$I)
2, 3, 4	(9)	q	2, 8($\to$E)
2, 3	(10)	$(p \to p) \to q$	4–9($\to$I)
1, 2, 3	(11)	q	1, 10($\to$E)
1, 2	(12)	$(p \to p) \to q$	3–11($\to$I)
1	(13)	$((p \to p) \to q) \to ((p \to p) \to q)$	2–12($\to$I)

The claimed implication then follows by one further application of ($\to$I), discharging assumption 1.

Thus (10) is a mingler, by Corollary 4.3. (The contrary had been claimed in Example 1.13 of Avron [4].)[27] Of course, since our interest in minglers is as instances of provably implying a self-implication, we have a shorter φ than (10) itself for which (10) provably implies $\varphi \to \varphi$, namely, take φ as the antecedent of (10), by the above proof. However, to the question implicitly raised above as to whether every mingler is equivalent to the converse of some theorem, we do not have the answer.

We have found two of the three axioms in the axiomatization of *BCI* are minglers, namely, C and I, so it is only fair to report on the status of the remaining axiom. For this purpose understand by B a representative instance of the schema. (Note that by Corollary 4.3, the following implies that B is not *BCI*-equivalent to any formula with a *BCI*-provable converse.)

Proposition 4.5 *B is not a mingler (in BCI).*

Proof The countermodel-finding program MaGIC created by Slaney can be used to test the formula $B \to (B \to B)$ for derivability from the axioms of *BCI*, and in response it produces (among others) a 6-element matrix validating all of *BCI* and an assignment on which the formula assumes an undesignated value—the sole undesignated value, in fact, the remaining 5 values all being designated. We do not reproduce the invalidating matrix here (since MaGIC is freely available). □

Faced with this disappointing result, we should recall that all we wanted was for every *BCI*-theorem φ to provably imply *some* self-implication, not necessarily the implication $\varphi \to \varphi$. We still do not know whether that is so for arbitrary *BCI*-theorems, or even, indeed, for the specific case of B. Recall that our interest in this question was that an affirmative answer would show a particularly simple way that μBCI coincided with BCI^*: this was the argument involving the passage from $\varphi \to (\varphi_0 \to \varphi_0)$ and $(\psi_0 \to \psi_0) \to \psi$ to $\varphi \to \psi$ via I^*, for provable φ, ψ. We have been exploring the possibility of showing that BCI provides for every provable φ, some φ_0 for which $\varphi \to (\varphi_0 \to \varphi_0)$. Now although this may be the simplest way of trying to show that $\mu BCI \subseteq BCI^*$, it is not the only way, and in particular the strategy tries to do much of the work within BCI. Yet I^* itself could be asked to pull its weight a little more. For instance, if we had successfully shown that B, along with C and I, provably implied a self-implication, it would still not be clear how this would extend to all theorems of BCI, the property in question (of provably implying a self-implication) not being clearly preserved by modus ponens in BCI. In BCI^*, however, matters stand otherwise, as was observed by Butchart.

Proposition 4.6 *In BCI^*, modus ponens preserves the property of provably implying a self-implication.*

Proof (Butchart) Let the premises of an application of modus ponens be $\varphi \to \psi$ and φ and suppose each provably implies a self-implication in BCI^*, say $\alpha \to \alpha$ and $\beta \to \beta$, respectively. By appeal to I^*, then, we may replace $\alpha \to \alpha$ with $\varphi \to \varphi$ and also $\beta \to \beta$ with $\psi \to \psi$, meaning that $(\varphi \to \psi) \to (\varphi \to \varphi)$ and also $\varphi \to (\psi \to \psi)$ are provable. From the latter we permute antecedents to obtain $\psi \to (\varphi \to \psi)$, which, with the former (by Transitivity) gives $\psi \to (\varphi \to \varphi)$, so our modus ponens conclusion from $\varphi \to \psi$ and φ, namely, ψ, does provably imply a self-implication on the hypothesis that each of the premises does. $\square$

Thus if B provably implies a self-implication in BCI, or for that matter if it does so in BCI^*, then the latter logic is none other than μBCI. We shall end as we began with extensions of BCI, by reporting on another enrichment of the pure implicational language. Instead of adding T to the language with its own axiom schema making it provably implied by every formula, we add another sentential constant governed by a single new axiom: this constant itself. The idea behind this venture was to see if in the resulting extension of BCI, this formula provably implied a self-implication, and indeed the author began by "asking" Slaney's MaGIC (see the proof of 4.5) if it could find a countermodel in this extension of BCI to a specific candidate for provability (with a self-implication as consequent), namely, the substitution instance of (5) above obtained by replacing each of p, q, in (5), by the new constant, which will be written as H:[28]

$$H \to [(H \to (H \to H)) \to (H \to (H \to H))]. \tag{11}$$

The point of the exercise was that if some formula or other of the form $H \to (\psi \to \psi)$ turned out to be provable in the extension of BCI with H as a new axiom, then, since all that was assumed about H was its provability, this would reveal in a general and uniform way how every BCI theorem provably implies a self-implication: just replace 'H' throughout the successful '$H \to (\psi \to \psi)$' by any theorem φ, to see φ's provably implying a self-implication. Of course one has to ask this of a specific candidate ψ, and (11) embodies the hypothesis that $H \to (H \to H)$ will serve as ψ. Interestingly enough, MaGIC offers no countermodels to (11) as a theorem of $BCI + H$, suggesting either that any suitable matrix, even if finite, might be too large to lie within the space of matrices searched, or else that (11) is indeed provable. The latter seemed unlikely in view of a concerted effort by the author, who was accordingly pleased to have the matter resolved when Avron pointed out that the infinite matrix for *BCI* given in [3] invalidates (11) on an assignment which gives H a designated value. It follows that (11) is not provable in $BCI + H$. The matrix in question has as elements the integers and as designated elements the positive integers, with $x \to y = 1 - x + y$. Assigning H the value 2 then gives the undesignated value 0 to (11). (Alternatively, we can consider the integer matrix of Meyer and Slaney [30], with the same set of values, but now with all nonnegative integers designated and implication handled by the simpler $x \to y = y - x$. Indeed this special case of a generalization of Avron's procedure is noted: see Theorem 4.5 and Note 4.6 in Avron [3]. When attention is restricted, as here, to pure implicational formulas, the set of valid formulas in all cases of the generalization is the same. We continue with Avron's procedure as described above, though with suitable adjustments the argument could be conducted using the Meyer-Slaney matrix.)[29] Better still, this matrix shows that no formula of the form $H \to (\psi \to \psi)$ is provable in $BCI + H$, since $\psi \to \psi$ will always receive the value 1, so if H is assigned the (designated) value 2, $H \to (\psi \to \psi)$ receives the undesignated value $0 = (1 - 2 + 1)$, showing its unprovability (in $BCI + H$).

What does this show about whether every *BCI*-theorem provably implies a self-implication? $BCI + H$ is after all not *BCI*, and every *BCI*-theorem evaluates to 1 in Avron's matrix (is "strictly valid," as it is put in [3]), so the kind of assignment we needed above to show the unprovability of $H \to (\psi \to \psi)$, for any ψ, will not be available in *BCI* proper. So our problem about *BCI* and self-implications remains unsolved.[30] An incidental question thrown up by the discussion of $BCI + H$ and the infinite model(s) just considered arises over whether this logic has the finite model property. According to Buszkowski ([14], [15]), *BCI* itself does have the fmp—so it would be fascinating if the property were to be lost on addition of one new formula about which all that is said is that this formula is provable (though it would certainly explain why MaGIC might have failed to find a countermodel to (11)).

We conclude by collecting some of the questions that have arisen but gone unanswered in our discussion, and in particular its last two sections. Several of them take us somewhat beyond the theme of Curry's Paradox and concern structural features of *BCI* into which the pursuit of that theme had led us. (We do not repeat the two questions falling under this general heading that were posed in note 27, or the question just asked concerning $BCI + H$ and the finite model property.)

(1) Is every fixed point extension (in the sense of Section 3) of *BCI* conservative?
(2) In the remark after the proof of Proposition 3.3, we described the fixed point extension there treated as taking us "from BCI^* all the way up to *BCK*, at least." Do any fixed point extensions—for example, that considered in Proposition 3.3—of BCI^* take us nonconservatively *beyond* the logic *BCK*, and indeed, Is *BCK* itself always conservatively extended by arbitrary fixed point extensions?
(3) Is every mingler (in *BCI*) equivalent to the converse of a *BCI* theorem?
(4) Does (every instance of the schema) B provably imply a self-implication, in *BCI* (or in BCI^*)?

And more generally, and perhaps of greatest interest in its own right,

(5) Does every *BCI*-theorem provably imply a self-implication? (As we saw in Section 4, if the answer to this question is affirmative, then $BCI^* = \mu BCI$.)

Added in Press (i) The mistake in Raftery and van Alten [38] remarked on in note 30 has since been acknowledged by them: see the 2005 Corrigendum. (ii) An ingenious syntactic derivation establishing that $BCI^* = \mu BCI$, thereby settling one of our open questions, appears in a note by T. Kowalski and S. Butchart, dated 2005.

Notes

1. In Restall [39] and Rogerson and Restall [42], what we are calling the background logic is something which is extended by (NCA), the result being a kind of higher order logic. We prefer to think of matters somewhat differently, along the lines of the relationship between "pure" classical predicate logic and arbitrary first-order theories. However, this difference in outlook will vanish presently, when we abstract from the way (NCA) provides fixed points and pursue their upshot by considering sentential logics with added fixed point operators.

2. A somewhat similar formulation, though couched in terms of a rather unusual proof-system, appears as Theorem 7.1 of Petersen [34]. For bibliographical information to the seminal literature (including Curry's own contributions) we refer the reader to van Benthem [55].

3. Cf. Geach [16], van Benthem [55], Hazen [18].

4. No doubt even more familiar would be a notation in which $\mathsf{F}_\mathsf{E}\varphi_1 \ldots \varphi_n$ was written with parentheses, which we avoid here to maximize the contrast with (1) as originally formulated.

5. In particular, we have in mind the requirement that the relation $\equiv$ defined to hold between formulas φ, ψ, just in case $\varphi \leftrightarrow \psi$ is provable, should be a congruence relation on the algebra of formulas (with the various primitive connectives taken as fundamental operations), so that the (Tarski-)Lindenbaum algebra emerges as the quotient algebra modulo $\equiv$. Following, e.g., Makinson [27] we accordingly call a logic with this property *congruential*; it amounts to saying that when $\varphi \leftrightarrow \psi$ is provable, φ and ψ are freely interreplaceable in longer formulas, salva provabilitate. It is also convenient to have a more localized version of this property, and we shall say that a particular connective is congruential according to a logic if provably equivalent formulas are similarly interreplaceable within the scope of that connective. Now the condition of congruentiality is one that is not satisfied by the fixed point operators we shall consider below (see the paragraph following Example 3.2) and to that extent the "fixed point" terminology is accordingly somewhat metaphorical. (A notion of congruentiality equally worthy of the name would apply it to consequence relations $\vdash$ when $\varphi \vdash \psi$ and $\psi \vdash \varphi$ together imply that φ and ψ are interreplaceable in all consequence statements $\Gamma \vdash \chi$, however deeply embedded φ and ψ may be inside the formulas in $\Gamma \cup \{\chi\}$. This property too can be applied derivatively to individual connectives in the language of $\vdash$. But it is not the same property as that isolated by the earlier definition, the formula logic *BCI* considered in Section 3 below, being congruential, while the consequence relation we call $\vdash_{BCI}$ in note 22 below is not.)

6. Modal provability logic, already mentioned above, has its own supply of fixed points—though of a rather different nature from those considered here; see Boolos [7], esp. Chapter 8.

7. For example, Prior [37], Meyer, Routley and Dunn [29], Slaney [49]. (This last paper even blames the Sorites Paradox on contraction: p. 87.) The contraction axiom—or axiom schema, as we have stated it—appears as W in the list of labeled principles at the start of Section 3 below. In some (especially of the earlier) of our references, the word "absorption" is used in place of "contraction"—an unfortunate choice in view of the (unrelated) *absorption law* in lattice theory.

8. Restall [39] conjectured, plausibly enough in the light of work such as Shaw-Kwei [43], though as it turned out—see Rogerson and Butchart [41]—falsely in the end, that all would be well as long as no *definable* connective obeyed a modus ponens rule and also a contraction principle. It should also be noted that a reaction rather different from that of rejecting contraction is associated with Fitch; see the contributions by Myhill and (especially) Anderson in [2] for discussion and references. A more recent contribution in this general "don't blame contraction" tradition may be found in Aitken and Barrett [1].

9. Accordingly, as is noted in line 2 of the table on p. 21 of Kabziński [22], conjunction is intuitionistically definable in terms of implication and equivalence (taking the latter as primitive—something not usually done); thus any one of $\{\wedge, \rightarrow, \leftrightarrow\}$ is definable intuitionistically in terms of the other two.

10. This wording is admittedly tailored rather specifically to a context in which the contrast between classical and intuitionistic logic is especially salient. More generally, any logic with Peirce's Law but not Contraction would, as just observed, "trivialize naïve comprehension." The observation goes back to Bunder [13]; see further Rogerson and Restall [42]. This point is somewhat academic, however, since any logic closed under the rules of Suffixing ($\varphi \rightarrow \psi / (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$) and Transitivity ($\varphi \rightarrow \psi, \psi \rightarrow \chi / \varphi \rightarrow \chi$)—and thus any logic including the suffixing schema B' (= the prefixing schema B from Section 3 below, with antecedents commuted) and closed under modus ponens—delivers Contraction from Peirce, as the following argument from Meredith shows. (See Meredith and Prior [28], p. 213; we have translated the proof out of the equational presentation given there to make it clear that only the Peircean direction of the corresponding equivalence is actually used.) (1) $((p \rightarrow q) \rightarrow p) \rightarrow p$ (Peirce), so (2), from (1) by Suffixing $p \rightarrow q$: $(p \rightarrow (p \rightarrow q)) \rightarrow [((p \rightarrow q) \rightarrow p) \rightarrow (p \rightarrow q)]$; (3) $[((p \rightarrow q) \rightarrow p)) \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$ (more Peirce); then by Transitivity from (2) and (3), $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$. The striking deductive potency of Peirce's Law had been noted long before the reference last cited, in a 1948 observation of Łukasiewicz (appearing in English in pp. 306–10 of Borkowski [26] and further discussed in, e.g., Thomas [54], Sobociński [51]) and has also been noted since (e.g., in Meyer [31], which is an immediate corollary of Łukasiewicz's result).

11. On the intuitionistic nonequivalence of $\varphi \leftrightarrow (\varphi \leftrightarrow \psi)$ with ψ, see further note 28 of Humberstone [19].

12. More precisely, we have in mind here the relation—call it $\vdash_{CL}$—on any language (reference to which we do not record in the '$\vdash_{CL}$' notation) defined thus: for any set $\Gamma \cup \{\psi\}$ of formulas of that language, $\Gamma \vdash_{CL} \psi$ if and only if for some $\varphi_1, \ldots, \varphi_n \in \Gamma$ the formula $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ is a substitution instance of a truth-functional tautology. While it is usually deemed philosophically preferable to think of logics as consequence relations in the first place rather than as sets of formulas and isolate the "set of formulas" logic derivatively as the set of consequences of the empty set, we are here simply sticking with the majority of works listed in our bibliography and taking the consequence relation as secondary.

13. And we mean no interpretation according to which the logic obtained by adding (1) is sound, not just no interpretation according to which it is both sound and complete. (Cf. Humberstone [19] on translational embeddings vs. faithful translational embeddings.)

14. More explicitly, in all but the inconsistent case, we avoid, for each logic extending classical logic by one of the fixed point equivalences, ending up with a logic that is "contraclassical modulo the Boolean connectives." This means that in each case in which the extension is conservative, there is some way of translating the language with the new fixed point operator into the language with just the Boolean connectives (any functionally complete collection of which may be taken as primitive), where this translation translates the Boolean connectives by themselves, and the translations of theorems of the logic with F are all classical tautologies.

15. As in note 18 of Humberstone [19].

16. The most user-friendly proof-system for *BCI*, however, is a natural deduction system in the style of Lemmon [25], except that instead of sets of line numbers in the far left "dependency column" of a Lemmon proof, multisets are used to keep track of how many times a given assumption has been used, with the Conditional Proof rule—or as we shall call it "$\to$I(ntroduction)" when we illustrate this system in action in Example 4.4 below—removing only one occurrence of the discharged assumption's line number per application, thereby avoiding what would otherwise be tacit appeals to contraction. The corresponding elimination rule, we call $\to$E. Because the only connective we are concerned with is implication, there will be no need of the distinction between two kinds of "bunching" (due to Dunn) found in the Lemmon-style system of Slaney [49], and because the target logic is *BCI*, we will have no need either for the apparatus of structural rules to be found there.

17. For further details the monographs Restall [40] and Paoli [33] may be consulted, though the latter does not mention the *BCI*/*BCK* nomenclature (to be found in the former's §§2.7–2.8, as well as at p. 86). This nomenclature is due to Meredith, and the realization that the individual ingredients (B, C, etc.) led a double life as the principal types of (the eponymous) combinators and as well-loved implicational principles to Curry. That observation constituted the first stone in the edifice that was to become known as the Curry-Howard correspondence (or isomorphism). This is the only point in the present discussion at which we make any contact with the subject of the (coincidentally) similarly entitled Poernomo [35].

18. Though see also Slaney [48].

19. We take it to be part of the meaning of "sequent calculus" that with the exception of the Cut rule, formulas in premise-sequents survive intact or as proper subformulas of formulas in the conclusion sequent. (Many a sequent-logical proof-system accordingly fails to count as a sequent calculus, or Gentzen system, as these are often called. The distinction between sequent calculi and arbitrary sequent logics—proof systems with sequent-to-sequent rules—is not always observed, e.g., Slaney [49], p. 80, line 2.)

20. Prawitz [36] uses a normal form theorem for natural deduction proofs to obtain numerous results traditionally obtained in proof theory by an appeal to Cut Elimination for a sequent calculus and in Appendix B of that work shows that a natural deduction system

for (roughly speaking) naïve set theory proofs like that involved in Curry's Paradox cannot be reduced to normal form, which strongly suggests that there will be trouble with Cut Elimination for a corresponding sequent calculus. The exact nature of this parallel has been the subject of considerable discussion (e.g., Zucker [59]).

21. By a *representative instance* of a schema, we mean a formula instantiating the schema, in which distinct schematic letters are replaced by distinct propositional variables.

22. The extension of *BCI* by this rule is proposed in Kabziński [23] in the interests of turning the logic *BCI* into something for which the class of *BCI*-algebras would provide an appropriate algebraic semantics. Note that we give the new rule only in the course of describing an axiomatic system, which leaves open the question of what consequence relation is at issue, without which the question of whether the logic is algebraizable in the sense of Blok and Pigozzi [6] cannot be asked. Let $\vdash_{BCI}$ be the consequence relation defined by setting $\Gamma \vdash_{BCI} \varphi$ if and only if φ can be obtained by successive uses of modus ponens from instances of the schemas B, C, and I and formulas in Γ. This consequence relation, Theorem 5.9 of Blok and Pigozzi [6] shows not to be algebraizable. If we define $\Gamma \vdash_{\mu BCI} \varphi$ along similar lines but allowing not only modus ponens but also the new rule $\varphi, \psi/\varphi \to \psi$ to be used in derivations from Γ (and the axioms) we obtain the consequence relation that Kabziński has in mind, and this consequence relation is algebraizable, with its equivalent quasi-variety semantics being the class of *BCI*-algebras. The present author has in mind a different consequence relation—as is evident from the description "monothetic" (= to within interreplaceability, there is just one "thesis," or theorem)—namely, that $\vdash$ defined by saying $\Gamma \vdash \varphi$ just in case φ can be obtained by applications of modus ponens from formulas in Γ and theorems of μBCI. Thus the new rule is restricted in its application to applying to formulas provable outright. In other words—words from Smiley [50], in particular—we are considering the new rule as a rule of proof rather than a rule of inference. Since the consequences of the empty set are the same on either definition, this difference does not affect our discussion. (The author is uncertain as to whether the definition given is equivalent to defining $\vdash$ as the least consequence relation extending $\vdash_{BCI}$ and satisfying the condition that $\vdash \varphi \to \psi$, that is, $\varnothing \vdash \varphi \to \psi$, whenever $\vdash \varphi$ and $\vdash \psi$, and also as to whether this consequence relation is algebraizable, though a negative answer here seems likely in view of what Blok and Pigozzi's Theorem 5.9 says about the implicational fragment of **S5** à la Wajsberg.) Here, as in note 12, Γ ranges over *sets*, not, as in the sequent calculus of Section 3, over *multisets* of formulas; for an account of the relation between the provability of a sequent in the *BCI* sequent calculus and the consequence relation $\vdash_{BCI}$, and of an appropriate form of the Deduction Theorem for the latter relation, see Avron [3], p. 931. We add that "*BCI*- algebra" is used here in its traditional sense, as explained in Kabziński [23] and the references there cited, and not for the structures referred to by that name in Meyer and Ono [32]. Like Kabziński, Bunder also sought a closer alignment with *BCI*-algebras and suggested (Bunder [11], [12]) extending *BCI* accordingly. In the first of these papers, he considered I^* and a further schema, namely, $\varphi \to (\varphi \to (\chi \to \chi))$, while in the second, I^* disappears, presumably because it is easily derivable from the further schema, and only this schema survives. However, for the purposes of a rapprochement between logic and algebra this is not very helpful because the new schema is not valid on all *BCI*-algebras (as indeed Bunder [11] recognizes). (In both of these papers, Bunder writes in a way that conflates the threefold distinction between an individual algebra, a class of algebras, and the equational theory of a class of algebras, though his observations survive more careful reformulation.)

23. It was because of the greater simplicity of an axiomatic extension of *BCI* that *BCI** was cited in Proposition 3.3 rather than μBCI—for which the same result holds, by the same argument mutatis mutandis. Indeed, these may turn out, as we are now observing, to be the *same* logic, in which case the above use of "cited" means "cited under that description." (Even if they are distinct, the possibility would remain open of exhibiting μBCI as an axiomatic extension of *BCI*.)

24. Self-implications are often called *identities*, but as this usage might be confusing for those with an algebraic background, we avoid it here.

25. Of course, for the present case we know we can choose ψ_0 as just ψ itself, but this part of the reasoning extends to systems such as $BB'I$ (B' as in note 10) where that choice of ψ_0 can't be made but we can still find a suitable ψ_0 by taking either the antecedent or the consequent of ψ, which must be an implicational formula if it is to be provable.

26. One could already be aware of this fact independently of knowing that every formula provably implies a theorem of the type we are currently interested in—an instance of I, that is—from other examples such as the following (*BCI* and therefore) *BCIW* theorem with, thanks to W, a provable consequent: $p \to [(p \to (p \to q)) \to (p \to q)]$.

27. The mistake, which stands as an isolated assertion in Avron [4], does not affect the main results of that paper, of which the most interesting for the topic under discussion in the present paper is the following. Call a formula φ a *contractor* (in a given logic) if for all formulas ψ, the formula $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ is provable (in the logic). Then the result is that the contractors in *BCI* are precisely the theorems of *BCI*. It would be interesting to know if there is a nontheorem φ for which such an instance of W was provable *even for a single formula* ψ. A negative answer to this question would mean that the rule $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)/\varphi$ was (though not of course *derivable* in any of the proof systems we have considered) *admissible* for *BCI*. A related question arises over the admissibility of the rule $(\varphi \to \psi) \to \psi/\varphi$; using the cut-free sequent calculus for *BCI* from Section 3, the author had trouble over some cases in obtaining an affirmative answer to this question, which others more ingenious may see how to circumvent.

28. Since the first letter of the word "theorem" is reserved for a special role (whether the "Ackermann constant" t—no relation to the "t" for "term" in (NCA) as formulated in Section 1—or the "Church constant" T from earlier in this section), its second letter was pressed into service in this capacity.

29. The matrix in question is not just a matrix for the logic Meyer and Slaney consider, but a characteristic matrix—for their Abelian logic, whose implicational fragment can be axiomatized (somewhat redundantly) by adding the converse of (9), or rather (given how we have been doing things) the converse of the corresponding schema. For the extension of this system to accommodate disjunction, conjunction, and so on, the nonnegative integers, rather than just 0, have to be taken as designated elements. The implicational fragment was rediscovered in Kabziński [24]) under the potentially confusing name *BCII*, and with some rule-of-inference vs. rule-of-proof issues arising (as in note 22) which we need not go into here. The same fragment had previously been studied by Meredith in the 1950s and Kalman in the 1960s and 1970s; see the reference to Forder's axiom in line 13 on p. 221 of Meredith and Prior [28]. (Note that the Corrigendum must be consulted because of a misprint in this line.) The issues we have been concerned with for *BCI*, arising over the effect of replacing I with I^*, do not arise for "Abelian *BCI*"

since this logic is easily seen to be monothetic (like *BCK*), especially in the light of Meyer-Slaney's characteristic matrix with 0 as sole designated element.

30. After the present paper was accepted for publication, my attention was drawn by Matthew Spinks to Raftery and van Alten [38], which appears to provide a solution to this problem. The authors' Proposition 22 claims, inter alia, that the same consequence relation is characterized twice over as (i) the least extension $\vdash$ of $\vdash_{BCI}$ (see note 22 above) such that $\vdash (\varphi \to \varphi) \to (\psi \to \psi)$ for all formulas φ, ψ, and (ii) as the least extension $\vdash$ of $\vdash_{BCI}$ satisfying the condition that $\varphi, \psi \vdash \varphi \to \psi$ for all φ, ψ. That is, against the background of *BCI* logic, what we have called I^* (and Raftery and van Alten call (P)) and the rule mentioned apropos of Kabziński in note 22 (which Raftery and van Alten call (G)) are equivalent. As explained in that note, the latter is much stronger that the condition of monotheticity, which demands only that $\vdash \varphi$ and $\vdash \psi$ should imply $\vdash \varphi \to \psi$, so certainly, if Raftery and van Alten's result were correct, it would show I^* to guarantee monotheticity. But despite the putative proof offered, the result is not correct, as one may see using the Meyer-Slaney/Avron integer matrix mentioned above, with reverse subtraction interpreting $\to$ and the nonnegative integers as designated elements. This is a matrix for $\vdash_{BCI}$ on which all instances of I^* are valid, but it invalidates a representative instance of the condition $\varphi, \psi \vdash \varphi \to \psi$ obtained by taking φ and ψ as p and q, respectively: assign these variables the respective values 1 and 0, both designated, and the right-hand formula receives the undesignated value -1.

References

[1] Aitken, W., and J. A. Barrett, "Computer implication and the Curry paradox," *Journal of Philosophical Logic*, vol. 33 (2004), pp. 631–37. Zbl 1064.03005. MR 2101635. 123

[2] Anderson, A. R., R. B. Marcus, and R. M. Martin, editors, *The Logical Enterprise*, Yale University Press, New Haven, 1975. Zbl 0344.00006. 123

[3] Avron, A., "Some properties of linear logic proved by semantic methods," *Journal of Logic and Computation*, vol. 4 (1994), pp. 929–38. Zbl 0814.03006. MR 1311370. 121, 125

[4] Avron, A., "Formulas for which contraction is admissible," *Logic Journal of the IGPL*, vol. 6 (1998), pp. 43–48. Zbl 0893.03018. MR 1600444. 119, 126

[5] Blamey, S., "Partial logic," pp. 261–354 in *Handbook of Philosophical Logic*, edited by D. M. Gabbay and F. Guenthner, vol. 5, Kluwer Academic Publishers, Dordrecht, 2d edition, 2002. 112

[6] Blok, W. J., and D. Pigozzi, "Algebraizable logics," *Memoirs of the American Mathematical Society*, vol. 77 (1989), no. 396. Zbl 0664.03042. MR 0973361. 125

[7] Boolos, G., *The Logic of Provability*, Cambridge University Press, Cambridge, 1993. Zbl 0891.03004. MR 1260008. 102, 122

[8] Brady, R. T., "The consistency of the axioms of abstraction and extensionality in a three-valued logic," *Notre Dame Journal of Formal Logic*, vol. 12 (1971), pp. 447–53. Zbl 0185.01402. MR 0297535. 110

[9] Brady, R. T., "The simple consistency of a set theory based on the logic CSQ," *Notre Dame Journal of Formal Logic*, vol. 24 (1983), pp. 431–49. Zbl 0488.03026. MR 0717905. 110

[10] Brady, R. T., "The non-triviality of dialectical set theory," pp. 437–71 in *Paraconsistent Logic: Essays on the Inconsistent*, edited by G. Priest, R. Routley, and J. Norman, Analytica, Philosophia Verlag GmbH, Munich, 1989. Zbl 0691.03038. MR 1077648. 110

[11] Bunder, M. W., "The answer to a problem of Iséki on *BCI*-algebras," *Mathematics Seminar Notes, Kobe University*, vol. 11 (1983), pp. 167–69. Zbl 0545.03041. MR 0749185. 125

[12] Bunder, M. W., "*BCK* and related algebras and their corresponding logics," *The Journal of Non-Classical Logic*, vol. 2 (1983), pp. 15–24. Zbl 0558.03033. MR 0822660. 115, 125

[13] Bunder, M. W., "Tautologies that, with an unrestricted comprehension axiom, lead to inconsistency or triviality," *The Journal of Non-Classical Logic*, vol. 3 (1986), pp. 5–12. Zbl 0623.03048. MR 0892265. 123

[14] Buszkowski, W., "The finite model property for *BCI* and related systems," *Studia Logica*, vol. 57 (1996), pp. 303–23. Zbl 0861.03020. MR 1416738. 121

[15] Buszkowski, W., "Finite models of some substructural logics," *Mathematical Logic Quarterly*, vol. 48 (2002), pp. 63–72. Zbl 0992.03029. MR 1874203. 121

[16] Geach, P. T., "On insolubilia," *Analysis*, vol. 15 (1954), pp. 71–72. reprinted in [17]. 122

[17] Geach, P. T., *Logic Matters*, University of California Press, Berkeley, 1972. Zbl 0551.03002. MR 0457137. 128

[18] Hazen, A., "A variation on a paradox," *Analysis*, vol. 50 (1990), pp. 7–8. Zbl 0943.03541. MR 1042366. 122

[19] Humberstone, L., "Contra-classical logics," *Australasian Journal of Philosophy*, vol. 78 (2000), pp. 437–74. 103, 107, 110, 123, 124

[20] Humberstone, L., "The pleasures of anticipation: Enriching intuitionistic logic," *Journal of Philosophical Logic*, vol. 30 (2001), pp. 395–438. Zbl 1002.03012. MR 1867138. 110

[21] Humberstone, L., "Implicational converses," *Logique et Analyse. Nouvelle Série*, vol. 45 (2002), pp. 61–79. Zbl 1051.03012. MR 2054329. 118

[22] Kabziński, J. K., "Basic properties of the equivalence," *Studia Logica*, vol. 41 (1982), pp. 17–40. Zbl 0528.03038. MR 0706734. 123

[23] Kabziński, J. K., "*BCI*-algebras from the point of view of logic," *Bulletin of the Section of Logic*, vol. 12 (1983), pp. 126–29. Zbl 0542.03044. MR 0966391. 125

[24] Kabziński, J. K., "Abelian groups and identity connective," *Bulletin of the Section of Logic*, vol. 22 (1993), pp. 66–71. Zbl 0809.03046. MR 1238125. 126

[25] Lemmon, E. J., *Beginning Logic*, 1st edition, edited by G. W. D. Berry, Nelson, London, 1965. Zbl 0158.24406. MR 0479901. 124

[26] Łukasiewicz, J., *Selected Works*, edited by L. Borkowski, Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1970. Zbl 0212.00902. MR 0294080. 123

[27] Makinson, D., "A warning about the choice of primitive operators in modal logic," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 193–96. Zbl 0266.02016. MR 0414321. 122

[28] Meredith, C. A., and A. N. Prior, "Equational logic," *Notre Dame Journal of Formal Logic*, vol. 9 (1968), pp. 212–26. Zbl 0175.25905. MR 0246753. 123, 126

[29] Meyer, R. K., R. Routley, and J. M. Dunn, "Curry's Paradox," *Analysis*, vol. 39 (1979), pp. 124–28. Zbl 0411.03019. 111, 123

[30] Meyer, R. K., and J. K. Slaney, "Abelian logic (from A to Z)," pp. 245–88 in *Paraconsistent Logic, Essays on the Inconsistent*, edited by G. Priest, R. Routley, and J. Norman, Analytica, Philosophia Verlag GmbH, Munich, 1989. Zbl 0694.03019. MR 1077648. 121

[31] Meyer, R. K., "Peirced clean through," *Bulletin of the Section of Logic*, vol. 19 (1990), pp. 100–101. Zbl 0713.03010. MR 1078002. 123

[32] Meyer, R. K., and H. Ono, "The finite model property for *BCK* and *BCIW*," *Studia Logica*, vol. 53 (1994), pp. 107–18. Zbl 0787.03020. MR 1271133. 125

[33] Paoli, F., *Substructural Logics: A Primer*, vol. 13 of *Trends in Logic—Studia Logica Library*, Kluwer Academic Publishers, Dordrecht, 2002. Zbl 1025.03002. MR 2039844. 124

[34] Petersen, U., "Logic without contraction as based on inclusion and unrestricted abstraction," *Studia Logica*, vol. 64 (2000), pp. 365–403. Zbl 0954.03023. MR 1766138. 110, 122

[35] Poernomo, I., *Variations on a Theme of Curry and Howard: Adapting the Curry-Howard Isomorphism for the Synthesis of Imperative and Structural Programs*, Ph.D. thesis, Monash University, 2003. 124

[36] Prawitz, D., *Natural Deduction. A Proof-Theoretical Study*, vol. 3 of *Stockholm Studies in Philosophy*, Almqvist & Wiksell, Stockholm, 1965. Zbl 0173.00205. MR 0193005. 124

[37] Prior, A. N., "Curry's paradox and three-valued logic," *Australasian Journal of Philosophy*, vol. 33 (1955), pp. 177–82. 123

[38] Raftery, J. G., and C. J. van Alten, "Residuation in commutative ordered monoids with minimal zero," *Reports on Mathematical Logic*, vol. 34 (2000), pp. 23–57. Corrigendum: *Reports on Mathematical Logic*, vol. 39 (2005), pp. 133–35. Zbl 0996.03040. MR 1836321. 122, 127

[39] Restall, G., "How to be really contraction free," *Studia Logica*, vol. 52 (1993), pp. 381–91. Zbl 0793.03003. MR 1242772. 122, 123

[40] Restall, G., *An Introduction to Substructural Logics*, Routledge, London, 2000. Zbl 1028.03018. 124

[41] Rogerson, S., and S. Butchart, "Naïve comprehension and contracting implications," *Studia Logica*, vol. 71 (2002), pp. 119–32. Zbl 1003.03026. MR 1910906. 107, 123

[42] Rogerson, S., and G. Restall, "Routes to triviality," *Journal of Philosophical Logic*, vol. 33 (2004), pp. 421–36. Zbl 1067.03040. MR 2081110. 122, 123

[43] Shaw-Kwei, M., "Logical paradoxes for many-valued systems," *The Journal of Symbolic Logic*, vol. 19 (1954), pp. 37–40. Zbl 0055.00503. MR 0060437. 123

[44] Skolem, T., "A set theory based on a certain 3-valued logic," *Mathematica Scandinavica*, vol. 8 (1960), pp. 127–36. Zbl 0096.24301. MR 0126373. 110

[45] Skolem, T., "Studies on the axiom of comprehension," *Notre Dame Journal of Formal Logic*, vol. 4 (1963), pp. 162–70. Zbl 0133.25805. MR 0168453. 110

[46] Skolem, T., "Investigations on a comprehension axiom without negation in the defining propositional functions," *Notre Dame Journal of Formal Logic*, vol. 1 (1960), pp. 13–22. Zbl 0116.01002. MR 0147374. 110

[47] Slaney, J., "A simple proof of consistency for a naive axiom of propositional comprehension," *The Journal of Non-Classical Logic*, vol. 4 (1987), pp. 23–31. Zbl 0638.03020. MR 0915672. 110

[48] Slaney, J. K., "RWX is not Curry paraconsistent," pp. 472–80 in *Paraconsistent Logic: Essays on the Inconsistent*, edited by G. Priest, R. Routley, and J. Norman, Analytica, Philosophia Verlag GmbH, Munich, 1989. Zbl 0695.03013. MR 1077648. 124

[49] Slaney, J. K., "A general logic," *Australasian Journal of Philosophy*, vol. 68 (1990), p. 74–88. 123, 124

[50] Smiley, T., "Relative necessity," *The Journal of Symbolic Logic*, vol. 28 (1963), pp. 113–34. Zbl 0125.00512. MR 0175765. 125

[51] Sobociński, B., "Note about Łukasiewicz's theorem concerning the system of axioms of the implicational propositional calculus," *Notre Dame Journal of Formal Logic*, vol. 19 (1978), pp. 457–60. Zbl 0368.02010. MR 0490999. 123

[52] Terui, K., "Light affine set theory: A naïve set theory of polynomial time," *Studia Logica*, vol. 77 (2004), pp. 9–40. Zbl 1048.03041. MR 2071847. 110

[53] Thistlewaite, P. B., M. A. McRobbie, and R. K. Meyer, *Automated Theorem-Proving in Nonclassical Logics*, Research Notes in Theoretical Computer Science, Pitman Publishing Ltd., London, 1988. Zbl 0682.68097. MR 0940018. 117

[54] Thomas, I., "A proof of a theorem of Łukasiewicz," *Notre Dame Journal of Formal Logic*, vol. 12 (1971), pp. 507–508. Zbl 0205.30303. MR 0295880. 123

[55] van Benthem, J. F. A. K., "Four paradoxes," *Journal of Philosophical Logic*, vol. 7 (1978), pp. 49–72. Zbl 0377.02002. MR 0472460. 103, 104, 122

[56] White, R. B., "The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz," *Journal of Philosophical Logic*, vol. 8 (1979), pp. 509–34. Zbl 0418.03037. MR 0551284. 110, 111

[57] White, R. B., "A demonstrably consistent type-free extension of the logic *BCK*," *Mathematica Japonica*, vol. 32 (1987), pp. 149–69. Zbl 0646.03019. MR 0886208. 110, 111

[58] White, R. B., "A consistent theory of attributes in a logic without contraction," *Studia Logica*, vol. 52 (1993), pp. 113–42. Zbl 0765.03015. MR 1204880. 110

[59] Zucker, J., "The correspondence between cut-elimination and normalization. Part I. Intuitionistic predicate logic," *Annals of Mathematical Logic*, vol. 7 (1974), pp. 1–112; errata, ibid., p. 156. Zbl 0298.02023. MR 0369035. 125

Acknowledgments

I thank Arnon Avron (see the discussion surrounding (11)) and Sam Butchart (Proposition 4.6) for their assistance especially over the material in Section 4. As well as contributing Proposition 4.6, Butchart customized MaGIC to run efficiently under DOS/Windows thereby rendering it usable by me, and he used it to decide $B \rightarrow (B \rightarrow B)$—a very long formula when written out explicitly—thereby also providing Proposition 4.5. Both Butchart and Avron raised the issue about $BCI + H$ and the finite model property. I am conscious also of benefitting from many discussions over several years on Curry's Paradox, Contraction, and related matters, with Su Rogerson. Thanks, finally, to the members of the Automated Reasoning Group for their active participation in a seminar at which this material was presented (November 2004) to the Computer Sciences Laboratory seminar series at RSISE, ANU (Canberra).

Department of Philosophy
School of Philosophy and Bioethics
Monash University
Victoria 3180
AUSTRALIA
Lloyd.Humberstone@arts.monash.edu.au