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# Ramsey Theory for Countable Binary Homogeneous Structures

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**Abstract** Countable homogeneous relational structures have been studied by many people. One area of focus is the Ramsey theory of such structures. After a review of background material, a partition theorem of Laflamme, Sauer, and Vuksanovic for countable homogeneous binary relational structures is discussed with a focus on the size of the set of unavoidable colors.

## 1 Introduction

The Rado graph, $\mathbb{RG} = (\omega, E_{\mathbb{RG}})$, is a special case of a countable homogeneous binary relational structure of degree 2. It is named for Richard Rado who described one construction of it in a paper published in 1964 [29] where he looked at questions relating to larger cardinals. It is universal in the sense that every finite graph is isomorphic to an induced subgraph. Erdős and Rényi [12] observed that a graph obtained by choosing edges independently with probability one half is isomorphic to the Rado graph with probability 1. So the Rado graph is frequently called the infinite *random graph*.

For every $n$ and every countable homogeneous binary relational structure $\mathbb{U} = (U; \mathcal{L})$ of finite degree $d$, Laflamme, Sauer, and Vuksanovic [26] describe a canonical equivalence relation on the $n$-element subsets and write $r_{\mathbb{U}}$ for the number of equivalence classes. An equivalence relation is *canonical* if its equivalence classes are *persistent*, that is, each equivalence class has a representative in every copy of the structure within itself, and the equivalence classes are *indivisible*, that is, for every finite partition of each equivalence class, there is a copy of the structure in itself such that all elements of the equivalence class in the copy lie in the same cell of the partition. The canonical partition is the partition into these equivalence classes. The work of Laflamme, Sauer, and Vuksanovic generalizes that of Erdős

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and Rado [13] who determined the canonical partitions of $n$ element sequences of natural numbers.

Any coloring $c : [U]^n \to r_{\mathbb{U}}$ which is monochromatic on the equivalence classes and takes different colors on different equivalence classes is an example witnessing $\mathbb{U} \nrightarrow [\mathbb{U}]^n_{r_{\mathbb{U}}}$, that is, a coloring of the $n$-element subsets for which there is no copy of $\mathbb{U}$ whose $n$-element subsets receive fewer colors under this coloring. An additional property of $r_{\mathbb{U}}$ is that $\mathbb{U} \to (\mathbb{U})^n_{<\omega/r_{\mathbb{U}}}$, namely, that for every coloring of the $n$-element subsets, there is a copy of $\mathbb{U}$ whose $n$-element subsets receive at most $r_{\mathbb{U}}$ colors under this coloring.

In Section 2, universal relational structures are discussed to provide a context for the result of Laflamme, Sauer, and Vuksanovic. In Section 3, connections are made to other combinatorial questions. In Section 4, a widely applied theorem of Milliken is reviewed, with an eye to the simplifications possible for trees of finite sequences under extension. In Section 5, an outline is given of the proof of the result of Laflamme, Sauer, and Vuksanovic, together with a proof of the canonization directly from Milliken's Theorem and their Strong Diagonalization. The goal is to highlight the key ingredients in the proof which spans two preprints each about twenty-five pages long. In Section 6, an algorithm is described for computing the critical value in their partition theorem and for colorings of cliques and anticliques in the Rado Graph.

## 2 Universal Relational Structures

The Rado graph is *homogeneous* in the sense that any isomorphism between finite substructures can be extended to an automorphism of the entire graph. Some authors call this property *ultrahomogeneity* and use *homogeneity* to refer to structures with the property that for any pair of isomorphic finite substructures, there is an automorphism of the entire structure that sends the first finite substructure onto the second. Thus a countable homogeneous structure is universal for the class of its finite substructures.

Fraïssé [15] introduced the notion of *age* of a relational structure $\mathbb{U}$, where the Age($\mathbb{U}$) of a structure is the class of all finite substructures over the same language which are embeddable in $\mathbb{U}$. He proved the following theorem which is a foundation of the classification of countable universal relational structures. For a general reference, see his book on the *Theory of Relations* [15].

**Theorem 2.1 (Fraïssé [15])** *Suppose $\mathfrak{C}$ is a class of structures for a language $\mathfrak{L}$. Then $\mathfrak{C}$ is the age of a countable homogeneous relational structure $\mathbb{U} = (U; \mathfrak{L})$ if and only if $\mathfrak{C}$ satisfies the following four conditions:*

1. *every isomorphic copy of a structure in $\mathfrak{C}$ is in $\mathfrak{C}$;*
2. *every induced substructure of a structure in $\mathfrak{C}$ is in $\mathfrak{C}$;*
3. *$\mathfrak{C}$ contains only countably many nonisomorphic structures;*
4. *$\mathfrak{C}$ has the* amalgamation property*: given $\mathbb{B}_1 = (B_1; \mathfrak{L}), \mathbb{B}_2 = (B_2; \mathfrak{L}) \in \mathfrak{C}$ and an isomorphism $f : A_1 \to A_2$ for $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, there is $\mathbb{C} = (C; \mathfrak{L}) \in \mathfrak{C}$ and embeddings $g_1 : B_1 \to C$, $g_2 : B_2 \to C$ such that for all $a \in A_1$, $g_1(a) = g_2(f_1(a))$. In other words, $\mathbb{B}_1$ and $\mathbb{B}_2$ are embedded in $\mathbb{C}$ in such a way that $A_1$ and $A_2$ are identified according to the isomorphism $f$.*

*If the four conditions above hold, then the structure* $\mathbb{U}$ *is called the* Fraïssé limit *of* $\mathcal{C}$.

Another example of a countable binary homogenous structure is the rationals with the usual order, $(\mathbb{Q}, <)$. It is the Fraïssé limit of the collection of finite totally ordered sets.

Droste and Kuske [9] have shown that if a class of countable relational structures contains an infinite $\omega$-categorical universal homogeneous structure $\mathbb{U}$, then $\mathbb{U}$ can be constructed probabilistically.

Lachlan [24] has described a combinatorial method for determining the homogeneous countable structures for a finite relational language which continues to serve as a basis for a program to explore homogeneous structures (see [6]).

Cherlin and Shi [7] have reduced the question of the existence of a universal countable structure for a finite relational language omitting a finite set of finite relational structures to the case of {0, 1}-vertex colored finite graphs.

Lachlan and Woodrow classified the countable homogeneous graphs. Here $K_n$ stands for a complete graph on $n$ vertices.

**Theorem 2.2 (Lachlan-Woodrow [25])** *Every countable homogeneous graph is isomorphic to one of the following:*

1. *the Random graph;*
2. *the Fraïssé limit (see Section 3) of the class of finite graphs omitting* $K_n$*, for fixed* $n \geq 3$*;*
3. *the disjoint union of* $m$ *complete graphs of size* $n$*, for* $m, n \leq \omega$ *with* $\max\{m, n\} = \omega$*;*
4. *the complements of the graphs in the previous item.*

Several other collections of countable homogeneous structures have been classified: Schmerl [31] has classified countable homogeneous partial orders (there are countably many of them); Lachlan [23] classified countable homogeneous tournaments ($(\mathbb{Q}, <)$ is an example; there are two finite and three infinite isomorphism types); Cherlin [5] classified the countable homogeneous directed graphs (there are continuum many).

Hrushovski [20] developed a way to build a constrained amalgamation of a class rather than a free amalgamation of the class in the Fraïssé limit. In [14], Evans gives an axiomatic framework for some of the $\aleph_0$-categorical structures constructed by Hrushovski. Baldwin [1] has a discussion of these constructions which result in what he calls generic structures and their connections to the Rado graph, and Baldwin and Holland [2] look at questions of model-completeness for them.

Universal structures for linear orderings have been studied for many cardinalities. Shelah [33], for example, showed the consistency of the existence of a universal order at $\aleph_1$ with the negation of the Continuum Hypothesis. Kojman and Shelah [22] show there can be a universal linear order at a regular cardinal $\lambda$ only if $\lambda = \lambda^{<\lambda}$ or if $\lambda = \mu^+$ and $2^{<\mu} \leq \lambda$. They also show that if a singular cardinal $\mu$ is not a strong limit and is not a fixed point of the $\aleph$ function, then there is no universal order in $\mu$.

The study of universal graphs continues, for example, with a recent paper by Džamonja and Shelah [10], in which it is shown consistent that there is a singular cardinal $\kappa$ of cofinality $\omega$ with $2^{\kappa^+}$ much larger than $\kappa^{++}$ but for which there is a collection of $\kappa^{++}$ graphs of cardinality $\kappa^+$ such that every graph of cardinality $\kappa^+$ embeds isomorphically into one of them.

Recently, Kechris, Todorčević, and Prestov [21] explored connections between the topological dynamics of automorphism groups, the Fraïssé theory of amalgamation, and the Ramsey theory of classes of finite structures. For a discussion of finite structural Ramsey theory, with many references, this paper is recommended.

## 3 Combinatorial Connections

Infinitary Ramsey theory may be regarded as the study of the relationship between small (or well-behaved) substructures of a larger structure and what they have to say about copies of the larger structure within itself.

Compactness and failures of compactness fit this description, matching the *coloring* or *partition* property of $\kappa \to (\kappa)^2_2$ for uncountable $\kappa$ with weakly compact cardinals. The arrow notation given here means that for any coloring $f : [\kappa]^2 \to 2$ of the (unordered) pairs from $\kappa$ with two colors, there is a subset of cardinality $\kappa$ which is *homogeneous* or *monochromatic* for one of the colors.

Two cardinal models with a universe of specified size and a distinguished unary predicate of a smaller specified size have been studied for some time. Furkhen [16] in 1965 had an early result on compactness for such models using ultraproducts. Shelah looked at two cardinal compactness in a paper [32] that appeared in 1971 and developed the notion of *identities* as the combinatorial essence needed to build such models. Continuing this study, Gilchrist and Shelah [17] looked at the types of colorings of finite subgraphs that must occur (they call them *identities*) when the complete graph on $\kappa$ vertices is colored with countably many colors where $\aleph_1 \le \kappa \le \aleph_\omega$. This particular example shows that singular cardinal combinatorics uses finite combinatorics.

Henson [19] proved that the Rado graph satisfies the pigeonhole principle, that is, every finite partition of the vertices has a block isomorphic to the whole graph. Cameron [4] showed that the Rado graph, $K_{\aleph_0}$ (the complete graph on countably many points) and its complement are, up to isomorphism, the only graphs with this property. Bonato and Delić describe a large collection of relational structures that satisfy the pigeonhole principle. Continuing that project, Bonato, Cameron, and Delić [3] classify the continuum many nonisomorphic countable tournaments that satisfy the pigeonhole principle and discuss orders, quasi orders, and oriented graphs satisfying it.

Galvin (see [15]) gave a canonization argument to show that $\mathbb{Q} \to (\mathbb{Q})^2_{<\omega,2}$. In his 1979 doctoral dissertation, Devlin [8] generalized the result and proved

$$\mathbb{Q} \to (\mathbb{Q})^n_{<\omega/t_n} \text{ and } \mathbb{Q} \nrightarrow [\mathbb{Q}]^n_{t_n},$$

where $t_n$ is the $n$th tangent number and may be computed using the generating function $\tan(x) = \sum_1^\infty t_n \frac{x^{2n-1}}{(2n-1)!}$. In 2002, Vuksanovic [35] gave a proof of Devlin's result using binary trees, where Devlin's proof used the language of category theory. While Devlin proved a partition theorem for the rationals, a dense linear order without endpoints, Laflamme, Sauer, and Vuksanovic proved a parallel partition theorem for countable homogeneous binary relational structures. For another perspective on the theorem of Laflamme, Sauer and Vuksanovic, the reader is referred to a paper by Vuksanovic [36] which gives a proof of the special case for the Rado graph and includes the computation of some small values.

## 4 Milliken's Theorem Revisited

A very useful tool in partition theory for countable structures is a Ramsey Theorem for weakly embedded subtrees proved by Milliken [28]. It will be used in Section 5 in a discussion of partitions of $n$-element subsets of a countable homogeneous binary structure. Some notation and definitions are necessary before we can state Milliken's result. One purpose of this section is to update the notation to fit with that of Laflamme, Sauer, and Vuksanovic.

Call a tree $S$ a $p$-*tree* if all its branches have order type $p$.

**Definition 4.1 (Definition 1.2 [28])** Suppose $S$ is a $p$-tree and $T$ is a $q$-tree for $0 \leq p \leq q \leq \omega$. We say $S$ is *embedded* in $T$ provided

1. $S \subseteq T$, and the partial order on $S$ is induced from $T$,
2. if $s \in S$ is nonmaximal in $S$, then for every immediate successor $t$ of $s$ in $T$, there is a single immediate successor of $s$ in $S$ which is a sucessor of $t$ in $T$ (allow the possibility that $s = t$).

We say $S$ is *weakly embedded* in $T$ provided condition 1 above holds and condition 2 is replaced by condition 3 below:

3. if $s \in S$ is nonmaximal in $S$ and $t$ is an immediate successor of $s$ in $T$, then there is at most one immediate successor of $s$ in $S$ which is a sucessor of $t$ in $T$.

We say $S$ is *strongly embedded* in $T$ provided $S$ is embedded in $T$, and

4. there is an increasing function $l : p \to q$ such that for all $i < p$, the $i$th level of $S$ is a subset of the $l(i)$th level of $T$.

A mild generalization of the notion of weakly embedded tree is obtained by replacing the requirement that $S$ be a $p$-tree with the requirement that it be rooted. It is frequently convenient to work with trees of finite sequences with entries from $\omega$. Let $\omega^{<\mathbb{N}}$ denote the tree of all such finite sequences partially ordered by $\subseteq$, and for $2 \leq d < \omega$, let $d^{<\mathbb{N}}$ denote the tree of all finite sequences with entries from $d = \{0, 1, \ldots, d-1\}$. Both kinds of trees are *regular*, since each node has the same number of immediate successors. There is a nice characterization of the weakly embedded subtrees of $d^{<\mathbb{N}}$ using *meets*: for any pair of incomparable elements $s$ and $t$, the *meet* of $s$ and $t$, denoted $s \wedge t$, is the unique maximal length initial segment common to both. The closure of a set $A$ under taking pairwise meets is denoted by $A^{\wedge}$.

**Lemma 4.2** *Suppose* $2 \leq d \leq \omega$ *and* $S \subseteq T = d^{<\mathbb{N}}$ *is nonempty. Then* $S$ *is a (rooted) weakly embedded subtree of* $T$ *if and only if* $S^{\wedge} \subseteq S$.

**Proof** For the first direction, suppose $S^{\wedge} \subseteq S$. Since $S$ is nonempty, it has an element of minimal length. Since it is closed under meets, there is only one element of this minimal length and it is a subset of every other element of $S$, so $S$ is a rooted induced subtree of $T$. To see that it is weakly embedded, suppose $s \in S$ is nonmaximal and $s^\frown\langle\delta\rangle$ is an immediate successor of $s$ in $T$. Assume toward a contradiction that $t$ and $u$ are different immediate successors of $s$ with $s^\frown\langle\delta\rangle \subseteq t$ and $s^\frown\langle\delta\rangle \subseteq u$. Then $s^\frown\langle\delta\rangle \subseteq t \wedge u \in S$, contradicting $t$ and $u$ being immediate successors of $s$ in $S$.

For the other direction, suppose $S \subseteq T$ is a rooted weakly embedded subtree. Let $t$ and $u$ be two different elements of $S$. If $t \subseteq u$ or $u \subseteq t$, then $t \wedge u \in \{t, u\}$. So assume $t$ and $u$ are incomparable. Since $S$ is rooted, the set of all $r$ in $S$ with $r \subseteq t$ and $r \subseteq u$ is the intersection of $S$ with the set of initial segments of $t \wedge u$. Since $S$ is rooted, this set is nonempty. Let $r^*$ be the longest initial segment of $t \wedge u$ in $S$. If $r^*$ is a proper initial segment of $t \wedge u$, then for some $\delta$, $r^{*\frown}\langle\delta\rangle \subseteq t \wedge u$. Let $t^* \subseteq t$ and $u^* \subseteq u$ be the shortest initial segments of $t$, and $u$, respectively, in $S$ that properly extend $r^*$. Then $t^*$ and $u^*$ are immediate successors of $r^*$ in $S$ extending $r^{*\frown}\langle\delta\rangle$, contradicting the fact that $S$ is weakly embedded. □

The following lemma is an immediate consequence of the definition of strongly embedded tree.

**Lemma 4.3** *Suppose $2 \le d < \omega$ and $T = d^{<\mathbb{N}}$. If $R$ is a strongly embedded subtree of $S$ and $S$ is a strongly embedded subtree of $T$, then $R$ is a strongly embedded subtree of $T$.*

There is a useful characterization of strongly embedded $\omega$-trees in $d^{<\mathbb{N}}$ for finite $d \ge 2$ using *passing number preserving* maps.

**Definition 4.4** For $z, x \in d^{<\mathbb{N}}$ with $|z| > |x|$, call $z(|x|)$ the *passing number* of $z$ at $x$. Call a function $f : T \to T$ *passing number preserving* or a *pnp map* if it preserves

1. length order: ($|x| < |y|$ implies $|f(x)| < |f(y)|$) and
2. passing numbers: ($|x| < |y|$ implies $f(y)(|f(x)|) = y(|x|)$).

**Lemma 4.5** *Suppose $2 \le d < \omega$ and $S \subseteq T = d^{<\mathbb{N}}$. Then $S$ is a strongly embedded subtree of $T$ if and only if there is an extension and passing number preserving bijection $h$ from $T$ to $S$ that carries levels to levels.*

**Proof** If there is a such a map $h : T \to T$ with $h[T] = S$, then $S$ is an induced subtree of $T$, every node $s = h(t)$ in $S$ has a unique extension of $s^\frown\langle\delta\rangle$, namely, $h(t^\frown\langle\delta\rangle)$, and the function $l : \omega \to \omega$ such that the $i$th level of $S$, which is the image under $h$ of the $i$th level of $T$, is a subset of the $l(i)$th level of $T$, must be increasing because as a pnp map, $h$ preserves length order.

For the other direction, suppose $S$ is a strongly embedded tree and $l : \omega \to \omega$ is the increasing function so that the $i$th level of $S$ is a subset of the $l(i)$th level of $T$. Define $g : S \to T$ as follows: for all $s \in S$ of the $i$th level of $S$, let $g(s) = s \circ l \restriction i$. By induction on $i$, show that $g$ maps the $i$th level of $S$ onto the $i$th level of $T$. By condition 2 of the definition of embedded tree (Definition 4.1), $g$ is injective. The reader may check that $h = g^{-1}$ is the desired extension and passing number preserving bijection from $T$ to $S$ that carries levels to levels. □

One more definition is necessary before we can state Milliken's Theorem. It is stated here for subtrees of $d^{<\mathbb{N}}$ in which the extended order is the lexicographic order $<_{\text{lex}}$ which is defined on incomparable elements $a$ and $b$ by $a <_{\text{lex}} b$ if and only if $a(|a \wedge b|) < b(|a \wedge b|)$.

**Definition 4.6 (Definition 4.1 [28])** For any finite $d \ge 2$ and any subtrees $A$ and $B$ weakly embedded in $S = d^{<\mathbb{N}}$, we say $A$ and $B$ *have the same embedding type*, in symbols, $A \sim_{\text{Em}} B$, provided the following conditions hold:

1. there is an order isomorphism $f : A \to B$, that is, a bijection satisfying $a \subsetneq a'$ if and only if $f(a) \subsetneq f(a')$;

2. if $a$ is on the $n$th level of $S$, $a'$ is on the $n'$th level of $S$, $f(a)$ is on the $m$th level of $S$, and $f(a')$ is on the $m'$th level of $S$, then $n < n'$ if and only if $m < m'$;
3. suppose $A$ has an element $e$ on the $n$th level of $S$ with $f(e)$ on the $m$th level of $S$; further suppose $a$ is an element of the $n$th level of $S$ with some proper extension $c$ in $A$; then we require that for each $\delta < e$, $a^\frown\langle\delta\rangle \subseteq c$ if and only if $(f(a)\restriction m)^\frown\langle\delta\rangle \subseteq f(c)$.

In other words, there is an extension and passing number preserving bijection $f : A \to B$.

Here is Milliken's Ramsey theorem for weakly embedded subtrees of a finite regular $\omega$-tree. His proof uses the Laver-Pincus version of the Halpern-Läuchli Theorem.

**Theorem 4.7 (Milliken's Theorem [28])** *Suppose $2 \leq d < \omega$ and $T = d^{<\mathbb{N}}$. For any $m < \omega$, any finite weakly embedded subtree $P \subseteq T$, any strongly embedded tree $T' \subseteq T$, and any coloring $d : \left\{ Q \subseteq T' : Q \sim_{Em} P \right\} \to M$, there are $k < m$ and a strongly embedded subtree $T'' \subseteq T'$ such that for all $R \subseteq T''$, if $R \sim_{Em} P$ then $d(R) = k$.*

## 5 Partition Theorems

This section starts with a discussion of a partition theorem for the edges of the Rado graph and then has an overview of the proof of the Laflamme, Sauer, and Vuksanovic theorem.

The Rado Graph RG $= (\omega, E_{\mathrm{RG}})$ is a particularly nice example of a countable homogeneous structure. Its edge relation is a binary symmetric irreflexive relation (no loops).

For each $n < \omega$, define $u_n : n \to 2$ by $u_n(i) = 1$ if and only if $\{i, n\} \in E_{\mathrm{RG}}$. Notice that the mapping $n \mapsto u_n$ is an embedding of the Rado graph in the binary tree $2^{<\mathbb{N}}$.

Call an edge $\{i, n\}_< \in E_{\mathrm{RG}}$ an "up" edge if $u_i \not\subseteq u_n$ and $u_i <_{\mathrm{lex}} u_n$; call it a "down" edge if $u_i \not\subseteq u_n$ and $u_n <_{\mathrm{lex}} u_i$. Here $<_{\mathrm{lex}}$ is the lexicographic order on the binary sequences. Erdős, Hajnal, and Pósa showed that every copy of the Rado graph in itself must have both kinds of edges.

**Theorem 5.1 (Erdős, Hajnal, Pósa [11])** *In every copy of the Rado Graph inside itself, there are both "up" edges and "down" edges:*

$$RG \nrightarrow [RG]_2^{\mathrm{edge}}.$$

The above result appeared in 1975. In a preprint dated 2003, Laflamme, Sauer, and Vuksanovic generalized this result to colorings of $n$-element subsets rather than just edges, and to countable homogeneous binary relational structures of finite degree $d$. Such structures may be embedded in the regular $d$-branching tree $d^{<\mathbb{N}}$, just as Erdős, Hajnal, and Pósa embedded the Rado graph into the complete binary tree of finite sequences of zeros and ones.

**Theorem 5.2 (Laflamme, Sauer, Vuksanovic [26])** *If $\mathbb{U}$ is a countable universal binary homogeneous structure of degree $d < \omega$, then there is some $r_n(d)$ such that*

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/r_n(d)} \text{ and } \mathbb{U} \nrightarrow [\mathbb{U}]^n_{r_n(d)}.$$

The remainder of this section is a brief outline of the proof done with a broad brush.

The first step in the proof is a translation of the problem from one about relational structures into one about subsets of a tree of sequences (see Section 7 of [26] for further details). Suppose $\mathbb{U} = (U, R_0, \ldots, R_p)$ is a countable homogeneous binary relational structure where the $R_i$s are the binary relations and for all $u \in U$ and $m \leq p$, $\neg R_m(u, u)$. Add a new binary relation $<$ whose interpretation is a well-order of order type $\omega$. Thus $\mathbb{U}$ is isomorphic to a structure with universe $\omega$ in which the order relation is interpreted as $\in$, say $\mathbb{U}' = (\omega, R'_0, \ldots, R'_p, \in)$. Thus without loss of generality, assume that $\mathbb{U}$ has universe $\omega$. The type of a pair $i < n$ is the structure $\tau$ with domain $\{0, 1\}$ isomorphic to the restriction of $\mathbb{U}$ to $\{i, n\}$. List the types of two element sets of $\mathbb{U}$ as $\tau_0, \tau_1, \ldots, \tau_{d-1}$. For each $n$, define $u_n : n \to d$ by $u_n(i) = c$ if and only if $\{ i, n \}$ has type $\tau_c$. We call $d$ the *degree* of the structure.

Let $T$ be the regular $d$-branching tree of finite sequences with entries from $\{ 0, 1, \ldots, d-1 \}$. The mapping $\sigma : \omega \to d^{<\mathbb{N}}$ defined by $n \mapsto u_n$ is an embedding into a cofinal subset of the regular $d$-branching tree of finite sequences whose entries come from $d = \{ 0, 1, \ldots, d-1 \}$. It is cofinal, that is, for every $s \in d^{<\mathbb{N}}$, there is some $t \in d^{<\mathbb{N}}$ with $s \subseteq t$, by the homogeneity property.

Information about the type of a two element structure $\{ i, n \} \subseteq \omega$ inside $\mathbb{U}$ is coded in the value $u_n(i) = u_n(|u_i|)$. The theorem quoted below appears in Section 7 of [26].

**Theorem 5.3 (Translation Theorem)** *A function $G : \omega \to \omega$ is an isomorphism of $\mathbb{U}$ into itself if and only if $\sigma \circ G \circ \sigma^{-1}$ is a pnp map of $\sigma[\omega]$ to $\sigma[\omega]$.*

The second step in the proof is an examination of tree structure. In the proof of the Erdős-Hajnal-Pósa result, the lexicographic order on incomparable elements of $2^{<\mathbb{N}}$ played an important role. Call a subset $A \subseteq d^{<\mathbb{N}}$ an *antichain* if its elements are pairwise incomparable. A subset $D$ is *diagonal* if it is an antichain, $(D^{\wedge}, \subseteq)$ is a binary tree, and different elements of $D^{\wedge}$ have different lengths.

We have already defined lexicographic order on incomparable pairs, $<_{\text{lex}}$. The subset relation, $\subseteq$, is another partial order on $d^{<\mathbb{N}}$, and it is the tree order.

**Definition 5.4** Call a bijective pnp map $f : R \to S$ a *similarity* if it satisfies the following conditions for all $x, y, u, v \in R$:

1. $x \wedge y \subseteq u \wedge v$ if and only if $f(x) \wedge f(y) \subseteq f(u) \wedge f(v)$; and
2. $|x \wedge y| < |u \wedge v|$ if and only if $|f(x) \wedge f(y)| < |f(u) \wedge f(v)|$; and
3. if $x <_{\text{lex}} y$, then $f(x) <_{\text{lex}} f(y)$.

Call $R$ and $S$ *similar* and write $R \sim S$, if there is a similarity $f : R \to S$. Then similarity is an equivalence relation on $T$. A similarity $f : R \to S$ extends via $f(x \wedge y) = f(x) \wedge f(y)$ uniquely to a bijection $f^*$ of $R^{\wedge}$ to $S^{\wedge}$. Note that $f^*$ preserves the tree order (extension) and length order, but may fail to be a similarity by failing to preserve passing numbers. If $f^*$ does preserve passing numbers, then $R^{\wedge} \sim_{\text{Em}} S^{\wedge}$.

The next theorem, a consequence of Theorems 4.1 and 5.1 of [26], says that every similarity type of diagonal set occurs in the image under $\sigma$ of every copy of $\mathbb{U}$ inside itself. Section 4 of [26] devoted to this result is about two and a half preprint pages.

**Theorem 5.5 (Persistence Theorem)** *For any diagonal set $D \subseteq T$ and any pnp map $f : \sigma[\omega] \to \sigma[\omega]$, there is a diagonal set $E \subset f[\sigma[\omega]]$ with $D \sim E$.*

With these definitions and theorems in hand, the square bracket partition relation can be proved. This consequence of the cited theorem was chosen since it does not require additional definitions to state. Also the color classes of the coloring defined in the proof sketch is the canonical partition defined by Laflamme, Sauer, and Vuksanovic and alluded to in the introduction.

**Theorem 5.6 (Representation Theorem, 7.6 [26])** *$\mathbb{U} \nrightarrow [\mathbb{U}]^n_r$ where $r = r_n(d)$ is the number of $\sim$-equivalence classes of n-element diagonal subsets of $T$.*

**Proof Sketch** Since the meet closures of the $n$-element diagonal sets have size $2n-1$, there are only finitely many similarity classes. Enumerate them as

$$[E_1]_\sim, [E_2]_\sim, \ldots, [E_r]_\sim.$$

Define a coloring of $[\omega]^n$ by $c(A) = j$ if $\sigma[A]$ is diagonal with $\sigma[A] \sim E_j$, and $c(A) = 0$ otherwise. By the Representation Theorem 5.6 and the Persistence Theorem 5.5, no color is omitted in any subcopy of $\mathbb{U}$. □

The above theorem places a lower bound on the number of colors realized in every copy of $\mathbb{U}$ in some coloring of its $n$-tuples. For the Rado Graph, $(\mathbb{RG}, E_{\mathbb{RG}})$, the critical or *canonical* partition of 2-element sets into equivalence classes of diagonal similarity described in the proof above, is into "up edges", "down edges", "up nonedges", "down nonedges".

Next turn to the question of an upper bound. That is, we seek a fixed number of colors, so that every coloring of the $n$-element subsets of $\mathbb{U}$ with finitely many colors has a copy in which no more than the fixed number of colors are realized.

For a diagonal set $D$, the passing numbers of longer elements of $D$ at shorter elements of $D$ are important, but it is not clear at first glance exactly what other information from the embedding into the tree is important when looking at copies of the structure into itself and their images under the embedding $\sigma$.

**Definition 5.7** Sauer [30] singles out for special attention the *strongly diagonal sets*, which are diagonal subsets $F$ of $d^{<\mathbb{N}}$ with two additional properties for all $x, y, z \in F$ with $x \neq y$:

1. $|x \wedge y| < |z|$ and $x \wedge y \nsubseteq z$ implies $z(|x \wedge y|) = 0$;
2. $x(|x \wedge y|) = 0$ or $x(|x \wedge y|) = 1$.

Observe that every subset of a strongly diagonal set is strongly diagonal. Also, similar strongly diagonal sets have the same embedding type. The next lemma is proved in [30], which phrases it as follows: "If $f$ is a similarity of the strongly diagonal set $F$ to the strongly diagonal set $G$, then $f$ is a strong similarity."

**Lemma 5.8** *For any strongly diagonal sets $A, B \subseteq T$, if $A \sim B$, then $A^\wedge \sim_{Em} B^\wedge$.*

**Proof** Suppose $A, B \subseteq T$ are strongly diagonal subsets which are similar and $f : A \to B$ is the witnessing similarity. Define $f^* : A^\wedge \to B^\wedge$ extending $f$ by $f^*(x \wedge y) = f(x) \wedge f(y)$. As noted in the discussion following the definition of similar, $f^*$ preserves extension and length order. Thus to show $A^\wedge \sim_{\mathrm{Em}} B^\wedge$, it is enough to show $f^*$ preserves passing numbers.

By the definition of similarity, $f$ and hence $f^*$ preserves passing numbers for elements of $A$. It suffices to show that for all $x, y, z \in A$, if $|x \wedge y| < |z|$, then $f(z)(|f(x) \wedge f(y)|) = z(|x \wedge y|)$.

Suppose $x, y$ are two different elements of $A$, and by renaming if necessary, assume $x <_{\text{lex}} y$. Then $x(|x \wedge y|) = 0$ and $y(|x \wedge y|) = 1$, by definition of strongly diagonal. Consequently, $f(x)(|f(x) \wedge f(y)|) = 0$ and $f(y)(|f(x)gf(y)|) = 1$, since $f$ preserves the lexicographic order.

If $z$ is an element of $A$ with $|x \wedge y| < |z|$ and $x \wedge y \subseteq z$, then since $A^{\wedge}$ is binary, one of $(x \wedge y)^\frown\langle 0\rangle$ and $(x \wedge y)^\frown\langle 1\rangle$ is an initial segment of $z$, so $x \wedge y = x \wedge z$ or $x \wedge y = z \wedge y$. Thus by the argument of the above paragraph $f(z)(|f(x) \wedge f(y)|) = z(|x \wedge y|)$.

If $z$ is an element of $A$ with $|x \wedge y| < |z|$ and $x \wedge y \not\subseteq z$, then $z(|x \wedge y|) = 0$, since $A$ is strongly diagonal. Consequently, $f(z)(|f(x) \wedge f(y)|) = 0$ since $f$ preserves extension and $B$ is strongly diagonal. □

The theorem below (which is proved in [30] and quoted in [26]) says that the entire tree $T$ can be embedded in $\sigma[\omega]$ via an injective pnp map whose image is strongly diagonal and carries diagonal sets to similar diagonal sets. The latter property is listed last in the statement below and follows from the fact that the mapping $f$ is a pnp map satisfying conditions 1–3. Section 4 of [30] devoted to the proof of this theorem is about six preprint pages long.

**Theorem 5.9 (Diagonalization Theorem)** *For any cofinal set $S \subseteq T$ there is an injective pnp map $g : T \to S$ such that*

1. *the image, $g[T]$, is strongly diagonal;*
2. *$g$ preserves lexicograpic order of incomparable elements;*
3. *for all $x, y, u, v \in T$, $|x \wedge y| < |u \wedge v|$ implies $|g(x) \wedge g(y)| < |g(u) \wedge g(v)|$; and*
4. *for all diagonal $D \subset T$, the image $g[D]$ is diagonal and $D \sim g[D]$.*

*Call such a mapping a* strong diagonalization of $T$ into $S$.

The next theorem corresponds to Corollary 5.3 of [26] but is slightly stronger and is stated without the notation of that result. For completeness, a proof is given using results quoted above.

**Theorem 5.10 (Indivisibility)** *Suppose $S \subseteq T$ is a cofinal subset and $d' : [S]^n \to m$ is any coloring. Then there is a strong diagonalization $f : T \to S$ such that for all $A, B \in [f[T]]^n$, if $A \sim B$, then $d'(A) = d'(B)$.*

**Proof** Let $g : T \to S$ be a strong diagonalization obtained from Theorem 5.9. Define $e : [T]^n \to m$ by $e(A) = d'(g[A])$. Let $[E_1]_\sim, [E_2]_\sim, \dots, [E_r]_\sim$ list the $\sim$-equivalence classes of diagonal subsets of $T$. Without loss of generality, we may assume that each $E_i$ is strongly diagonal.

Apply Milliken's Theorem (Theorem 4.7) to $e$ and $(E_1{}^{\wedge}, \subseteq)$, $(E_2{}^{\wedge}, \subseteq)$, $\dots$, $(E_r{}^{\wedge}, \subseteq)$ to get a decreasing sequence of strongly embedded trees $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_r$ and a list of colors $k_1, k_2, \dots, k_r$ such that if $R \in [T_i]^n$ is diagonal and $R^{\wedge} \sim_{\text{Em}} E_i{}^{\wedge}$, then $e(R^{\wedge}) = k_i$.

Since $T_r$ is a strongly embedded subtree of $T$, there is an extension and passing number preserving map $h : T \to T$ with $T_r = h[T]$.

Let $f = g \circ h \circ g$. Then $f$ is an injective pnp map, since the composition of injective pnp maps is also an injective pnp map. Note that $f[T]$ is strongly diagonal since it is a subset of a strongly diagonal set, $g[T]$. Observe that $f$ preserves the lexicographic order of incomparable elements since both $g$ and $h$ do.

Since $h$ is an extension and passing number preserving bijection from $T$ to $T_r$, it preserves meets: for all $z, w \in T$, $h(z \wedge w) = h(z) \wedge h(w)$. Consequently $h[g[T]]$ is strongly diagonal, since $g[T]$ is strongly diagonal. Moreover, the function $h$ satisfies condition 3 of Theorem 5.9 since it is a pnp map and its image is closed under meets.

To see that $f$ satisfies condition 3 of Theorem 5.9, suppose $x, y, u, v \in T$ are such that $|x \wedge y| < |u \wedge v|$. Then $|g(x) \wedge g(y)| < |g(u) \wedge g(v)|$ by condition 3. Since $h$ is a pnp map that preserves meets, $|h(g(x)) \wedge h(g(y))| = |h(g(x) \wedge g(y))| < |h(g(u) \wedge g(v))| = |h(g(u)) \wedge h(g(v))|$. Since $g$ satisfies condition 3, it follows that $f = g \circ h \circ g$ satisfies condition 3.

As noted above, an injective pnp mapping which satisfies the first three conditions of Theorem 5.9 also satisfies the fourth, so $f$ is a strong diagonalization of $T$ into $S$.

To complete the proof of the theorem, consider an $n$-element subset $F \subseteq f[T]$ with $F = f[C]$ for some $n$-element set $C$. Let $E_i$ be such that $F \sim E_i$. Since $F$ and $E_i$ are both strongly diagonal, their meet closures have the same embedding type, $F^\wedge \sim_{\mathrm{Em}} E_i{}^\wedge$.

Now $D := h[g[C]]$ is strongly diagonal as a subset of a strongly diagonal set. Since $D \subseteq T_r$ and $h$ preserves meets, the meet closure, $D^\wedge$, is also a subset of $T_r$. Moreover, $D \sim F$ since $g$ is a strong diagonalization. Thus their meet closures have the same embedding type, $D^\wedge \sim_{\mathrm{Em}} F^\wedge \sim_{\mathrm{Em}} E_i{}^\wedge$. It follows that $e(D) = k_i = d'(g[D])$, so $d'(F) = k_i$.

Since any two similar subsets of $f[T]$ are both similar to some $E_i$, the theorem follows. □

**Theorem 5.11 (Limitation on Colors)** *Let $r = r_n(d)$ be the number of similarity types of $n$-element diagonal subsets of $T$. Then*

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/r}.$$

**Proof** Suppose $c : [\omega]^n \to m$ is an arbitrary coloring. Let $S = \sigma[\omega]$. Then $S$ is cofinal. Define $d' : [S]^n \to m$ by $d'(A) = c(\sigma[A])$.

Let $f : T \to S$ be the strong diagonalization of Theorem 5.10. Then there are at most $r_n(d)$ colors realized by $n$-element subsets of $f[T] \subseteq S$. Let $U = \sigma^{-1}[f[T]]$ and set $G = \sigma^{-1} \circ f \circ \sigma$. Then $G$ maps $\omega$ to $U$ and by the Translation Theorem 5.3, $G$ is an isomorphism of $\mathbb{U}$ into itself. The number of colors realized by $c$ on $n$-element subsets of $U$ is the same as the number of colors realized by $d'$ on $\sigma[U] = f[T]$. □

Now Theorem 5.2 follows from Theorems 5.6 and 5.11.

## 6 Counting Similarity Classes

This section discusses an approach to counting the number of similarity classes of $n$-element diagonal subsets of the regular $d$-branching tree $T$. By Theorem 5.10 and the proof of Theorem 5.6, there is a strongly diagonal set in each similarity class of $n$-element diagonal subsets of $T$.

If $A$ is an antichain and $l$ is the increasing enumeration of $\{\, |t| : t \in A^\wedge \,\}$, then let $\mathrm{clp}(A)$ be the downward closed tree whose leaves are $\{\, s \circ l : s \in A \,\}$. Note that $(A^{\mathrm{lev}}, \subseteq)$ and $(\mathrm{clp}(A), \subseteq)$ are isomorphic via a pnp map that takes levels to levels. Observe that $\mathrm{clp}(A)$ is a finite subtree closed under initial segments with the additional property that each level contains at least one node that is either a meet of two nodes of the tree or a leaf of the tree. Call any $P \subseteq T$ with these properties a *(strong)*

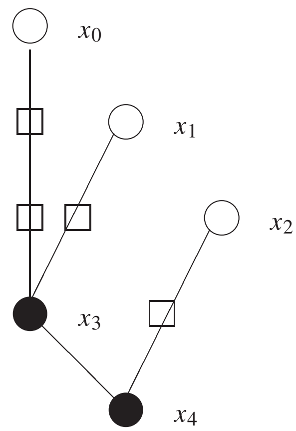


**Figure 1** A sample similarity tree with meet indicator sequence.

*similarity tree*. This name is motivated by the fact that if $A$ and $D$ are similar strongly diagonal subsets of $T$, then $\text{clp}(A) = \text{clp}(D)$.

Thus to compute the number $r_n(d)$ of similarity classes, it is enough to compute the number of similarity trees whose set of leaves is an $n$-element strongly diagonal set. Call such trees *strongly diagonal similarity trees*. Figure 1 is a drawing of a strongly diagonal similarity tree with three leaves, where the leaf nodes are indicated by open circles, the meets by solid circles, and restrictions to these lengths by open boxes. Since a strongly diagonal set $D$ with $n$-elements has binary meet closure, one can show that $|D^\wedge| = 2n - 1$.

For the purposes of counting strongly diagonal similarity trees, it turns out to be useful to define a statistic called a *meet indicator sequence*: if $D$ is a diagonal set and $x_0, x_1, \ldots, x_{2n-2}$ enumerates its meet closure in decreasing order of length, then $\mu = \mu_D$ is defined by $\mu(i) = +1$ if $x_i \in D$, that is, it is a leaf of $D^\wedge$ and $\mu(i) = -1$ if $x_i \in D^\wedge \setminus D$, that is, it is a meet of $D$. For the tree $R$ pictured in Figure 1, the meet indicator sequence of its set of leaves is $\mu_R = \langle +1, +1, +1, -1, -1\rangle$. For any sequence $\nu : m \to \{+1, -1\}$, the *tally of* $\nu$ is the function $\tau = \tau_\nu : m \to \omega$ defined by $\tau(0) = 0$ and for $j > 0$, $\tau(j) = \sum_{i<j} \nu(j)$.

For the tree $R$ pictured in Figure 2, the tally of its meet indicator sequence is $\langle 0, 1, 2, 3, 2\rangle$. Starting from the top, the $i$th value is the number of elements on the level of the tree one higher than that of $x_i$. This connection holds in general for a strongly diagonal similarity tree $S$ with leaves $D$, since the size of each level is increased or decreased by one, depending on whether the node on that level in $D^\wedge$ is a leaf or a meet. Hence for meet indicator sequences, their partial sums are all positive and the sum of the entire sequence is one. Such sequences have been studied.

A sequence $\rho : (2n + 1) \to \{-1, +1\}$ is a 2-*Raney sequence of length* $2n + 1$ if all of its partial sums are positive and its total sum is $+1$. Let $\mathcal{R}(n)$ denote the set of 2-Raney sequences of length $2n + 1$. For all $n < \omega$, the number of sequences in $\mathcal{R}(n)$ is a Catalan number:

$$|\mathcal{R}(n)| = C(n) = \binom{2n}{n}\frac{1}{n+1}.$$

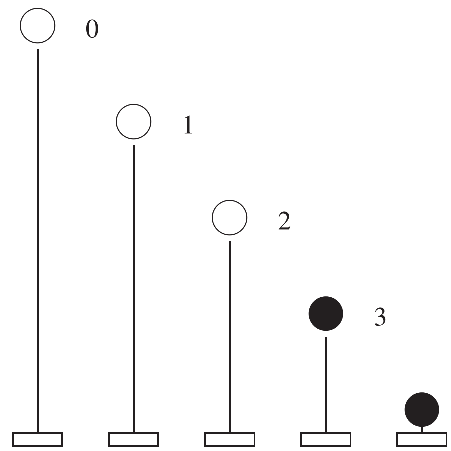


**Figure 2** Ingredients from which to build a similarity tree.

For more information on 2-Raney sequences, see *Concrete Mathematics* [18], pp. 345–47. They are closely related to the *ballot sequences* discussed in *Enumerative Combinatorics*, Vol. 2 [34] (see p. 173 for a definition). One can show (see Proposition 9.3 [27]) that for any $\nu : (2n + 1) \to \{-1, +1\}$, there is some diagonal set $D$ with meet indicator sequence $\nu$ if and only if $\nu$ is a 2-Raney sequence.

To illustrate the counting technique, suppose we are given the 2-Raney sequence $\nu = \langle +1, +1, +1, -1, -1 \rangle$. Imagine setting the stage (see Figure 2) with five sticks of lengths 4, 3, 2, 1, 0 where the $i$th one is topped with open circles if $\nu(i) = +1$ and a solid circle otherwise.

Stage 0: there is nothing to be done, since there are no elements longer than $x_0$.

Stage 1: there is a single element one longer than $x_1$ and $x_1$ is a leaf, so we get to choose a passing number for $x_0$ at $x_1$. There are $d^1$ ways to do so.

Stage 2: there are two elements one longer than $x_2$ and $x_2$ is a leaf, so we get to choose a passing number for $x_0$ at $x_2$ and for $x_1$ at $x_2$. There are $d^2$ ways to do so.

Stage 3: there are three elements one longer than $x_3$, and we get to choose one, say $x_2$, to be the sucessor of $x_3$ with passing number 0 at $x_3$ and another, say $x_0{\restriction}2$, to be the sucessor of $x_3$ with passing number 1 at $x_3$. There are $3 \cdot 2$ ways to do so. With our specified choices, the scene changes as illustrated in Figure 3, where most of the passing numbers are omitted.

Stage 4: there are two elements one longer than $x_4$, and we get to choose one, say $x_1{\restriction}1$, to be the sucessor of $x_4$ with passing number 0 at $x_4$ and another, say $x_3$, to be the sucessor of $x_4$ with passing number 1 at $x_3$. There are $2 \cdot 1$ ways to do so. The completed tree we constructed is illustrated in Figure 4.

Now let us count the ways that such a tree can be built using $\nu = \langle +1, +1, +1, -1, -1 \rangle$ as a guide. For $j > 0$ with $\nu(j) > 0$, we choose $d^{\tau_\nu(j)}$

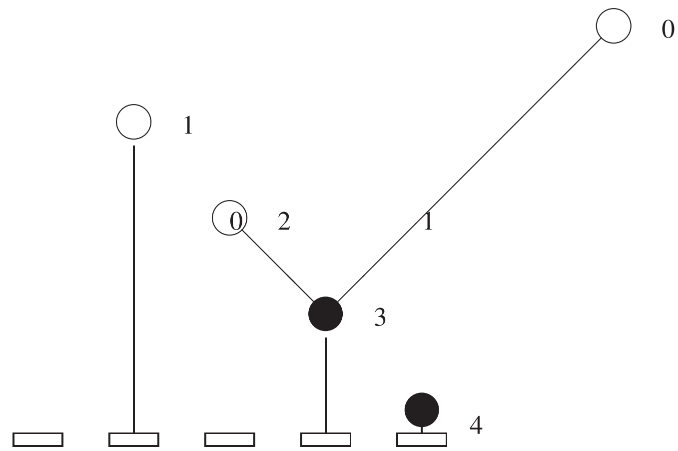


**Figure 3** Stage 3 of the construction.

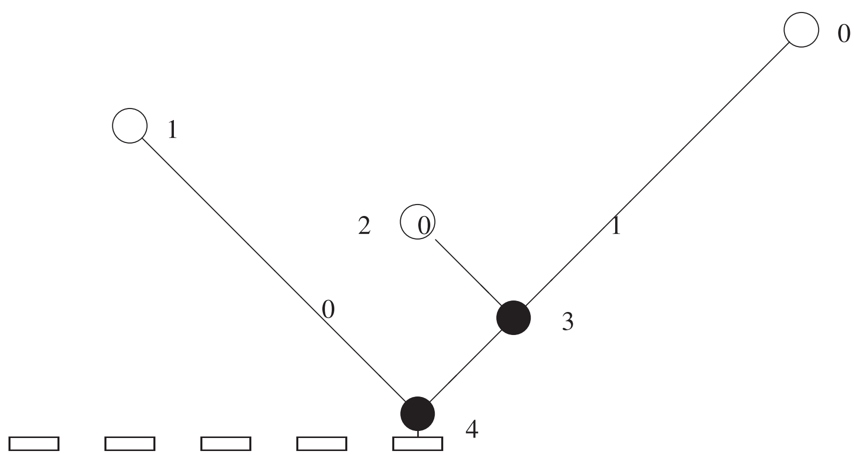


**Figure 4** Stage 4 of the construction.

passing numbers; for $j > 0$ with $\nu(j) < 0$, we choose in one of $\tau_\nu(j)(\tau_\nu(j) - 1)$ ways, the extensions of the meet at that level. Altogether there are $d^{\tau(0)+\tau(1)+\tau(2)} = d^3$ ways to choose the passing numbers of leaves and $[\tau(3)(\tau(3) - 1)][\tau(4)(\tau(4 - 1)]$ ways to choose the extensions for a total of $[3 \cdot 2][2 \cdot 1] = 12$. Altogether we have $12d^3$ choices.

**Theorem 6.1 (See Theorem 9.4 [27])** *For positive $n < \omega$, $d$ with $2 \leq d < \omega$, and $Q(\mu) := \sum_{j<2n \wedge \mu(j)>0} \tau_\mu(j)$, the number of similarity classes of diagonal sets is*

$$\alpha_{n+1}(d) = \sum_{\mu \in \mathcal{R}(n)} d^{\,Q(\mu)} \prod_{\substack{j \leq 2n \\ \mu(j)<0}} \tau_\mu(j)(\tau_\mu(j) - 1)$$

$$= \sum_{\mu \in \mathcal{R}(n)} \prod_{j \leq 2n} \theta_\mu(j),$$

*where* $\theta_\mu(j) = d^{\tau_\mu(j)}$ *if* $\mu(j) > 0$ *and* $\theta_\mu(j) = \tau_\mu(j)(\tau_\mu(j) - 1)$ *if* $\mu(j) < 0$.

Here are some small values:

(i) $r_2(d) = 2d$;
(ii) $r_3(d) = 12d^3 + 4d^2$;
(iii) $r_4(d) = 144d^6 + 72d^5 + 48d^4 + 8d^3$;
(iv) $r_5(d) = 2880d^{10} + 1728d^9 + 1723d^8 + 1008d^7 + 432d^6 + 144d^5 + 16d^4$.

**Corollary 6.2 (Corollary 9.9 [27])** *The polynomial* $r_{n+1}(d)$ *has the following properties:*

(i) *the degree of* $r_{n+1}(d)$ *is* $\frac{n(n+1)}{2}$;
(ii) *the leading coefficient of* $r_{n+1}(d)$ *is* $n!(n+1)!$;
(iii) *the lowest degree term is* $2^n d^n$; *and*
(iv) $n!(n+1)!d^{n(n+1)/2} \leq r_{n+1}(d) < (2n)!d^{n(n+1)/2}$.

**Example 6.3** The procedure for computing $r_n(d)$ can be turned into a Maple procedure. Using it, we find the number of equivalence classes of the canonical partition of the Rado graph, namely, the values of $r_n(2)$ for $n = 1, \ldots, 10$:

| | |
|---|---|
| 1 : | 1 |
| 2 : | 4 |
| 3 : | 112 |
| 4 : | 12352 |
| 5 : | 4437760 |
| 6 : | 4686103552 |
| 7 : | 13624250626048 |
| 8 : | 104218697796173824 |
| 9 : | 2028257407393613676544 |
| 10 : | 97849915247810309454561280 |

There is a connection with Devlin's Theorem mentioned in Section 3:

$$\mathbb{Q} \to (\mathbb{Q})^n_{<\omega/t_n} \text{ and } \mathbb{Q} \nrightarrow [\mathbb{Q}]^n_{t_n}.$$

Namely, $r_n(1) = t_n$ is the $n$th tangent number for all $n \geq 2$. Recall that a clique is a set of vertices all of whose pairs are joined. Such an induced subgraph is also called a complete graph, denoted $K_n$ if the subgraph has $n$ vertices. Moreover, one can show that for colorings of the $n$-element cliques of the Rado graph, the critical value is again $t_n$, and the same result is true with anticliques or independent sets in place of cliques.

$$\mathrm{RG} \to (\mathrm{RG})^{K_n}_{<\omega/t_n} \text{ and } \mathrm{RG} \nrightarrow [\mathrm{RG}]^{K_n}_{t_n}.$$

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