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Adding Closed Unbounded Subsets of ω_2 with Finite Forcing

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Abstract An outline is given of the proof that the consistency of a κ^+-Mahlo cardinal implies that of the statement that $I[\omega_2]$ does not include any stationary subsets of $\mathrm{Cof}(\omega_1)$. An additional discussion of the techniques of this proof includes their use to obtain a model with no ω_2-Aronszajn tree and to add an ω_2-Souslin tree with finite conditions.

1 Introduction

In [15], Definition 2.1 Shelah defined the approachability ideal $I[\kappa^+]$ as follows.

Definition 1.1 For any sequence $A = \langle a_\alpha : \alpha < \kappa^+ \rangle$ of sets, let $B(A)$ be the set of ordinals $\lambda < \kappa^+$ such that there is a set $c \subset \lambda$ with

(i) $\mathrm{otp}(c) = \mathrm{cf}(\lambda)$,
(ii) $\bigcup c = \lambda$, and
(iii) $\{ c \cap \xi : \xi < \lambda \} \subset \{ a_\alpha : \alpha < \lambda \}$.

Then $I[\kappa^+]$ is the set of subsets of κ^+ which are contained, up to a nonstationary set, in some set $B(A)$.

Shelah proved in [15], Theorem 4.4 that if κ is regular then $\kappa^+ \cap \mathrm{Cof}(<\kappa) \in I[\kappa^+]$, where we write $\mathrm{Cof}(\eta)$, or $\mathrm{Cof}(<\eta)$, for the set of ordinals ν such that $\mathrm{cf}(\nu) = \eta$ or $\mathrm{cf}(\nu) < \eta$, respectively. It is consistent that $\mathrm{Cof}(\kappa) \notin I[\kappa^+]$: an example is given by the model of [11] in which there are no ω_2-Aronszajn trees, and others are given by Proposition 2.11 and Theorem 4.1 of this paper. Shelah asked whether it is consistent that every subset of $\mathrm{Cof}(\omega_1)$ in $I[\omega_2]$ is nonstationary. The following theorem answers this question.

Theorem 1.2 *If it is consistent that there is a cardinal κ which is κ^+-Mahlo, then it is consistent that $I[\omega_2]$ does not contain any stationary subset of* $\mathrm{Cof}(\omega_1)$.

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The fact that a κ^+-Mahlo cardinal κ is necessary is due to Shelah, and a proof is given in [12], Theorem 13. In this paper we will outline the proof of Theorem 1.2, describe the techniques involved, and discuss some of their variations and limitations. A full proof of Theorem 1.2 is given in [13].

2 Adding One Closed Unbounded Set

A strategy for the proof of Theorem 1.2 is straightforward: Let κ be κ^+-Mahlo, and for each $\alpha < \kappa^+$ define $B_\alpha := \{\lambda < \kappa : \lambda \text{ is } f_\alpha(\lambda)\text{-Mahlo}\}$, where f_α is some function chosen so that $\alpha = [f_\alpha]_{\mathrm{NS}}$. The statement that κ is κ^+-Mahlo implies that each of the sets B_α is stationary, and the forcing will add κ^+ many new closed unbounded sets $D_\alpha \subset B_\alpha$. The forcing should preserve ω_1 and κ while collapsing all intermediate cardinals so that κ becomes ω_2 in the generic extension. A further constraint is given by the following observation of Shelah, which implies that the forcing must add new reals.

Proposition 2.1 *If $2^\omega \leq \omega_2$ then $I[\omega_2]$ contains a stationary subset of* $\mathrm{Cof}(\omega_1)$.

Proof Let $A = \langle a_\nu : \nu < \omega_2 \rangle$ enumerate $[\omega_2]^\omega$. To see that $B(A) \cap \mathrm{Cof}(\omega_1)$ is stationary, let $C \subset \omega_2$ be closed and unbounded and pick a chain $\langle M_\nu : \nu < \omega_1 \rangle$ of elementary submodels of H_{ω_3} of size ω_1 such that $\omega_1 \cup \{A, C\} \in M_0$ and $\langle M_\alpha : \alpha < \nu \rangle \in M_{\nu+1}$ for each $\nu < \omega_1$.

Set $\alpha_\nu = \sup(M_\nu \cap \omega_2)$ and $\alpha = \bigcup_{\nu<\omega_1} \alpha_\nu$. Then $\alpha \in C \cap \mathrm{Cof}(\omega_1)$ since $C \in M_0$, so it will be sufficient to show that $\alpha \in B(A)$. To this end, set $c = \{\alpha_\nu : \nu < \omega_1\}$ and note that if $\xi < \omega_1$ then $c \cap \alpha_\xi = \{\alpha_\nu : \nu < \xi\} \in M_{\xi+1}$ since $\langle M_\nu : \nu < \xi \rangle \in M_{\xi+1}$. Since $A \in M_0 \subset M_{\xi+1}$ we have $[\omega_2]^\omega \cap M_{\xi+1} = \{a_\nu : \nu \in M_{\xi+1}\} \subset \{a_\nu : \nu < \alpha_{\xi+1}\}$, so the set c witnesses that $\alpha \in B(A)$. □

We begin the search for the appropriate forcing by studying the known methods for adding a new closed, unbounded subset D of ω_1. There are two of these: one with finite conditions and one with countable conditions. The one using finite conditions first appeared in [3], page 926, and the form we describe is essentially due to Abraham [1]. The conditions are pairs $p = (I^p, O^p)$, where $I^p \in [\omega_1]^{<\omega}$ and O^p is a finite set of half open intervals $(\eta', \eta]$ satisfying the constraint that if $\lambda \in I^p$ and $(\eta', \eta] \in O^p$ then $\lambda \notin (\eta', \eta]$. A condition p forces $\lambda \in \dot{D}$ if and only if $\lambda \in I^p$ and $p \Vdash \lambda \notin \dot{D}$ if and only if $\lambda \in (\eta', \eta]$ for some $(\eta', \eta] \in O^p$. The forcing using countable conditions, which first appeared in [2], has as conditions closed, bounded subsets c of ω_1 ordered by end extension.

These two methods suggest three possible ways to force a new closed unbounded subset of ω_2:

1. use finite conditions $p = (I^p, O^p)$ as in the forcing at ω_1, except that the ordinals are in ω_2 instead of ω_1;
2. use countable conditions $p = (I^p, O^p)$; this is like the first alternative, except that I^p and O^p are countable;
3. generalize the second alternative for ω_1 by using as conditions closed, bounded subsets of ω_2.

The third alternative is the most common method and the obvious choice; however neither it nor the second alternative add new reals, and Proposition 2.1 implies that ω_3 new reals are needed. Furthermore both of these alternatives use the continuum hypothesis, so it is difficult to devise a suitable iterated forcing which alternates either forcing with a separate forcing to add the required new reals.

This leaves the first alternative, but that collapses ω_1. To see this, let D be the closed and unbounded set added by this forcing and define, for $\xi < \omega_2$, $\sigma(\xi) = \sup\{\, \nu < \omega_1 : \omega_1 \cdot \xi + \nu \in D \,\}$. If p is any condition then

$$\begin{aligned} &p \Vdash \sigma(\xi) = \omega_1 && \text{if } \omega_1 \cdot (\xi + 1) \in I^p, \\ &p \Vdash \sigma(\xi) = 0 && \text{if } \exists (\eta', \eta] \in O^p \ (\eta' < \omega_1 \cdot \xi \ \ \& \ \ \omega_1 \cdot (\xi+1) \le \eta), \text{ and} \\ &p \Vdash \sigma(\xi) \le \nu && \text{if } \exists (\eta', \eta] \in O^p \ (\omega_1 \cdot \xi + \nu < \eta' \ \ \& \ \ \omega_1 \cdot (\xi+1) \le \eta) \end{aligned}$$

for each $\nu < \omega_1$. For any other ordinal ξ, and for any ordinal $\nu < \omega_1$, there is $p' < p$ such that $p' \Vdash \nu < \sigma(\xi) < \omega_1$. Hence it is forced that there are unboundedly many ordinals $\xi < \omega_2$ such that $0 < \sigma(\xi) < \omega_1$, and if we let A be the set containing the first ω-many of these ordinals then $\{\, \sigma(\xi) : \xi \in A \,\}$ is unbounded in ω_1.

In order to avoid this collapse, we modify the forcing by using a variation of Todorčević's method of forcing with models as side conditions. This forcing P_{ω_2} to add a closed unbounded subset of ω_2 was independently discovered by Friedman [4], and his proof has been translated by Morgan [14] into Koszmider's technique of using a morass for forcing with models as side conditions.

We state the following definition under the assumption $V = L$ and take the countable models in the side condition $\mathcal{A}^p$ to be Σ_1-elementary substructures of L_{ω_2}. This can be extended to more general contexts, for example, by taking the models to be substructures of $L_{\omega_2}[A]$ where $A \subset \omega_2$ codes 2^ω.

If $V = L$ then the countable models M which are used in Definition 2.2 in the set $\mathcal{A}^p$ of side conditions may be taken to be Σ_1-elementary substructures of L_{ω_2}. More generally, if we assume $2^\omega \le \omega_1$ then we could take the models to be of the form $L_{\omega_2}[A]$ where A is a fixed subset of ω_2 such that $2^\omega \subset L_{\omega_2}[A]$.

Definition 2.2 The conditions in the forcing P_{ω_2} are triples $p = (I^p, O^p, \mathcal{A}^p)$. The sets I^p and O^p are as in the finite forcing for ω_1 except that the ordinals are taken from ω_2, and $\mathcal{A}^p$ is a finite set of countable models M.

A condition $p = (I^p, O^p, \mathcal{A}^p)$ must satisfy the following conditions:

1. if $\lambda \in I^p$ and $(\eta', \eta] \in O^p$ then $\lambda \notin (\eta', \eta]$;
2. if $(\eta', \eta] \in O^p$ and $M \in \mathcal{A}^p$ then either $\eta', \eta \in M$ or else $(\eta', \eta] \cap M = \varnothing$;
3. suppose $M, M' \in \mathcal{A}^p$;
   (a) either $M \cap M' \in M$ or else $M \cap M' = M \cap L_\delta$, where $\delta = \sup(M \cap M')$,
   (b) $\lim(M) \cap \lim(M') = \lim(M \cap M')$.

The set P_{ω_2} is ordered by $(I', O', \mathcal{A}') \le (I, O, \mathcal{A})$ if $I' \supset I$, $O' \supset O$, and $\mathcal{A}' \supset \mathcal{A}$.

Of course, clause 3(a) also holds with M and M' switched. Clause 1 is taken directly from the forcing at ω_1. Clause 2 states that if $M \in \mathcal{A}$ then any requirement $(\eta', \eta] \in O^p$ either is a member of M or else does not affect the forcing $P \cap M$ in M. Clause 3 is more complex, but it is motivated by similar considerations which are made precise in the next definition.

Definition 2.3 If P is a forcing order and X is a model then a condition p is *strongly X-generic* if p forces that $\dot{G} \cap X$ is a V-generic subset of $P \cap X$, where $\dot{G}$ is a name for the generic subset.

This should be contrasted with Shelah's notion of an X-generic condition, which only requires that p forces that $\dot{G} \cap X$ is X-generic. The importance of strong genericity in our construction is largely due to Lemma 2.10 below, but in the meantime we will need the following combinatorial characterization. We assume that the forcing P is closed under meets; that is, any pair p, q of compatible conditions in P has a greatest lower bound $p \wedge q \in P$.

Proposition 2.4 *A condition $p \in P$ is strongly X-generic if and only if for all conditions q compatible with p there is a condition $q|X \in P \cap X$ such that every condition $r \leq q|X$ in $P \cap X$ is compatible with $q \wedge p$.* $\square$

Thus, strong X-genericity of a condition p means that any effect which a condition $q \leq p$ has on the forcing inside X can be specified by a condition $q|X$ which is a member of X. In Lemma 2.6 we will see that $(\{\lambda\}, \varnothing, \varnothing)$ is strongly L_λ-generic, and $(\varnothing, \varnothing, \{M\})$ is strongly M-generic. Clause 2 is necessary for the latter: any interval $(\eta', \eta] \in O^q$ such that $M \cap (\eta', \eta] \neq \varnothing$ will affect the forcing in M, and hence must be included in $O^{q|M}$.

Figure 1 Clause 2 of Definition 2.2

Clause 3(b) asserts that two models M and M' in $\mathcal{A}^p$ look like Figure 2, with a common part $M \cap M'$ at the bottom and finitely many disjoint intervals above. The figure shows the second alternative of clause 3(a), in which $M \cap M'$ is an initial segment of both M and M'.

For Theorem 1.2 we need to modify Definition 2.2 in two ways. We need new closed unbounded subsets of the κ^+-Mahlo cardinal κ instead of ω_2^V. For this we simply require the cardinals in I^p and endpoints of intervals from O^p be taken from κ, and (assuming V = L) we take the models $M \in \mathcal{A}^p$ to be Σ_1-elementary substructures of L_λ for some inaccessible cardinal $\lambda \leq \kappa$. The other change is that the new closed unbounded set is to be a subset of a given stationary set B, or rather of the set $B^* := B \cup \operatorname{Cof}(\omega)$. To obtain this forcing P_B we modify Definition 2.2 as follows. The countable models M used in the forcing P_B may be taken to be as in Definition 2.2, except that $M \prec_1 L_\kappa$ (or $M \prec_1 L_\kappa[A]$) instead of $M \prec_1 L_{\omega_2}$. For the forcing of Section 3 we use the additional assumption that M is closed under cardinal successor, which can be ensured by using the set of cardinals as a predicate.

Definition 2.5 P_B is the set of triples $p = (I^p, O^p, \mathcal{A}^p)$ satisfying the three clauses of Definition 2.2 such that

1. if $\lambda \in I^p$ and $\operatorname{cf}(\lambda) > \omega$ then $\lambda \in B$;
2. if $M \in \mathcal{A}^p$ and $\lambda \in I^p \cap \sup(M)$ then $\min(M \setminus \lambda) \in B$;
3. if $M, M' \in \mathcal{A}^p$ and $\delta = \sup(M \cap M')$ then $\min(M \setminus \delta) \in B$ if $\delta \notin M$, and $\min(M \setminus \lambda) \in B$ whenever either $\delta < \lambda \in M'$ and $\lambda < \sup(M)$.

The last two clauses of Definition 2.5 are pictured in Figure 2, where the black circles represent ordinals which are required by these two clauses to be in B. To see the significance of these clauses, note that no interval $(\eta', \eta] \in O^p$ can contain any of the black ordinals: in the case of clause 2 the interval would have to be a member of M, and hence would also include λ; while in the case of clause 3 the interval would have to be a member of $M \cap M'$, and that is impossible since $\eta > \sup(M \cap M')$. As a consequence, if μ is one of the black ordinals then no condition $p_0 \leq p$ can force that $D \cap \mu$ is bounded in μ. Hence p forces that μ is a limit point of D, and since D is intended to be closed it follows that p should force that $\mu \in D$. Since $\mathrm{cf}(\mu) > \omega$ it follows that μ must be in B.

This consideration also explains clause 3(b) of Definition 2.2: since all of the black ordinals are forced by p to be in D, and hence are effectively in I^p, it is necessary that there be only finitely many of them.

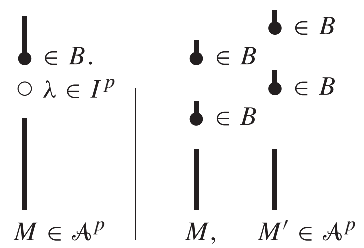

Figure 2 Clauses 2 and 3 of Definition 2.5

Figure 2 can also be viewed in connection with strong genericity: the inclusion of λ in I^p has consequences for the forcing inside M, since there are intervals $(\eta', \eta] \in M$ which include λ. Strong genericity requires that there be a condition $p|M$ in $P \cap M$ with the same consequences, and this requires that μ be included in $I^{p|M}$. Thus, again, μ should be in B. Similarly, in the case of clause 2, the inclusion of M' in $\mathcal{A}^p$ has consequences for the forcing inside M. Some of these consequences are enforced by including $M \cap M'$ in $\mathcal{A}^{p|M}$ when this is a member of M, and the rest are enforced by including the black ordinals in $I^{p|M}$.

Lemma 2.6 *If $\lambda \in B$ is inaccessible then the condition $(\{\lambda\}, \varnothing, \varnothing)$ is strongly L_λ-generic, witnessed by the function mapping $q = (I^q, O^q, \mathcal{A}^q) \leq (\{\lambda\}, \varnothing, \varnothing)$ to*

$$q|L_\lambda := \big(I^q \cap L_\lambda, O^q \cap L_\lambda, \{\, M' \cap L_\lambda : M' \in \mathcal{A}^q \,\}\big).$$

Any condition of the form $(\varnothing, \varnothing, \{M\})$ is strongly M-generic, witnessed by the function mapping $q = (I^q, O^q, \mathcal{A}^q) \leq (\varnothing, \varnothing, \{M\})$ to

$$q|M := \big((I^q \cap M) \cup I', O^q \cap M, \{\, M \cap M' : M' \in \mathcal{A}^q \;\;\&\;\; M \cap M' \in M \,\}\big),$$

where I' is the set of ordinals in M specified in clauses 2 and 3 of Definition 2.5.

It is not difficult to show that the indicated functions witness strong genericity, but it is somewhat tedious to verify that the triple $q|M$ is actually a condition.

Lemma 2.7 *If $p = (I^p, O^p, \mathcal{A}^p)$ is a condition, then so is $p' := (I', O^p, \mathcal{A}^p)$, where I' the smallest set of ordinals which contains I^p and all the ordinals required to be in B by clauses 2 and 3 of Definition 2.5, together with $\sup(M)$ and the ordinals $\sup(M \cap \lambda)$ for each $\lambda \in I'$.*

To give the flavor of what is involved, we present the case with the fewest subcases, namely, the case proving that p' satisfies clause 2 of Definition 2.5.

Proof of one case Suppose that $\lambda \in I^p$ and $M \in \mathcal{A}^p$. In order to verify that $\eta := \min(M \setminus \lambda)$ can be added to I^p, we need to verify that $\mu := \min(M' \setminus \eta) \in B$ for each $M' \in \mathcal{A}^p$. The argument involves three subcases:

1. If $M' \cap [\lambda, \eta) = \varnothing$ then $\mu = \min(M' \setminus \lambda)$, which is required to be in B since $\lambda \in I^p$ and $M' \in \mathcal{A}^p$.

2. If $\eta \geq \sup(M \cap M')$ then μ is required to be in B by clause 3 of Definition 2.5, since M and M' are in $\mathcal{A}^p$.

3. If $\eta < \sup(M \cap M')$ and $M' \cap [\lambda, \eta) \neq \varnothing$ then $M' \cap M$ is not an initial segment of M', so it must be that $M' \cap M \in M'$ and $M' \cap M$ is an initial segment of M. In particular $\eta \in M'$, so $\mu = \eta \in B$. $\square$

As an easy consequence of Lemma 2.6 we can verify that the generic extension has the desired cardinals.

Lemma 2.8 *The forcing P_B is proper and hence preserves ω_1. If $B \subset \kappa$ is a stationary set of inaccessible cardinals then P_B also preserves κ.*

Proof The first statement is immediate. To prove the second statement, suppose $p \Vdash \dot{h}\colon \omega_1 \to \kappa$. Since B is stationary, there is $X \prec L_{\kappa^+}$ such that $\{p, \dot{h}\} \subset X$ and $X \cap L_\kappa = L_\lambda$ for some inaccessible cardinal $\lambda \in B$. Then Lemma 2.6 implies that $p' := (I^p \cup \{\lambda\}, O^p, \mathcal{A}^p)$ is a condition extending p such that $p' \Vdash \operatorname{range}(\dot{h}) \subset \lambda$. $\square$

Another crucial fact is that the forcing P_B behaves like the forcing at ω_1.

Lemma 2.9 *Suppose that p is a condition, and p' is as given by Lemma 2.7. Then for any ordinal $\lambda < \kappa$*

$$p \Vdash \lambda \in \dot{D} \text{ if and only if } \lambda \in I^{p'}$$
$$p \Vdash \lambda \notin \dot{D} \quad \text{if and only if} \quad \exists(\eta', \eta] \in O^p \; (\lambda \in (\eta', \eta]).$$

Sketch of proof The first statement is the definition of D. To prove the second statement, we need to show that if $\lambda \notin I^{p'}$ then there is an interval $(\mu', \mu]$ with $\mu' < \lambda \leq \mu$ such that $(I^{p'}, O^p \cup \{(\mu', \mu]\}, \mathcal{A}^p)$ is a condition. By taking the interval $(\mu', \mu]$ small enough we can arrange that $I^{p'} \cap (\mu', \mu] = \varnothing$ and that $M \cap (\mu', \mu] = \varnothing$ for each $M \in \mathcal{A}^p$ such that $\lambda \notin M$ and $M \cap \lambda$ is bounded in λ. Now if M and M' are any two of the remaining members of $\mathcal{A}^p$ then we must have $\lambda < \sup(M \cap M')$, and it follows that on the relevant interval either M and M' are equal, or else one is a member of the other. Thus we can take the interval $(\mu', \mu]$ to be a member of the smallest of these models, and hence a member of all of them. $\square$

Note that Lemma 2.9 implies that D is closed, and also implies that $\{\, \lambda : p \Vdash \lambda \in \dot{D} \,\}$ is equal to the set $I^{p'}$ of Lemma 2.7.

Before explaining how the forcing P_B is used in the proof of Theorem 1.2, we need to present a general lemma which explains the importance of strong genericity. This lemma can be compared with the main lemma in the original construction [11] of a model with no $\aleph_2$-Aronszajn tree, and also with Hamkins's "key lemma" of [6] and [7], Lemma 13, which states that forcing with a δ-closure point satisfies the δ^+ approximation property. All of these can be easily proved using the idea of Lemma 2.10.

Lemma 2.10 *Suppose that P is a forcing notion with meets, $G \subset P$ is generic, and X is a model having a strongly X-generic condition $p \in G$ such that the witnessing function satisfies $(q \wedge q')|X = q|X \wedge q'|X$ whenever the conditions q, q', and p are compatible. Further suppose that the set of countable models M having strongly M-generic conditions is stationary; that is, for any cardinal θ with $P \in H_\theta$ and any set $a \in H_\theta$ there is a countable model $M \prec H_\theta$ with $a \in M$ such that for each $p \in P \cap M$ there is a strongly M-generic condition $q \leq p$.*

Let $h\colon \mu \to V$ be a function in $V[G]$ such that $h{\restriction}x \in V[G \cap X]$ for every $x \in ([\mu]^\omega)^V$. Then $h \in V[G \cap X]$.

Proof Pick a countable model $M \prec H_\theta$, for some θ large enough, so that $\dot{h}, X, p, P \in M$; and let $q \leq p$ be strongly M-generic. By extending q if necessary we can assume that there is a $P \cap X$-term $\dot\sigma$ such that $q \Vdash \dot h \restriction \sup(M \cap \omega_1) = \dot\sigma$. I claim that

$$M \models \forall r \leq q|M \forall \nu < \mu \forall x \left(r \Vdash \dot h(\nu) = x \implies (r|X \wedge q|M) \Vdash \dot h(\nu) = x\right). \quad (1)$$

It will follow by elementarity that the same sentence holds in V, which implies that h can be computed from $G \cap X$.

If the sentence (1) does not hold for some $\nu, x \in M$ and some $r \leq q|M$ in $P \cap M$ then there is a condition $r' \leq (r|X \wedge q|M)$ in $P \cap M$ such that $r' \Vdash \dot h(\nu) \neq x$. Now

$$r \wedge q \Vdash \dot\sigma(\nu) = \dot h(\nu) = x$$

so $(r \wedge q)|X \Vdash \dot\sigma(\nu) = x$; and similarly $(r' \wedge q)|X \Vdash \dot\sigma(\nu) \neq x$. But this is impossible since

$$(r' \wedge q)|X = r'|X \wedge q|X \leq r|X \wedge q|X = (r \wedge q)|X,$$

where the inequality uses the observation that $r' \leq r$ implies that $r' = r' \wedge r$ and hence $r'|X = (r' \wedge r)|X = r'|X \wedge r|X \leq r|X$. □

Finally we are able to verify that the forcing P_B provides one step in the construction of a model where $I[\omega_2]$ includes no stationary subsets of $\mathrm{Cof}(\omega_1)$.

Proposition 2.11 *If $B \subset \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary and $G \subset P_B$ is generic then the following two statements are true in $V[G]$:*

$$\mathrm{Cof}(\omega_1) \setminus B \text{ is nonstationary} \quad (2)$$

$$\{\lambda \in B : B \cap \lambda \text{ is not stationary in } V\} \in I[\omega_2]. \quad (3)$$

Furthermore, if in V $\{\lambda \in B : B \cap \lambda \text{ is stationary}\}$ is stationary, then the following property holds in $V[G]$:

$$\{\lambda \in B : B \cap \lambda \text{ is stationary in } V\} \notin I[\omega_2]. \quad (4)$$

Proof Statement (2) is immediate, since the closed unbounded set $D = \dot D^G$ is contained in $B \cup \mathrm{Cof}(\omega)$.

For statement (3), suppose that $\lambda \in B$ but $B \cap \lambda$ is nonstationary in λ, and let C_λ be a closed and unbounded subset of λ such that $C_\lambda \cap B = \varnothing$. Then $c := C_\lambda \cap D$ is closed and unbounded in λ and must have order type ω_1 since $D \cap \mathrm{Cof}(\omega_1) \subset B$ implies that $C_\lambda \cap D$ has no members of uncountable cofinality. Furthermore, if $\beta < \lambda$ then $c \cap \beta \in L_{\beta'}[C_\lambda, D \cap \beta]$, where $\beta' = \beta^{+L} < \lambda$. It follows that $B_0 := \{\lambda \in B : \lambda \cap B \text{ is nonstationary}\}$ is in $B(A)$ where A is an enumeration of $[\kappa]^\omega \cap L[\langle C_\lambda : \lambda \in B_0\rangle, D]$.

For the final statement (4), suppose we are given a name $\dot A$ for a sequence $A = \langle a_\nu : \nu < \kappa\rangle$ of sets in $[\kappa]^\omega$, and let E be the closed unbounded set of ordinals $\lambda < \kappa$ such that $(\{\lambda\}, \varnothing, \varnothing) \Vdash \forall \nu < \lambda \; \dot A \restriction \nu \in V[G \cap V_\lambda]$. I claim that $B \cap \lambda$ is nonstationary for any ordinal $\lambda \in B \cap E \cap D \cap B(A)$. To see this, let $c \subset \lambda$ witness that $\lambda \in B(A)$, so that c is a cofinal set of order type ω_1 and $c \cap \beta \in \{a_\nu : \nu < \lambda\} \subset V[G \cap L_\lambda]$ for each $\beta < \lambda$. Since $\lambda \in D \cap B$ the strongly L_λ-generic condition $(\{\lambda\}, \varnothing, \varnothing)$ is a member of G, and hence Lemma 2.10 implies that $c \in V[G \cap \lambda]$. However, $G \cap L_\lambda$ is a generic subset of $P_B \cap L_\lambda = P_{B \cap \lambda}$, and because of the set c this forcing collapses λ. It follows by Lemma 2.8 that $B \cap \lambda$ is a nonstationary subset of λ. □

3 Adding κ^+-many Closed Unbounded Sets

Assume that κ is Mahlo and let $G \subset P_B$ be generic where B is the set of inaccessible cardinals below κ. Then according to Lemma 2.11 the set $\{\,\lambda : \mathrm{cf}^V(\lambda) = \omega_1\,\}$, which generates the restriction of $I[\kappa]^V$ to $\mathrm{Cof}(\omega_1)$, becomes nonstationary in $V[G]$; however, the set of inaccessible but non-Mahlo cardinals of V is a new stationary member of $I[\omega_2]^{V[G]}$. If κ is at least 2-Mahlo then the set of Mahlo cardinals below κ is a stationary set which is not a member of $I[\omega_2]^{V[G]}$, and this allows the process to be repeated. After κ^+ many repetitions every member of $I[\omega_2]$ will have been made nonstationary.

This summary is a little misleading, as the forcing used to prove Theorem 1.2 is not an iterated forcing, but rather is more like a product forcing. Before describing the forcing, we recall what we expect it to accomplish: we defined $B_\alpha = \{\,\lambda < \kappa : \lambda \text{ is } f_\alpha(\lambda)\text{-Mahlo}\,\}$, and we intend to add a closed, unbounded subset $D_\alpha \subset B_\alpha$ for each $\alpha < \kappa^+$. The sequence of sets $\langle\, D_\alpha : \alpha < \kappa^+\,\rangle$ will be continuously diagonally decreasing, that is, $D_{\alpha+1} \subset D_\alpha$ and $D_\alpha = \triangle_{\alpha'<\alpha} D_{\alpha'} := \{\,\lambda < \kappa : \lambda \in \bigcap\{\,D_{\alpha'} : \alpha' \in \pi_\alpha``\lambda\,\}\,\}$ for limit ordinals α. Here f_α is a function representing α modulo the nonstationary ideal, $[f_\alpha]_{\mathrm{NS}} = \alpha$, and π_α is a function mapping κ onto α.

The sequences of functions $\langle\, f_\alpha : \alpha < \kappa^+\,\rangle$ and $\langle\, \pi_\alpha : \alpha < \kappa^+\,\rangle$ used in the last paragraph must be fixed in advance. The construction used to do so is somewhat delicate, using $\square_\kappa$ and minimal walks to find sets $A_{\alpha,\xi} \subset \alpha$ for each ordinal $\xi < \lambda$ so that (among other things)

(i) $\alpha = \bigcup_{\xi<\lambda} A_{\alpha,\xi}$,
(ii) $\left|A_{\alpha,\xi}\right| = |\xi|$,
(iii) $\xi' < \xi \implies A_{\alpha,\xi'} \subset A_{\alpha,\xi}$, and
(iv) if $\alpha' \in A_{\alpha,\xi} \cup \lim(A_{\alpha,\xi})$ then $A_{\alpha',\xi} = A_{\alpha,\xi} \cap \alpha'$.

The coherence property (iv) is critical: it gives us a tree ordering $\prec_\xi$ on κ^+ for each $\xi < \kappa$, defined by putting $\alpha' \prec_\xi \alpha$ if $\alpha' \in A_{\alpha,\xi}$.

The full forcing P^* has conditions which are, as in the forcing P_B to add a single closed unbounded set, triples $p = (I^p, O^p, \mathcal{A}^p)$ of finite sets. The first two coordinates are straightforward: a member of I^p is a pair (α, λ) which forces that $\lambda \in D_\alpha$, and hence $\lambda \in D_{\alpha'}$ for all $\alpha' \prec_\lambda \alpha$, while a member of O^p is a pair $(\alpha, (\eta', \eta])$ which forces that $(\eta', \eta] \cap D_\alpha = \varnothing$, and hence $\lambda \notin D_{\alpha'}$ whenever $\eta' < \lambda \le \eta$ and $\alpha \prec_\lambda \alpha'$.

The definition of the third component, $\mathcal{A}^p$, is more complicated, and this note will only attempt to give a first approximation, by taking the members of $\mathcal{A}^p$ to be pairs (M, α) where $M \subset L_\kappa$ is as in the forcing P_B and $\alpha < \kappa^+$. The effect of having the pair (M, α) in $\mathcal{A}^p$ is that M is used as in Section 2 to control the forcing for $D_{\alpha'}$, for each ordinal $\alpha' \in \pi_\alpha``M$.

The approximation given above does cover the models for which strong genericity holds.

Lemma 3.1 *Suppose $N_2 \prec L_{\kappa^+}$ is transitive and contains κ, $N_1 \prec N_2$ is a model with $\kappa \in N$ and $N \cap L_\kappa$ is transitive, and $N_0 \prec N_2$ is a countable model. Set $\alpha_2 := \sup(N_2)$, $\alpha_1 := \sup(N_1)$, and $\alpha_0 := \sup(N_0)$, and assume that $N_i \cap \lim(C_{\alpha_i})$ is cofinal in α_i for $i = 0, 1, 2$. Then*

$$\big(\varnothing, \varnothing, \varnothing\big) \qquad \textit{is strongly } N_2\textit{-generic,} \tag{5}$$
$$\big((\alpha_1, \sup(N_1 \cap \kappa)), \varnothing, \varnothing\big) \qquad \textit{is strongly } N_1\textit{-generic.} \tag{6}$$
$$\big(\varnothing, \varnothing, \{(N_0 \cap L_\kappa, \alpha_0)\}\big) \qquad \textit{is strongly } N_0\textit{-generic,} \tag{7}$$

It is easy to see that there are stationarily many models of each of the three types. Clause (5) implies that the full forcing P has the κ^+-chain condition and hence preserves κ^+ and larger cardinals. Clause (6) implies that P does not collapse κ, and with a little more analysis it shows that the forcing to add just α_2 new closed, unbounded subsets of κ does not collapse κ so long as κ is at least $\alpha_2 + 1$-Mahlo. Finally, clause (7) implies that P does not collapse ω_1.

We will conclude this section by indicating briefly how Lemma 3.1 is used to complete the proof of Theorem 1.2. Suppose that $A = \langle a_\nu : \nu < \kappa \rangle$ is a sequence of countable subsets of ω_2 in $V[G]$; we will show that $B(A) \cap \mathrm{Cof}(\omega_1)$ is nonstationary. Let $\dot{A}$ be a name for A, and fix N_2 as in Lemma 3.1 so that $\dot{A}$ is in N_2. Then by clause (5) it is forced that $\dot{A} \in V[G \cap N_2]$, and that $G \cap N_2$ is a generic subset of $P \cap N_2$.

Set $\alpha := \sup(N_2)$. For $\lambda < \kappa$ let $N_1(\lambda)$ be the Skolem hull in N_2 of $\lambda \cup \{\kappa, \dot{A}, C_\alpha\}$, and let E be the closed unbounded subset of cardinals $\lambda < \kappa$ such that $\lambda = N_1(\lambda) \cap \kappa$. For $\lambda \in E$ set $\alpha(\lambda) := \sup(N_1(\lambda))$, and note that $C_{\alpha(\lambda)} = C_\alpha \cap \alpha(\lambda) \subset N_1(\lambda)$. We claim that $\mathrm{Cof}(\omega_1) \cap E \cap D_{\alpha+1} \cap B(A) = \varnothing$. To see this, let λ be a cardinal of uncountable cofinality in $E \cap D_{\alpha+1}$. Then $\{(\alpha, \lambda), \varnothing, \varnothing\} \in G$, and by clause (5) it follows that $A \restriction \lambda \in V[G \cap N_1(\lambda)]$. This implies, as in the proof of Lemma 2.11, that $\lambda \notin B(A)$: if $c \subset \lambda$ is a set of order type ω_1 such that every initial segment is in $A \restriction \lambda = \{a_\nu : \nu < \lambda\}$ then Lemma 2.10 implies that $c \in V[G \cap N_1(\lambda)]$. Hence λ is collapsed in $V[G \cap N_1(\lambda)]$; however, $P \cap N_1(\lambda)$ is isomorphic to the forcing to add $f_\alpha(\lambda)$ new closed unbounded subsets of λ, and this does not collapse λ because $\lambda \in D_{\alpha+1} \cap \mathrm{Cof}(\omega_1) \subset B_{\alpha+1}$, so λ is $f_\alpha(\lambda) + 1$-Mahlo.

4 Variations, Limitations, and Generalizations

We will begin this section by mentioning three other applications of the techniques of Section 2 and (in one case) Section 3: a simpler model with no ω_2-Aronszajn trees, a forcing which adds an ω_2-Souslin tree with finite conditions, and a model in which $I[\omega_2]$ is not 1-generated. Following this we will discuss some limitations of the method, and a final section will then look at possible generalizations of this method to larger cardinals.

4.1 Aronszajn trees We begin by showing how the use of models as side conditions, together with strong genericity, gives an alternative model for the main theorem of [11]. It may be noted that the model in Section 2 also gives such a model; however, the following proof is substantially simpler than either of these: the forcing is done using only the side conditions.

Theorem 4.1 *If κ is a Mahlo cardinal, then there is a generic extension in which there are no special ω_2-Aronszajn trees. If κ is a weakly compact cardinal, then there is a generic extension in which there are no ω_2-Aronszajn trees.*

Proof For this proof, we will use the term "model" to mean a countable set $M \prec_1 L_\kappa$. The conditions of P_κ are finite sets $\mathcal{A}$ of models such that for each

$M, M' \in \mathcal{A}$ one of the following conditions holds:

$$M \cap M' \in M \quad \text{or} \quad \exists\lambda \in M\ M \cap M' = M \cap L_\lambda \quad \text{or} \quad M \cap M' = M. \tag{8}$$

The ordering on P_κ is given by $\mathcal{A}' \leq \mathcal{A}$ if and only if $\mathcal{A}' \supset \mathcal{A}$.

First we observe that this forcing does collapse all ordinals $\nu < \kappa$ onto ω_1, since if G is a generic set then condition (8) implies that the set $X = \{ M \cap \nu : M \in G \;\&\; \nu \in M \}$ is linearly ordered by $\in$. Since each member of X is countable and $\nu = \bigcup X$, this implies that $|\nu| \leq \omega_1$ in the generic extension.

Proposition 4.2 *If $L_\lambda \prec_1 L_\kappa$ and $\mathrm{cf}(\lambda) > \omega$ then $\varnothing$ is strongly L_λ-generic, and any condition $\{M\}$ is strongly M-generic.*

Proof The function used to verify strong genericity is the following.

$$\mathcal{A}|X = \{ M \cap X : M \in \mathcal{A} \;\&\; M \cap X \in X \}$$

where X is either L_λ or M. The verification that $\mathcal{A}|X$ is a condition uses the observation that if M' is any member of $\mathcal{A}$ then ${}^{\omega}(M' \cap X) \cap M' \subset (M' \cap X)$.

Now suppose that $\mathcal{B} \leq \mathcal{A}|X$ is in $X \cap P_\kappa$; we will show that $\mathcal{A} \cup \mathcal{B}$ is a condition. To verify the condition (8), suppose that $N \in \mathcal{A}$ and $N' \in \mathcal{B}$. Then $N \cap N' = (N \cap X) \cap N'$. If $X = L_\lambda$, or if $X = M$ and $M \cap N \in M$, we have $N \cap X \in \mathcal{B}$, so N and N' satisfy (8) because $N \cap X$ and N' do so in $\mathcal{B}$. In the remaining case, when $X = M$ and $M \cap N$ is an initial segment of M, the fact that $N' \in M$ implies that $N \cap N'$ is an initial segment of N', so $N \cap N'$ is a member of $N \cap M$ and hence of N. $\square$

It follows that this forcing preserves the cardinals ω_1 and κ: ω_1 because the forcing is proper, and κ because the strong L_λ-genericity, together with the inaccessibility of κ, implies that the forcing has the κ-chain condition.

We now proceed exactly as in [11]. First suppose that κ is Mahlo and that the condition $\mathcal{A}$ forces "$\dot{T}$ is an ω_2-Aronszajn tree with specializing function $\dot{\sigma} : T \to \omega_1$". Pick $X \prec L_{\kappa^+}$ so that $\{\mathcal{A}, \dot{\sigma}, \dot{T}\} \in X$ and $X \cap L_\kappa = L_\lambda$ for some inaccessible cardinal $\lambda < \kappa$. Then $T \restriction \lambda$ and $\sigma \restriction (T \restriction \lambda)$ are each in $V[G \cap X]$. Pick any node of T of height λ, let b be the branch of T below that node, and let $\tau = (\sigma \restriction g)^{-1}$; that is, τ is the partial function defined by $\tau(\nu) = n$ if n is a node in b such that $\sigma(n) = \nu$. Then every initial segment of b and σ is in $V[G \cap X]$, and hence $\tau \restriction \xi \in V[G \cap X]$ for any $\xi < \omega_1$. It follows by Lemma 2.10 that $\tau \in V[G \cap X]$, but this is impossible because $P_\kappa \cap L_\lambda = P_\lambda$, which has the λ-chain condition because of the inaccessibility of λ.

Now suppose that κ is weakly compact, and $\mathcal{A}$ forces "$\dot{T}$ is a ω_2-Aronszajn tree". By the Π^1_1 indescribability of κ there is $X \prec L_{\kappa^+}$ with $\{\mathcal{A}, \dot{T}\} \subset X$ such that $X \cap L_\kappa = L_\lambda$ for some inaccessible cardinal $\lambda < \kappa$ and $\mathcal{A}$ forces in $P \cap X$ that "$\dot{T}$ has no branch of length λ". Since T has height κ it has a node at level λ, which determines a branch b through $T \restriction \lambda$. Every initial segment of b is determined by a node of $T \restriction \lambda$, and hence is in $V[G \cap X]$. It follows by Lemma 2.10 that $b \in V[G \cap X]$, but this contradicts the choice of X. $\square$

4.2 Forcing a Souslin Tree We now give a forcing P which adds a ω_2-Souslin tree using finite conditions.

Definition 4.3 Conditions in P are triples $p = (d, <, \mathcal{A})$ such that

1. d is a finite subset of $\omega_2 \times \omega_1$, and $<$ is a tree order on $d \cup \{0\}$ with root 0 such that $\alpha' < \alpha$ whenever $(\alpha', \nu') < (\alpha, \nu)$;

2. if $M \in \mathcal{A}$, $\alpha, \alpha' \in M \cup \lim(M)$, $\nu \in M$, and $(\alpha', \nu') < (\alpha, \nu)$ then $\nu' \in M$;
3. if $M \in \mathcal{A}$, and (α, ν) and (α', ν') are in $M \cap d$, then their meet, the largest node in the tree $(d, <)$ which is below both (α, ν) and (α', ν'), is also in M;
4. if $\alpha \in M \in \mathcal{A}$, $\alpha' = \sup(M \cap \alpha)$, and $\nu \in M$ then $(\alpha', \nu) <_p (\alpha, \nu)$ if either pair is in d;
5. if $M, M' \in \mathcal{A}$ then
   (a) either $M \cap M' \in M$ or $M \cap M'$ is an initial segment of M, and
   (b) $\lim(M) \cap \lim(M') = \lim(M \cap M')$.

The order on P is $(d', <', \mathcal{A}') \leq (d, <, \mathcal{A})$ if $d' \supset d$, $<' \supset <$, and $\mathcal{A}' \supset \mathcal{A}$.

We will write $\operatorname{levels}(d) = \{\alpha : \exists \nu\, (\alpha, \nu) \in d\}$, and we will use $b^p_{\alpha,\nu}$ for the branch below (α, ν), that is, $b^p_{\alpha,\nu}(\alpha') = \nu'$ if and only if $(\alpha', \nu') < (\alpha, \nu)$.

Definition 4.4 $p = (d, <, \mathcal{A})$ is *complete* if $\operatorname{domain}(b^p_{\alpha,\nu}) = \operatorname{levels}(d) \cap \alpha$ for all $(\alpha, \nu) \in d$, and $\sup(M \cap \lambda) \in \operatorname{levels}(d)$ and $\min(M \setminus \lambda) \in \operatorname{levels}(d)$ for each $\lambda \in \operatorname{levels}(d)$ and $M \in \mathcal{A}$.

A somewhat complicated combinatorial proof using induction on $\alpha < \omega_2$ proves the following lemma.

Lemma 4.5 *For any condition p and any pair $(\alpha, \nu) \in \omega_2 \times \omega_1$ there is a complete condition $p' \leq p$ with $(\alpha, \nu) \in d^{p'}$.*

If the proof to the previous lemma is done carefully it is then straightforward to prove the following strong genericity lemma.

Lemma 4.6 *Any condition $(\emptyset, \emptyset, \{M\})$ is strongly M-generic. If $X \prec_1 L_{\omega_2}$ is transitive and ω-closed then $(\emptyset, \emptyset, \emptyset)$ is strongly X-generic, with witnessing function*

$$(d, <, \mathcal{A})|X := \big(d \cap X, < \cap X, \{\, M \cap X : M \in \mathcal{A}\,\}\big).$$

Furthermore, suppose that $p = (d, <, \mathcal{A})$ is complete and $\lambda \in \operatorname{levels}(d)$, and that $r = (d^r, <^r, \mathcal{A}^r) \leq p|X$ is in $P \cap X$. Set $\lambda' := \sup(\operatorname{levels}(d) \cap X)$, and suppose that $(\lambda, \nu) \in d$ and $(\alpha, \xi) \in d^r$ with $(\lambda', b^p_{\lambda,\nu}(\lambda')) <^r (\alpha, \xi)$. Then there is a common extension q of p and r such that $(\lambda', b^p_{\lambda,\nu}(\lambda')) <^q (\alpha, \xi) <^q (\lambda, \nu)$.

Theorem 4.7 *Suppose that $G \subset P$ is generic and $T = \bigcup\{\, <^p : p \in G\,\}$. Then T is an ω_2-Souslin tree with domain $\omega_2 \times \omega_1 \cup \{0\}$.*

Proof It is straightforward to prove that T is a tree with domain $\omega_2 \times \omega_1 \cup \{0\}$, so it suffices to prove that every antichain in T has size at most ω_1. For this, let p be a condition forcing that $\dot{A}$ is an antichain, let $X \prec L_{\omega_3}$ be a model of size ω_1 with $\{\dot{A}, p\} \cup \omega_1 \subset X$, and let $\lambda := \sup(X \cap \omega_2)$.

I claim that $p \Vdash \dot{A} \subset X$. To see this it will be sufficient to verify that every node $(\lambda, \nu) \in T$ is comparable with some node $(\alpha, \xi) \in A \cap X$. To this end suppose $q \leq p$ is a complete condition with $(\lambda, \nu) \in d^q$ and set $\lambda' := \sup(X \cap \operatorname{levels}(d^q))$. Since q is complete there is some ν' such that $(\lambda', \nu') <^q (\lambda, \nu)$. Now choose $r \leq q|X$ in $P \cap X$ such that r forces for some (α, ξ) that $(\alpha, \xi) \in \dot{A}$ and (α, ξ) is comparable with (λ', ν'). Then Lemma 4.6 implies that there is a common extension q' of r and q such that $(\alpha, \xi) <^{q'} (\lambda, \nu)$. □

4.3 Generating $I[\omega_2]$ The third application is closer to the construction of Sections 2 and 3.

Theorem 4.8 *If it is consistent that there is a κ^+-Mahlo cardinal then it is consistent that there is no set $B \in I[\omega_2]$ such that $I[\omega_2]$ is generated by $\{B\} \cup \mathrm{NS}_{\omega_2}$.*

In Shelah's terminology this means that $I[\omega_2]$ is not 1-generated. Notice that this implies that $I[\omega_2]$ is not generated by fewer than ω_3 of its members, since the diagonal union of any ω_2 members of $I[\omega_2]$ is also a member of $I[\omega_2]$.

The proof of Theorem 4.8 will be given in detail elsewhere, and we will only give a brief discussion here. The ideal $I[\omega_2]$ in the generic extension is generated by the sets $\kappa \setminus B_\alpha$ for $\alpha < \kappa^+$. Consider Lemma 2.11, which states that adding the set D_α in the forcing of Section 3 makes $\kappa \setminus B_\alpha$ nonstationary and adds the set $B_{\alpha+1} \setminus B_\alpha$ to $I[\omega_2]$. In that model the first effect was intended; however, the second was an undesirable side effect which could be mitigated, using the third clause stating that $B_{\alpha+1} \notin I[\omega_2]$, by adding a closed and unbounded subset $D_{\alpha+1}$ of $B_{\alpha+1}$. In the current argument the second effect is desired, and it is the first which is an undesirable side effect. To avoid this the forcing P_B of Section 2 is replaced with a forcing which adds a $\square$-like sequence $\vec{C}$ of closed unbounded subsets of ordinals $\lambda \in B_\alpha$. This sequence, like the generic closed unbounded set C of Section 2, adds $\{\, \lambda \in B : B \cap \lambda \text{ is not stationary in } \lambda \,\}$ to $I[\omega_2]$, but unlike C it preserves all stationary subsets of κ.

4.4 Limitations Some consideration of the limitations of these techniques may help to guide a search for future applications. The first of these is the fact which motivated their discovery: the construction makes the continuum hypothesis false. Sy Friedman has conjectured that this might be avoided by using a morass as discussed later to require that $M \cong M'$ whenever M and M' are members of $\mathcal{A}$ such that $M \cap \omega_1 = M' \cap \omega_1$, and then designing the conditions so that this isomorphism extends to one between $M[G \cap M]$ and $M'[G \cap M']$; however, this proposal appears to be difficult to carry out.

A related fact, that every new countable set in $V[G]$ is added by a Cohen real, essentially rules out the use of these techniques for questions involving the topology of the reals. This is a consequence of the fact that there is a stationary set of countable models M having strongly generic conditions: if $\dot{f}$ is a name for a countable sequence, M is a countable model with $\dot{f} \in M$, and p is strongly M-generic, then $p \Vdash \dot{f} \in V[M \cap G]$. The set $M \cap G$ is a generic subset of $P \cap M$, which is an atomless countable forcing and hence is equivalent to Cohen forcing.

A third limitation is that the new closed unbounded set C is a subset of a stationary set of the form $B \cup \mathrm{Cof}(\omega)$, that is, any ordinal of countable cofinality is allowed to be a member of C. This restriction can be slightly weakened: essentially the same construction can be used to add a closed unbounded subset of any stationary set $B \subset \omega_2$, provided that B is mutually stationary in the sense that there is a stationary set of countable models M such that $\sup(M \cap \lambda) \in B$ for every $\lambda \in M \cap B$. Work of Stanley [17] indicates that the general problem of deciding which subsets of ω_2 can contain a closed unbounded set in a larger model does not have an easy answer; for example, Friedman [5] uses Stanley's results to show that if $0^\sharp$ exists then $0^\sharp$ is constructible from the set of $B \in \mathcal{P}^L(\omega_2^L)$ such that $\omega_2 \setminus B$ is nonstationary in some model M satisfying $\omega_2^M = \omega_2^L$.

5 Larger Cardinals

There are two possible approaches for extending these techniques to larger cardinals. The first is relatively straightforward, at least for cardinals of the form κ^{++} where κ is regular, but seems unlikely to be useful for singular cardinals. It is still not known if the second approach will work, even for simple applications, but if it does then it would be more promising for questions involving singular cardinals.

The more straightforward approach is to increase the size of the conditions: for a regular cardinal κ we can add a closed unbounded subset of κ^{++} by using as conditions triples $(I, O, \mathcal{A})$ which are as before except that I, O, and $\mathcal{A}$ each have cardinality less than κ and the members $M \in \mathcal{A}$ satisfy $|M| = \kappa$ and ${}^{<\kappa}M \subset M$. The techniques of Section 3 have not been checked in this case, but by using a μ^+-Mahlo cardinal $\mu > \kappa$ this should give a generic extension in which cardinals up to κ^+ are preserved, μ becomes κ^{++}, and $I[\kappa^{++}]$ contains no nonstationary subsets of $\mathrm{Cof}(\kappa^+)$.

It is unclear whether this idea could be modified to work at κ^+ where κ is a regular limit cardinal. It seems quite unlikely that it could be applied at κ^+, or even at κ^{++}, when κ is singular.

The approach described above would not add new bounded subsets of κ, but would make $2^\kappa = \kappa^{++}$. A second possible approach, which would add a closed unbounded subset of a successor cardinal κ^+ while making $2^\omega = \kappa^+$, would use finite conditions as in Section 2, but would allow $\mathcal{A}$ to include models of any cardinality less than κ. Preliminary investigations of this possibility have suggested that the allowable sets $\mathcal{A}$ would need to have properties similar to those coming up in higher gap morasses, and it might even be simplest to explicitly take the models from such a morass. Little progress has been made so far using this approach, which is complicated by the fact that no description of the sets from even a gap 2 morass is known.

The use of models which are members, or at least resemble members, of a morass, has a number of precedents in the use of forcing with models as side conditions: indeed most published work using this technique, except those in which the cardinal ω_2 is collapsed, can be viewed as taking models from an ordinary gap-1 morass. A clear example explicitly using sets from a simplified morass is given by Koszmider in [9]. Koszmider's approach is very similar to that used in this paper: on the one hand the forcing in Section 2 can be done using a morass in his style, and on the other hand the author has written a note [10] using the method of this paper to prove Koszmider's result.

There seems to be no obvious reason why this technique could not be applied, through the use of a gap-ω morass, to the study of $I[\omega_\omega^+]$. It might seem that the third of the limitations described earlier would block such an approach: we noted that in order to use the forcing of Section 2 to add a new closed, unbounded subset of a set $B \subset \omega_2$ the set B must essentially contain $\mathrm{Cof}(\omega) \cap \omega_2$. Under the proposed extension to the successor μ^+ of a singular cardinal, B would have to include all ordinals of any cofinality less than μ, and this would seem to say that B must contain essentially all of μ^+. If we start with enough cardinal strength, however, it may be possible to operate as in Section 2, adding a closed unbounded subset of an appropriate cardinal κ larger than μ while collapsing the intermediate cardinals so that κ becomes μ^+. It might be hoped that in the resulting model $I[\mu^+]$ would be generated by the set $\{\,\lambda < \kappa : \mathrm{cf}^V(\lambda) < \mu\,\}$, while $\{\,\lambda < \kappa : \mathrm{cf}^V(\lambda) = \lambda \;\;\&\;\; \mathrm{cf}^{V[G]}(\lambda) = \eta\,\}$ is stationary for each regular cardinal $\eta < \mu$. Such a model would fit in nicely with the fact, due to Shelah ([16], also see Kojman [8]) that if μ is singular then for every regular cardinal $\eta < \mu$ there is a stationary subset of $\mu^+ \cap \mathrm{Cof}(\mu)$ in $I[\mu^+]$.

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