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Some Open Problems in Mutual Stationarity Involving Inner Model Theory: A Commentary

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Abstract We discuss some of the relationships between the notion of "mutual stationarity" of Foreman and Magidor and measurability in inner models. The general thrust of these is that very general mutual stationarity properties on small cardinals, such as the $\aleph_n$s, is a large cardinal property. A number of open problems, theorems, and conjectures are stated.

1 Introduction

In this note we are interested in certain problems in the theory of mutual stationarity (as Foreman and Magidor have defined it in [2]). There are many open questions, but we restrict ourselves here to a few that have used—or are likely to use—inner model theory in a deep way to establish the strengths of the various principles in question. When we say 'deep' we mean only that in the relevant proofs more is needed than simply quoting a Covering Lemma or the like. Indeed the purpose of this note is to sketch how one obtains just $O^{\#}$ from some of the stated principles. Once this has been done, the framework will exist for anyone with a sufficient knowledge of core model and inner model theory (see, for example, Zeman [5]) to work through the extra complexities from iteration theory. Furthermore, with knowledge of the construction of global square sequences in core models, one can obtain stronger results. For Question 2.2 below, a lower bound has been determined in collaboration with Koepke (see [4] and [3]).

2 The Open Problems

Let $\langle \kappa_\alpha | \alpha < \delta \rangle$ for $\delta < \kappa_0$ be an ascending sequence of uncountable regular cardinals. Let $\kappa \stackrel{\text{def}}{=} \sup \kappa_\alpha$. For λ a regular cardinal let $\text{cof}_\lambda \stackrel{\text{def}}{=} \{\alpha \in \text{On} \mid \text{cf}(\alpha) = \lambda\}$.

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Definition 2.1 Let $\mathcal{S} = \langle S_\alpha | \alpha < \delta\rangle$ be a sequence of stationary sets, with $S_\alpha \subseteq \kappa_\alpha \cap \mathrm{cof}_\lambda$ ($\lambda < \kappa_0$). Then we say that $\mathcal{S}$ is 'mutually stationary' if the following set S is stationary:

$$S \stackrel{\text{def}}{=} \{X \in [\kappa]^\lambda \;\mid\; \sup(X \cap \kappa_\alpha) \in S_\alpha\}.$$

Let '$\mathrm{MS}(\langle \kappa_\alpha\rangle_{\alpha<\delta}, \lambda)$' abbreviate 'For all sequences $\mathcal{S} = \langle S_\alpha | \alpha < \delta\rangle$, with $S_\alpha \subseteq \kappa_\alpha \cap \mathrm{cof}_\lambda$, and S_α stationary⟩, $\mathcal{S}$ is mutually stationary'.

We are therefore asking whether all sequences of prior, independently chosen stationary sets of the appropriate kind form mutually stationary sequences.

Question 2.2 *What is the consistency strength of* $\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$*(+CH)?*

Let $\langle\lambda_n\rangle_{n<\omega}$ be an ascending sequence of regular cardinals, with infinitely many not Mahlo. Let $\gamma < \lambda_0$ be regular.

Question 2.3 *What is the consistency strength of* $\mathrm{MS}(\langle\lambda_a\rangle_{n<\omega}, \gamma)$*?*

Background: $\mathrm{MS}(\langle\kappa_n\rangle_{n<\omega}, \gamma)$ for any $\gamma < \kappa_0$ (and more) can hold in a Prikry generic extension of a model with a measurable cardinal (Cummings et al. [1], Theorem 5.4). (In this model the κ_n come from a tail of the generic Prikry sequence.) This is an equiconsistency [4]. The questions above are concerned with keeping the cardinals "small" in either sense. (In the Prikry generic model mentioned above the κ_n remain Ramsey, hence weakly compact, and hence Mahlo.) No upper bounds on consistency strength are known for such sequences of "small" κ_n. Lower bounds are in the order of inner models with measures of Mitchell order ν for every $\nu < \sup\{\omega_n\}$ (respectively, $\nu < \sup\{\kappa_n\}$).

We will sketch a proof of the following theorem.

Theorem 2.4 ([4]) $\mathrm{ZFC} \vdash \mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1) \longrightarrow O^\#$ *exists.*

This indicates that $\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$ must be a large cardinal property. Before sketching the proof we remark that Foreman and Magidor have shown that in L $\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$ fails.

Theorem 2.5 ([2], $V = L$) *Let* $T_n^h \subseteq \omega_n$ *be defined by*

$$T_n^h = \{\alpha < \omega_n \cap \mathrm{cof}_{\omega_1} \mid \mathrm{defcol}(\alpha) = h\}$$

where defcol *of an ordinal* α *is the least* h *so that* α *is cofinalized by a* $\Sigma_h(L_{\beta(\alpha)})$ *function, where in turn* $\beta(\alpha)$ *is the least* β *so that* $L_{\beta+1} \models$*"*α *is singular". Let* $f : \omega \longrightarrow \omega$*. Then*

- *(i)* *each* T_n^h *is stationary,*
- *(ii)* $\mathcal{S} = \langle S_n^{f(n)}\rangle$ *is mutually stationary iff* f *is eventually constant; hence,*
- *(iii)* $\neg\,\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$.

Their argument can be seen to work in any of the current extender models of the form $L[E]$ model for E a coherent sequence of extenders such as those described in [5].

Theorem 2.6 ($V = L[E]$) *If the definitions of* $\beta(\alpha)$ *and* defcol *are amended to those appropriate for the* $L[E]$ *hierarchy, then (i)–(iii) still hold.*

Proof The essential ingredient of the above argument for Theorem 2.5 is the Condensation Lemma for L. In general, Condensation will fail for $L[E]$ hierarchies. But note that we are only looking at the very bottom part of this hierarchy inside of $L[E]$. Below $\aleph_\omega$ all cardinals are small (!) and we do in fact have the requisite condensation for the simple reason that any failure of condensation, when some hull is collapsed, requires that transitivized collapse to contain a local inaccessible cardinal. But none such can appear in any hull of $H_{\aleph_\omega}$. □

Sketch of Proof of Theorem 2.4 We wish to step outside of the inner model L. We suppose that $\neg O^\#$. We may thus use both the Jensen Covering Lemma for L and the fact that L has a Global Square sequence $\Box$, which is then—by Covering—a Global Square sequence in V.

Definition 2.7 Let $\text{Sing} = \{\beta \in \text{Ord} \mid \lim(\beta) \wedge \text{cof}(\beta) < \beta\}$ be the class of singular limit ordinals. *Global square* ($\Box$) is the assertion that there is a system $(C_\beta)_{\beta \in \text{Sing}}$ satisfying

(a) C_β is a closed cofinal subset of β,
(b) $\text{otp}(C_\beta) < \beta$, and
(c) if $\overline{\beta}$ is a limit point of C_β then $\overline{\beta} \in \text{Sing}$ and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$.

Under our assumption of $\neg O^\#$, the Covering Lemma implies that with $\omega_2 = \omega_2^V$, $\text{Sing} \backslash \omega_2 = (\text{Sing} \backslash \omega_2)^L$. Since the other clauses in the definition of $\Box$ are absolute, the $\Box$ sequence defined in L is truly a $\Box$ sequence for V.

Lemma 2.8 *Let κ be a regular cardinal $\geq \aleph_2$ and λ a regular cardinal $< \kappa$. Then for every ordinal θ such that $\theta^+ < \kappa$, the set*

$$\{\beta \in \text{Cof}_\lambda \cap \kappa \mid \text{otp}(C_\beta) \geq \theta\}$$

is stationary in κ.

Proof Let $C \subseteq \kappa$ be closed unbounded in κ. Let $\mu = \max(\lambda, \theta^+)$ which is an uncountable regular cardinal $< \kappa$. Take a singular limit point γ of C of cofinality μ. Then $C \cap C_\gamma$ is closed unbounded in γ of ordertype $\geq \mu$. Take β to be a singular limit point of $C \cap C_\gamma$ such that $\text{cof}(\beta) = \lambda$ and $\text{otp}(C \cap C_\gamma \cap \beta) \geq \theta$. By the coherency property (Definition 2.7(c)), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap \{\beta \in \text{Cof}_\lambda \cap \kappa \mid \text{otp}(C_\beta) \geq \theta\} \neq \emptyset$. □

Note that $(S_n)_{n<\omega}$ with

$$S_n = \{\beta \in \text{Cof}_{\omega_1} \cap \aleph_{n+3} \mid \text{otp}(C_\beta) \geq \aleph_{n+1}\}$$

is a sequence of stationary sets to which we could apply the MS-principle. However if we let $\mathfrak{A}$ be a first-order structure of countable type with $\omega_\omega + 1 \cup H_{\aleph_\omega} \subseteq \mathfrak{A}$, and if there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ such that $\forall n < \omega\ \chi_n \stackrel{\text{def}}{=} \sup(|\mathfrak{B}| \cap \omega_{n+3}) \in S_n$, this would require that the element C_{χ_n} of L's global $\Box$ sequence be defined in a radically different manner from that of C_{χ_m} for $m \neq n$. Indeed as m goes to infinity, the order types of the C_{χ_m} must also go to ω_ω.

We may also assume—if we add ordinals $\zeta \leq \omega_2$ into the domain $|\mathfrak{B}|$—that $\text{card}\{|\mathfrak{B}|\} = \aleph_2$ (we may assume that we can do this without increasing the values χ_n if $\mathfrak{A}$ is sufficiently rich). Now let $\pi : \mathfrak{H} \longleftrightarrow \mathfrak{B}$ be the transitivization of $\mathfrak{B}$, and let $L^{\mathfrak{H}} = L_\gamma$, by the Condensation Lemma for L. Let $\pi(\beta_n) = \omega_n$ for $n \leq \omega$. Each β_n ($n > 2$) is regular in L_γ for $n < \omega$ but is in cof_{ω_1}. Such β_n cannot be regular in L (as otherwise $\mathrm{cf}(\beta_n) \geq \omega_2$ by the Covering Lemma again). Hence there is a least $\delta \geq \gamma$ where, for some $m, h < \omega$ $n > m \longrightarrow \beta_n$ is *singularized* by some function $\Sigma_h(L_\delta)$—where we assume h chosen least so that there is a function at this level of definability which maps some ordinal $\overline{\beta} < \beta_n$ cofinally into β_n, in terms of the notion defined above $h = \mathrm{defcol}(\beta_n)$.

Thus precisely at $\delta + 1$, L sees that a tail of the β_n is made singular by functions of the same order of complexity. L_δ is thus a form of a "singularizing" structure for a tail of the β_n. However it is at the same time the requisite structure for defining the C_{β_n}—sequences that go into L's $\square$ sequence. If one goes into the proof of $\square$ at this point one sees that there is $m < \omega$ and a single fixed ordinal $\alpha_0 < \beta_m$ so that for $n > m$ $\mathrm{ot}(C_{\beta_n}) \leq \alpha_0$. Now arguments of the type used in the Jensen Covering Lemma show, again for some $m \leq m_1 < \omega$, that $n > m_1 \longrightarrow \mathrm{ot}(C_{\chi_n}) < \pi(\alpha_0)$ by demonstrating that the lift-up of the singularizing structure for β_n is that of χ_n. Note that $\pi(\alpha_0)$ is a *fixed* ordinal between these various lifted-up structures. This contradicts our assumption on increasing order types for the S_n sets. $\square$

This argument can be repeated using the Weak Covering Lemmas due to Mitchell over models with measurables of varying Mitchell orders. (Obtaining an inner model with a measure is also proven in [4]; for larger models see [3].)

If $A \subseteq \omega$, then let us define $S_A \stackrel{\mathrm{def}}{=}$

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \mid \ \forall m > 1 \ \ \mathrm{cf}(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega & \text{if } m \in A \\ \omega_1 & \text{if } m \notin A \end{array} \right\}.$$

Question 2.9 *What is the consistency strength of "For some $A \subseteq \omega$ which infinitely often both contains and omits successor pairs $n, n+1$ of integers, S_A is stationary"?*

Background: One could ask many variants of this question. If the set A simply alternates—for example if $A = \mathrm{Evens}$—then it has been shown (by Magidor) equiconsistent that this S_A is stationary, with the existence of infinitely many measurable cardinals. If one raises the cofinalities to be ω_1 and ω_2, and takes $A = \mathrm{Evens}$, one gets more (perhaps unsurprisingly).

Theorem 2.10 $\mathrm{ZFC} + 2^{\aleph_0} < \aleph_\omega \vdash$ *"If $A \subseteq \omega$ is infinite and coinfinite, and $S_A \stackrel{\mathrm{def}}{=}$*

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \mid \ \forall m > 1 \ \ \mathrm{cf}(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega_1 & \textit{if } m \in A \\ \omega_2 & \textit{if } m \notin A \end{array} \right\}$$

is stationary, then there is an inner model with infinitely many measurables of Mitchell order ω_1."

It is, however, much harder to arrange other patterns of alternation of cofinality. As an upper bound for Question 2.9, we have from [1] (Theorem 6.7) the consistency of ZFC, together with the existence of infinitely many supercompact cardinals. (In fact, this cited theorem produces a model from this hypothesis where all such qualifying S_A are stationary for arbitrary $A \subseteq \omega$.)

As a lower bound for Question 2.9, current methods lead us to believe the following conjecture.

Conjecture 2.11 *If for a single A (as in Question 2.9) S_A is stationary, then there is an inner model with a strong cardinal.*

Why should this be a large cardinal property? Fix such an A with S_A stationary. Let $\mathfrak{B} \prec \mathfrak{A}$ as before, with $\omega_\omega \subseteq |\mathfrak{A}|$ and $\omega_2 \subseteq B$. Let $\mathfrak{H}$ be the transitivization of $\mathfrak{B}$

as above. Again let us suppose $\neg O^{\#}$. Let L_γ be as above and let δ again be least so that definably over L_δ a tail of the β_n are definably made singular. Reasonably elementary arguments show that all the successor cardinals of L_γ have—again, on a tail—the same cofinality (inherited from δ). So, consider the set of cardinals β_n^{+L}. By our assumption on A, S_A infinitely many of the β_n^{+L} must satisfy $\beta_n^{+L} < \beta_n^{+\mathfrak{K}}$ (otherwise we would have alternating cofinalities depending on the cofinalities of the $\beta_n^{+\mathfrak{K}}$); but for such n, as $\mathfrak{H} \models$ "*The Covering Lemma holds over L*," we have that $\mathfrak{H} \models$ "$\mathrm{cf}(\beta_n^{+L}) = \beta_n$." Take a particular case: suppose $\mathrm{cf}(\delta) = \omega_1$ and both $m, m+1$ sufficiently large in A. Then $\beta_m^{+L} < \beta_m^{+\mathfrak{K}} = \beta_{m+1}$, as the former has cofinality ω_1. But then it has the cofinality of β_m by Covering applied inside $\mathfrak{H}$! Contradiction! One argues similarly in the other case.

Why is the conjecture phrased in terms of a strong cardinal rather than something stronger, such as Woodins? Because to perform the argument in the case of core models and preserve cofinalities in the objects being iterated—while using the comparison lemma—we need to work in the world of linear iterations (cf. [5] Chapter 8); further, we use that any universal weasel must be an iterate of the true K (Jensen, cf. [5] Theorem 7.4.9).

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