Notre Dame Journal of Formal Logic
Volume 47, Number 1, 2006

An Uncountably Categorical Theory Whose Only Computably Presentable Model Is Saturated

Denis R. Hirschfeldt, Bakhadyr Khoussainov, and Pavel Semukhin

Abstract We build an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the saturated one. As a tool, we introduce a notion related to limitwise monotonic functions.

1 Introduction

An important theme in computable model theory is the study of computable models of complete first-order theories. More precisely, given a complete first-order theory T, one would like to know which models of T have computable copies and which do not. A special case of interest is when T is an $\aleph_1$-categorical theory. In this paper we are interested in computable models of $\aleph_1$-categorical theories, and we always assume that these theories are not $\aleph_0$-categorical. In addition, since we are interested in computable models, all the structures in this paper are countable.

We assume that all languages we consider are computable. A complete theory T in a language $\mathcal{L}$ is $\aleph_1$*-categorical* if any two models of T of power $\aleph_1$ are isomorphic. We say that a model $\mathcal{A}$ of T is *computable* if its domain and its atomic diagram are computable. A model $\mathcal{A}$ is *computably presentable* if it is isomorphic to a computable model, which is called a *computable presentation* of $\mathcal{A}$. The reader is referred to [2] for the basics of computable model theory and to Soare [12] for the basics of computability theory.

In [1], Baldwin and Lachlan developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They showed that the countable models of an $\aleph_1$-categorical theory T can be listed in an $\omega + 1$ chain

$$\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega,$$

Received December 19, 2004; accepted April 6, 2005; printed March 22, 2006
2000 Mathematics Subject Classification: Primary, 03045; Secondary, 03C57
Keywords: computable structure, $\aleph_1$-categoricity

where the embeddings are elementary, $\mathcal{A}_0$ is the prime model of T, and $\mathcal{A}_\omega$ is the saturated model of T. Based on the theory developed by Baldwin and Lachlan, Harrington [4] and Khisamiev [5] proved that if an $\aleph_1$-categorical theory T is decidable then all the countable models of T have computable presentations. Thus, for decidable $\aleph_1$-categorical theories the question of which models of T have computable presentations is fully settled. However, the situation is far from clear when the theory T is not decidable. The following definition is given in [9].

Definition 1.1 Let T be an $\aleph_1$-categorical theory and let $\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega$ be the countable models of T. The *spectrum of computable models* of T is the set $\{i : \mathcal{A}_i \text{ has a computable presentation}\}$. If $X \subseteq \omega + 1$ is the spectrum of computable models of some $\aleph_1$-categorical theory, then we say that X is *realized as a spectrum*.

There has been some previous work on the possible spectra of computable models of (undecidable) $\aleph_1$-categorical theories. For example, Nies [11] gave an upper bound of $\Sigma^0_3(\varnothing^\omega)$ for the complexity of the sets realized as spectra. Interestingly, the following are the only subsets of $\omega + 1$ known to be realizable as spectra: the empty set, $\omega + 1$ itself ([4], [5]), the initial segments $\{0, \ldots, n\}$, where $n \in \omega$ ([3], [10]), the sets $(\omega + 1) \setminus \{0\}$ and ω ([9]), and the intervals $\{1, \ldots, n\}$, where $n \in \omega$ ([11]). Our main result adds $\{\omega\}$ to this list by showing that there exists an $\aleph_1$-categorical theory whose only computably presentable model is the saturated one.

This paper is organized as follows. Section 2 contains the proof of a computability-theoretic result that will be used in constructing the desired theory. In Section 3 we introduce the basic building blocks of the models of this theory, which are called cubes. Finally, Section 4 contains the proof our main result.

2 A Computability-Theoretic Result

Limitwise monotonic functions were introduced by Khisamiev ([6], [7], [8]) and have found a number of applications in computable model theory. In particular, Khoussainov, Nies, and Shore [9] used them to show that $(\omega + 1) \setminus \{0\}$ is realized as a spectrum. We now introduce a related notion.

Let $[\omega]^{<\omega}$ denote the collection of all finite sets of natural numbers, and let ∞ be a special symbol. We define the class of S-limitwise monotonic functions from ω to $[\omega]^{<\omega} \cup \{\infty\}$, where S is an infinite set. This class captures the idea of a family $A_0, A_1, \ldots$ of uniformly c.e. sets, each of which is either finite or equal to S (represented by the symbol ∞), such that we can enumerate the set of i for which $A_i = S$.

Definition 2.1 Let S be an infinite set of natural numbers. An *S-limitwise monotonic function* is a function $f : \omega \to [\omega]^{<\omega} \cup \{\infty\}$ for which there is a computable function $g : \omega \times \omega \to [\omega]^{<\omega} \cup \{\infty\}$ such that

1. $f(n) = \lim_s g(n, s)$ for all n, and
2. for all $n, s \in \omega$, the following properties hold:
   (a) if $g(n, s + 1) \neq \infty$ then $g(n, s) \subseteq g(n, s + 1)$,
   (b) if $g(n, s) = \infty$ then $g(n, s + 1) = \infty$, and
   (c) if $g(n, s) \neq \infty$ and $g(n, s + 1) = \infty$ then $g(n, s) \subset S$.

We refer to g as a *witness* to f being S-limitwise monotonic.

Note that if f is an S-limitwise monotonic function then its witness g can be chosen to be primitive recursive.

Definition 2.2 A collection of finite sets is S-*monotonically approximable* if it is equal to $\{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f.

The main result of this section is the following computability-theoretic proposition, which shows that there is an infinite set S and a family of sets that is not S-monotonically approximable and has certain properties that will allow us to code it into a model of an $\aleph_1$-categorical structure.

Proposition 2.3 *There exists an infinite c.e. set S and uniformly c.e. sets $A_0, A_1, \ldots$ with the following properties:*

1. *each A_i is either finite or equal to S,*
2. *if $x \in S$ then $x \in A_i$ for almost all i,*
3. *if $x \notin S$ then $x \in A_i$ for only finitely many i,*
4. *if A_i is finite then there is a $k \in A_i$ such that $k \notin A_j$ for all $j \neq i$, and*
5. *$\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable.*

Proof Let $g_0, g_1, \ldots$ be an effective enumeration of all primitive recursive functions from $\omega \times \omega$ to $\omega^{<\omega} \cup \{\infty\}$ such that for all $n, s \in \omega$, if $g_e(n, s+1) \neq \infty$ then $g(n,s) \subseteq g(n, s+1)$, and if $g(n,s) = \infty$ then $g(n, s+1) = \infty$.

We want to build S and $A_0, A_1, \ldots$ to satisfy (1)–(4) and the requirements $\mathcal{R}_e$ stating that if g_e is a witness to some function f being S-limitwise monotonic, then $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable via f.

For each e, we define a procedure for enumerating A_e. We think of the procedures as alternating their steps, with the eth procedure taking place at stages of the form $\langle e, k\rangle$, which we call *e-stages*. All procedures may enumerate elements into S. The eth procedure is designed to satisfy $\mathcal{R}_e$ by ensuring that if g_e is a witness to some function f being S-limitwise monotonic and every $f(n) \neq \infty$ is equal to some A_i, then A_e is finite and not equal to $f(n)$ for any n. The eth procedure works as follows.

Let $A_e[s]$ and $S[s]$ denote the set of all numbers enumerated into A_e and S, respectively, by the end of stage s. The main idea is to find an appropriate number n_e such that if $\lim_s g_e(n,s) = A_e$ for some n then $n = n_e$, and let $A_e[s]$ always contain an element not in $g_e(n_e, s)$, thus ensuring that either A_e is finite but $\lim_s g_e(n_e, s) \neq A_e$ or $g_e(n_e, s)$ is eternally playing catch-up, and hence does not come to a limit.

At the first e-stage s, put $\langle e, 0\rangle$, $\langle e, 1\rangle$, and all elements of $S[s]$ into A_e. Let $m_{e,s} = 1$ and let n_e be undefined. (For each e-stage t, we will let $m_{e,t}$ be the largest m such that $\langle e, m\rangle \in A_e[t]$.)

At any other e-stage s, proceed as follows. Let t be the previous e-stage. If n_e is undefined and there is an $n \leqslant s$ such that $g_e(n,s) = A_e[t]$, then let $n_e = n$. If n_e is now defined and $g_e(n_e, s) = A_e[t]$ then put $\langle e, m_{e,t} - 1\rangle$ into S, put $\langle e, m_{e,t} + 1\rangle$ and all elements of $S[s]$ into A_e, and let $m_{e,s} = m_{e,t} + 1$. Otherwise, let $m_{e,s} = m_{e,t}$ and do nothing else.

This finishes the description of the eth procedure. Running all procedures concurrently, as described above, we build a uniformly c.e. collection of sets $A_0, A_1, \ldots$ and a c.e. set S. Now our goal is to show that these sets satisfy the properties in the statement of the proposition.

Since at every stage s at which we put numbers into A_e, we put $S[s]$ into A_e and the second largest element of $A_e[s-1]$ into S, every infinite A_e is equal to S. This shows that the first property in the proposition holds.

Since for each e we put $S[s]$ into A_e, where s is the first e-stage, every element of S is in cofinitely many A_e. This shows that the second property in the proposition holds.

Since the only way a number of the form $\langle e, k\rangle$ can enter A_i for $i \neq e$ is if it first enters S, every number that is in infinitely many A_i must be in S. This shows that the third property in the proposition holds.

If A_e is finite, then $m = \lim_s m_{e,s}$ exists, and $\langle e, m\rangle$ is in A_e but not in A_j for $j \neq e$. This shows that the fourth property in the proposition holds.

We now show that the last property in the proposition holds. Assume for a contradiction that $\{A_i : |A_i| < \omega\} = \{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f witnessed by g_e. Then n_e must eventually be defined, since otherwise A_e is finite but not in the range of f.

First suppose that $f(n_e) \neq \infty$. At the e-stage s_0 at which n_e is defined, $g_e(n_e, s_0)$ contains $\langle e, 0\rangle$ and $\langle e, 1\rangle$. If there is no e-stage $s_1 > s_0$ at which $g_e(n_e, s_1) = A_e[s_0]$, then $f(n_e)$ cannot equal any of the A_i, since A_e is then the only one of our sets that contains $\langle e, 1\rangle$, and $\langle e, 1\rangle \in g_e(n_e, s_0)$. So there must be such an e-stage s_1. Note that $g_e(n_e, s_1)$ contains $\langle e, 2\rangle$. By the same argument, there must be an e-stage $s_2 > s_1$ such that $g_e(n_e, s_2) = A_e[s_1]$, and this set contains $\langle e, 3\rangle$. Proceeding in this way, we see that $g_e(n_e, s)$ never reaches a limit.

Now suppose that $f(n_e) = \infty$. Let s_0 be the least s such that $g_e(n_e, s) = \infty$, and let t be the largest e-stage less than s_0. It is easy to check that $\left\langle e, m_{e,t} - 1\right\rangle \in g(n_e, t)$ but $\left\langle e, m_{e,t} - 1\right\rangle \notin S[t]$. We never put $\left\langle e, m_{e,t} - 1\right\rangle$ into S after stage t, so in fact $\left\langle e, m_{e,t} - 1\right\rangle \notin S$. Since $g_e(n_e, t) \subseteq g_e(n_e, s_0 - 1)$, we have $g_e(n_e, s_0 - 1) \not\subset S$, contradicting the choice of g_e. $\square$

3 Cubes

In this section we introduce a special family of structures which we call cubes. These will be used in the next section to build an $\aleph_1$-categorical theory. They generalize the n-cubes and ω-cubes used in [9].

We work in the language $\mathcal{L} = \{P_i : i \in \omega\}$, where each P_i is a binary predicate symbol. We will define structures for sublanguages $\mathcal{L}'$ of $\mathcal{L}$. Any such structure can be thought of as an $\mathcal{L}$-structure by interpreting the P_i not contained in $\mathcal{L}'$ by the empty set. We denote the domain of a structure denoted by a calligraphic letter such as $\mathcal{A}$ by the corresponding roman letter A.

We begin with the following inductive definition of the finite cubes.

Definition 3.1 *Base case*. For $n \in \omega$, an (n)-*cube* is a structure $\mathcal{A} = (\{a, b\}; P_n^{\mathcal{A}})$, where $P_n^{\mathcal{A}}(x, y)$ holds if and only if $x \neq y$.

Inductive Step. Now suppose we have defined σ-cubes for a nonrepeating sequence $\sigma = (n_1, \ldots, n_k)$, and let $n_{k+1} \notin \sigma$. An $(n_1, \ldots, n_k, n_{k+1})$-*cube* is a structure $\mathcal{C}$ defined in the following way. Take two σ-cubes $\mathcal{A}$ and $\mathcal{B}$ such that $A \cap B = \varnothing$ and let $f : \mathcal{A} \to \mathcal{B}$ be an isomorphism. Let $\mathcal{C}$ be the structure

$$(A \cup B; P_{n_1}^{\mathcal{A}} \cup P_{n_1}^{\mathcal{B}}, \ldots, P_{n_k}^{\mathcal{A}} \cup P_{n_k}^{\mathcal{B}}, P_{n_{k+1}}^{\mathcal{C}}),$$

where $P_{n_{k+1}}^{\mathcal{C}}(x, y)$ holds if and only if $f(x) = y$ or $f^{-1}(x) = y$.

Example 3.2 Let σ be a finite nonrepeating sequence. Consider $A = \mathbb{Z}_2^{|\sigma|}$ as a vector space over $\mathbb{Z}_2$, with basis $b_1, \ldots, b_{|\sigma|}$. If we define the structure $\mathcal{A}$ with domain A by letting $P^{\mathcal{A}}_{\sigma(i)}(x, y)$ if and only if $x + b_i = y$, then $\mathcal{A}$ is a σ-cube.

The following property of finite cubes, which is easily checked by induction, shows that we could have taken Example 3.2 as the definition of a σ-cube.

Lemma 3.3 *Let σ be a finite nonrepeating sequence. Any two σ-cubes are isomorphic.*

Furthermore, we have the following stronger property.

Lemma 3.4 *If σ is a finite nonrepeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof Let $\mathcal{A}$ and $\mathcal{B}$ be a σ-cube and a τ-cube, respectively. By Lemma 3.3, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are constructed as in Example 3.2. Since τ is a permutation of σ, there is a bijection f such that $\sigma(i) = \tau(f(i))$. Let φ be the vector space isomorphism induced by taking b_i to $b_{f(i)}$. We then have

$$\begin{aligned} &P^{\mathcal{A}}_{\sigma(i)}(x, y) \text{ iff } x + b_i = y \text{ iff } \varphi(x) + \varphi(b_i) = \varphi(y) \\ &\qquad \text{iff } \varphi(x) + b_{f(i)} = \varphi(y) \text{ iff } P^{\mathcal{B}}_{\tau(f(i))}(\varphi(x), \varphi(y)) \text{ iff } P^{\mathcal{B}}_{\sigma(i)}(\varphi(x), \varphi(y)). \end{aligned}$$

Thus φ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. $\square$

So instead of "σ-cube", where $\sigma = (n_1, \ldots, n_k)$, we will write "$A$-cube", where $A = \{n_1, \ldots, n_k\}$. (This notation matches that of [9], if we make the usual set-theoretic identification of n with $\{0, \ldots, n-1\}$.)

We now define infinite cubes.

Definition 3.5 Let $\alpha = (n_0, n_1, \ldots)$ be an infinite nonrepeating sequence of natural numbers. An *α-cube* is a structure of the form $\bigcup_{i \in \omega} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an $\{n_0, \ldots, n_i\}$-cube, and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$.

As with finite sequences, the order of an infinite sequence α does not affect the isomorphism type of α-cubes, so we can talk about S-cubes, where S is an infinite set. To show that this is the case, we will use the following fact, which is easy to check. Suppose that $A \subset B \subset C$ are finite, $\mathcal{Z}$ is a C-cube, and $\mathcal{X} \subset \mathcal{Z}$ is an A-cube. Then there exists a B-cube $\mathcal{Y}$ such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$.

Lemma 3.6 *If σ is an infinite nonrepeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof Let $\sigma = (m_0, m_1, \ldots)$ be an infinite nonrepeating sequence, and let $\tau = (n_0, n_1, \ldots)$ be a permutation of σ. Let $s_i = \{m_0, \ldots, m_i\}$ and $t_i = \{n_0, \ldots, n_i\}$.

Let $\mathcal{A}$ be a σ-cube and let $\mathcal{B}$ be a τ-cube. Then $\mathcal{A} = \bigcup_{i \in \omega} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an s_i-cube, and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. Similarly, $\mathcal{B} = \bigcup_{i \in \omega} \mathcal{B}_i$, where each $\mathcal{B}_i$ is a t_i-cube, and $\mathcal{B}_i \subset \mathcal{B}_{i+1}$.

We build a sequence of finite partial isomorphisms $\varphi_0 \subseteq \varphi_1 \subseteq \cdots$ such that $A_i \subseteq \operatorname{dom} \varphi_{2i+1}$ and $B_i \subseteq \operatorname{rng} \varphi_{2i+2}$. We begin with $\varphi_0 = \varnothing$.

Given φ_{2i}, let $k \geqslant i$ be such that $A_k \supseteq \operatorname{dom} \varphi_{2i}$, and let l be such that $B_l \supseteq \operatorname{rng} \varphi_{2i}$ and $s_k \subseteq t_l$. Then there is an s_k-cube $\mathcal{C} \subseteq \mathcal{B}_l$ such that $\operatorname{rng} \varphi_{2i} \subseteq C$. Extend φ_{2i} to an isomorphism $\varphi_{2i+1} : \mathcal{A}_k \to \mathcal{C}$.

Given φ_{2i+1}, proceed in an analogous fashion to define a finite partial isomorphism φ_{2i+2} including B_i in its range. Now $\varphi = \bigcup_{i\in\omega} \varphi_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

4 The Main Theorem

In this section we prove the main result of this paper.

Theorem 4.1 *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the saturated one.*

Proof Let $\{A_i\}_{i\in\omega}$ and S be as in Proposition 2.3. Fix an enumeration of $\{A_i\}_{i\in\omega}$ such that at each stage exactly one element is enumerated into some A_i. (For instance, we can take the enumeration given in the proof of Proposition 2.3.) Construct a computable model $\mathcal{M}_\omega = \bigcup_{n\in\omega} \mathcal{M}_\omega^n$ as follows. Begin with $\mathcal{M}_\omega^n[0] = \varnothing$ for all n. At stage $s+1$, if $A_n[s+1] \neq A_n[s]$ then extend $\mathcal{M}_\omega^n[s]$ to an $A_n[s+1]$-cube using fresh large numbers.

It is clear that this procedure can be carried out effectively so that $\mathcal{M}_\omega$ is computable. Furthermore, $\mathcal{M}_\omega$ is the disjoint union of one A_n-cube for each $n \in \omega$. In particular, every infinite cube in $\mathcal{M}_\omega$ is an S-cube.

Now let $T = \mathrm{Th}(\mathcal{M}_\omega)$ be the first-order theory of $\mathcal{M}_\omega$. We show that T is $\aleph_1$-categorical but not $\aleph_0$-categorical, $\mathcal{M}_\omega$ is saturated, and the only computably presentable model of T (up to isomorphism) is $\mathcal{M}_\omega$.

We begin by showing that T is $\aleph_1$-categorical. Since T includes sentences saying that for each n and x there is at most one y such that $P_n(x, y)$, we are free to use functional notation and write $P_n(x) = y$ instead of $P_n(x, y)$. For $n \in S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is not defined. For $n \notin S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is defined. Note that $k(n)$ is finite for all n.

It is easy to see that $\mathcal{M}_\omega$ satisfies the following list of statements, which can be written as an infinite set $\Sigma \subset T$ of first-order sentences:

1. For each n, the relation P_n is a partial one-to-one function and $P_n(x) = y \to P_n(y) = x$.
2. For all $n \neq m$ and all x, we have $P_n(x) \neq P_m(x)$ and $P_n(x) \neq x$.
3. For all $n \neq m$ and all x, if $P_n(x)$ and $P_m P_n(x)$ are defined, then $P_m(x)$ and $P_n P_m(x)$ are defined, and $P_n P_m(x) = P_m P_n(x)$.
4. For all k, all $n > n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, and all x, we have $P_{n_1}, \ldots, P_{n_k}(x) \neq P_n(x)$.
5. For each $n \in S$ there are exactly $k(n)$ many elements x for which $P_n(x)$ is not defined.
6. For each $n \notin S$ there are exactly $k(n)$ many elements x for which $P_n(x)$ is defined.
7. Let A_i be finite, and let $m \in A_i$ be such that $m \notin A_j$ for all $j \neq i$. Then there exists a finite A_i-cube $\mathcal{C}_i$ such that $\forall x\ (P_m(x) \text{ is defined} \to x \in \mathcal{C}_i)$. (Note that $m \notin S$ and $\mathcal{C}_i$ has $k(m)$ many elements, so together with Statements 3 and 6, this statement implies that $\mathcal{C}_i$ is not contained in a larger cube.)

Remark 4.2 Note that Statements 1 and 3 imply the following statement: for all $n \neq m$ and all u, if $P_n(u)$ and $P_m(u)$ are defined then $P_m P_n(u)$ and $P_n P_m(u)$ are defined and equal. To prove this let $v = P_n(u)$, which, by Statement 1, implies that

$P_n(v) = u$. Since $P_m P_n(v) = P_m(u)$ is defined, applying Statement 3 with $x = v$, we have that $P_m(v)$ and $P_n P_m(v)$ are defined, and $P_n P_m(v) = P_m P_n(v)$. If we let $w = P_m(v)$ then $P_m P_n(u) = w$. Since $P_n(w) = P_n P_m(v) = P_m P_n(v) = P_m(u)$, Statement 1 implies that $P_n P_m(u) = P_n P_n(w) = w$. Thus $P_m P_n(u) = P_n P_m(u)$.

Now suppose that $\mathcal{M}$ is a model of Σ. Let $A \subseteq \omega$ and $x \in M$. Using the statements above, it is easy to check that $\forall n \in A$ ($P_n^{\mathcal{M}}(x)$ is defined) if and only if x belongs to an A-cube. It is also clear that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are A-cubes in $\mathcal{M}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \varnothing$, then $\mathcal{C}_1 = \mathcal{C}_2$.

It now follows that $\mathcal{M}$ is the disjoint union of components $\mathcal{M}_0$ and $\mathcal{M}_1$, where $\mathcal{M}_0$ is the disjoint union of exactly one A_i-cube for each finite A_i. Let $x \in M_1$. If $n \in S$ then there are $k(n)$ elements in M_0 on which $P_n^{\mathcal{M}}$ is not defined. Statement 5 says that there are exactly $k(n)$ such elements in M. Hence $P_n^{\mathcal{M}}(x)$ is defined. Similarly, Statement 6 implies that if $n \notin S$ then $P_n^{\mathcal{M}}(x)$ is not defined. Therefore, x belongs to an S-cube. Thus, $\mathcal{M}_1$ is a disjoint union of S-cubes.

Let $\mathfrak{C}$ be the class of all structures that are the disjoint union of exactly one A_i-cube for each finite A_i and some finite or infinite number of S-cubes. Clearly, any structure in $\mathfrak{C}$ is a model of Σ, and we have shown that any model of Σ is in $\mathfrak{C}$. Let $\mathfrak{M}$ be a model of Σ. Each of the S-cubes in $\mathcal{M}$ is countable, so if $|M| = \aleph_1$, then there must be $\aleph_1$ many such S-cubes. Therefore, any two models of Σ of size $\aleph_1$ are isomorphic, and hence Σ is uncountably categorical. It now follows by the Łoś-Vaught Test that any model of Σ is a model of T. Thus T is uncountably categorical and, since $\mathfrak{C}$ contains infinitely many nonisomorphic countable structures, T is not countably categorical.

Lemma 4.3 *Let $\mathcal{M}$ be a computable model of T. Then $\mathcal{M}$ contains infinitely many S-cubes.*

Proof Assume for a contradiction that $\mathcal{M}$ contains a finite number r of S-cubes (which may be 0). We can assume without loss of generality that the domain of $\mathcal{M}$ is ω. Let $\mathcal{M}_s$ be the structure obtained by restricting the domain of $\mathcal{M}$ to $\{0, \dots, s\}$ and the language to $P_0, \dots, P_s$. Choose one element from each S-cube, say $c_1, \dots, c_r$. Define a computable function $g : \omega \times \omega \to [\omega]^{<\omega} \cup \{\infty\}$ as follows.

If $x > s$ then $g(x, s) = \varnothing$. If x is connected to some c_i in $\mathcal{M}_s$ then $g(x, s) = \infty$. Otherwise, $g(x, s)$ is the set of all $k \leqslant s$ for which there is a $y \leqslant s$ such that $P_k^{\mathcal{M}}(x, y)$.

Clearly, $g(x, s)$ is computable. Also, if x belongs to some A_i-cube in $\mathcal{M}$ then $g(x, s) \subseteq A_i$, and if $g(x, s) = \infty$ then x must belong to an S-cube. It is now easy to check that $f(x) = \lim_s g(x, s)$ is S-limitwise monotonic and $\{f(x) : f(x) \neq \infty\} = \{A_i : |A_i| < \omega\}$. But this contradicts the fact that $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable. □

Since $\mathcal{M}_\omega$ is computable, it contains infinitely many S-cubes, and therefore is saturated. Other countable models of T have only finitely many S-cubes, and hence do not have computable presentations. □

References

[1] Baldwin, J. T., and A. H. Lachlan, "On strongly minimal sets," *The Journal of Symbolic Logic*, vol. 36 (1971), pp. 79–96. Zbl 0217.30402. MR 0286642. 63

[2] Ershov, Yu. L., S. S. Goncharov, A. Nerode, J. B. Remmel, and V. W. Marek, editors, *Handbook of Recursive Mathematics. Vols. 1 and 2*, vols. 138 and 139 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1998. Zbl 0905.03001. MR 1673617. 63

[3] Goncharov, S. S., "Constructive models of $\aleph_1$-categorical theories," *Matematicheskie Zametki*, vol. 23 (1978), pp. 885–88. English translation: Mathematical Notes 23 (1978), pp. 486–88. Zbl 0403.03025. MR 502056. 64

[4] Harrington, L., "Recursively presentable prime models," *The Journal of Symbolic Logic*, vol. 39 (1974), pp. 305–309. Zbl 0332.02055. MR 0351804. 64

[5] Khisamiev, N. G., "On strongly constructive models of decidable theories," *Izvestiya Akademii Nauk Kazakhskoĭ SSR. Seriya Fiziko-Matematicheskaya*, (1974), pp. 83–84, 94. MR 0354344. 64

[6] Khisamiev, N. G., "A constructibility criterion for the direct product of cyclic p-groups," *Izvestiya Akademii Nauk Kazakhskoĭ SSR. Seriya Fiziko-Matematicheskaya*, (1981), pp. 51–55, 86. Zbl 0457.20001. MR 614069. 64

[7] Khisamiev, N. G., "Theory of Abelian groups with constructive models," *Sibirskiĭ Matematicheskiĭ Zhurnal*, vol. 27 (1986), pp. 572–85. Translation from Sib. Mat. Zh. 27, No.4(158), pp. 128–143, 215. Zbl 0625.03015. MR 867866. 64

[8] Khisamiev, N. G., "Constructive Abelian groups," pp. 1177–1231 in *Handbook of Recursive Mathematics, Vol. 2*, edited by Yu. L. Ershov, S. S. Goncharov, A. Nerode, J. B. Remmel, and V. W. Marek, vol. 139 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1998. Zbl 0940.03044. MR 1673602. 64

[9] Khoussainov, B., A. Nies, and R. A. Shore, "Computable models of theories with few models," *Notre Dame Journal of Formal Logic*, vol. 38 (1997), pp. 165–78. Zbl 0891.03013. MR 1489408. 64, 66, 67

[10] Kudaĭbergenov, K. Ž., "Constructivizable models of undecidable theories," *Sibirskiĭ Matematicheskiĭ Zhurnal*, vol. 21 (1980), pp. 155–58, 192. Zbl 0454.03011. MR 592228. 64

[11] Nies, A., "A new spectrum of recursive models," *Notre Dame Journal of Formal Logic*, vol. 40 (1999), pp. 307–14. Zbl 1007.03036. MR 1845630. 64

[12] Soare, R. I., *Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. Zbl 0623.03042. MR 882921. 63

Acknowledgments

This research was partially supported by the Marsden Fund of New Zealand. The first author's research was partially supported by the National Science Foundation of the USA under grant DMS-02-00465. The third author's research was partially supported by RFFR grant No. 02-01-00593 and Council for Grants under RF President, project NSh-2112.2003.1. We thank the referee for providing a simpler proof of Lemma 3.4.

Department of Mathematics
University of Chicago
5734 S University Ave
Chicago IL 60637
drh@math.uchicago.edu

Department of Computer Science
University of Auckland
Private Bag 92019 Auckland
NEW ZEALAND
bmk@cs.auckland.ac.nz

Department of Computer Science
University of Auckland
NEW ZEALAND
and
Department of Mathematics
Novosibirsk State University
RUSSIA
pavel@cs.auckland.ac.nz