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On Ideals Related to $I[\lambda]$

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Abstract We describe a recipe for generating normal ideals on successors of singular cardinals. We show that these ideals are related to many weakenings of $\square$ that have appeared in the literature. Our main purpose, however, is to provide an organized list of open questions related to these ideals.

1 Introduction

Throughout this note, we will let λ denote the successor of a singular cardinal μ. We will also let χ denote some regular cardinal much larger than λ; we will be concerned with elementary submodels of various expansions of $\langle H(\chi), \in, <_\chi \rangle$, where $<_\chi$ is some well-ordering of $H(\chi)$ (the sets hereditarily of cardinality $< \chi$).

Suppose that $M \prec \langle H(\chi), \in, <_\chi \rangle$ satisfies

1. $|M| = \mu$ and
2. $M \cap \lambda$ is an initial segment of λ.

The ordinal $\delta := M \cap \lambda$ lies in the interval (μ, λ) so, in particular, δ is singular with cofinality $< \mu$.

The ideals of concern to us have to do with asking about the extent to which the singularity of δ can be witnessed by a set "covered" by $M \cap [\lambda]^{<\mu}$. For example, is there a set $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ with every initial segment in M? Can we find such an A that is also closed and unbounded? What about if we demand only that every countable subset of A is covered by a set in $M \cap [\lambda]^{<\mu}$?

What follows is one way to systematically generate ideals associated to such questions. Our goal in this note is merely to demonstrate that many weakenings of $\square$ considered in the literature are instances of such a scheme and to point out some fairly general questions that ought to be investigated further.

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2 Ideals and Weakenings of $\square$

Definition 2.1 Let λ be a regular cardinal. A λ*-approximating sequence* is a sequence $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda\rangle$ such that

1. $\mathfrak{M}$ is a continuous $\in$-chain of elementary submodels of $\langle H(\chi), \in, <_\chi\rangle$,
2. $\langle M_j : j \leq i\rangle \in M_{i+1}$,
3. $\lambda \in M_0$,

and for each $\alpha < \lambda$,

4. $|M_\alpha| < \lambda$, and
5. $M_\alpha \cap \lambda$ is an initial segment of λ.

A λ-approximating sequence is said to be *over* x if $x \in \bigcup_{\alpha<\lambda} M_\alpha$.

Our recipe for generating normal ideals will use λ-approximating sequences. Each instance of the recipe depends on two things—how we want our ordinals singularized and how we want our singularizing sets to be covered. It is probably best to give an example to show what is meant.

Example 2.2 A set $S \subseteq \lambda$ is in I if there is a parameter $x \in H(\chi)$ such that for every λ-approximating sequence $\langle M_\alpha : \alpha < \lambda\rangle$ over x, there is a closed unbounded $E \subseteq \lambda$ such that for all $\delta \in E \cap S$,

1. $\lambda \cap M_\delta = \delta$,[1] and
2. there is an $A \subseteq \delta$ cofinal of order-type $< \delta$ such that every initial segment of A is in M_δ.

Standard tricks allow us to fix a single parameter x that always works—for example, we could let x be any λ-approximating sequence.

In the preceding example, we want δ singularized by an unbounded (as opposed to, say, a closed unbounded) set that is covered in a certain sense by M_δ. We can generate other ideals by varying the demands on the singularizing sets and how they are to be covered, but first let us show that our recipe does in fact generate normal ideals.

Claim 2.3 *The collection I is a normal ideal.*

Proof The proof is very easy—it is "the" proof that our recipe generates normal ideals. We note that I is easily shown to be an ideal, so we only worry about the normality.

Thus suppose that $\langle S_\alpha : \alpha < \lambda\rangle$ is a family of sets from I. Let x_α be the parameter witnessing S_α's membership in I, and let $\bar{x} = \langle x_\alpha : \alpha < \lambda\rangle$. We claim that $\bar{x}$ will certify that $S := \triangledown_{\alpha<\lambda} S_\alpha$ is in I.

Let $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence over $\bar{x}$. We note that $\mathfrak{M}$ is "over x_α" for each $\alpha < \lambda$ as $x_\alpha \in M_{\alpha+1}$. Thus for each α there is a closed unbounded $E_\alpha \subseteq \lambda$ that "works for" S_α in the definition of I.

Let $E = \triangle_{\alpha<\lambda} E_\alpha$ and consider $\delta \in E \cap S$. By definition there is an $\alpha < \delta$ such that $\delta \in S_\alpha$. Thus $\delta \in E_\alpha \cap S_\alpha$, and by the choice of E_α there is a cofinal $A \subseteq \delta$ of order-type $< \delta$ such that every initial segment of A is in M_δ. Since δ was an arbitrary member of $E \cap S$, it follows that $S \in I$ as advertised, and I is a normal ideal. $\square$

Notice that the above proof didn't really depend on the specifics of "covering" and "singularizing." The same proof works in general. We now show that the ideal I we constructed above is interesting—it coincides with the ideal $I[\lambda]$ first introduced by Shelah in [6]. We recall the definition.

Definition 2.4 Let λ be a regular cardinal. A set $S \subseteq \lambda$ is in $I[\lambda]$ if and only if there is a sequence $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$ and a closed unbounded $E \subseteq \lambda$ such that

1. $P_\alpha \subseteq \mathcal{P}(\alpha)$,
2. $|P_\alpha| < \lambda$, and
3. if $\delta \in E \cap S$, then there is an unbounded $A_\delta \subseteq \delta$ such that
   (a) $\mathrm{otp}(c) < \delta$ (so δ is singular), and
   (b) for $\gamma < \delta$, $c \cap \gamma \in \bigcup_{\beta<\delta} P_\beta$.

Claim 2.5 *The ideal I of Example 2.2 coincides with the ideal $I[\lambda]$.*

Proof Suppose that $S \in I$ as exemplified by $x \in H(\chi)$, and let $\langle M_\alpha : \alpha < \lambda \rangle$ be any λ-approximating sequence over x. Define $P_\alpha = M_\alpha \cap \mathcal{P}(\alpha)$ and it is easy to see that the sequence $\langle P_\alpha : \alpha < \lambda \rangle$ certifies S's membership in $I[\lambda]$. Conversely, suppose that $S \in I[\lambda]$ as witnessed by $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$ and E. Let $\langle M_\alpha : \alpha < \lambda \rangle$ be a λ-approximating sequence over $\bar{P}$. We note that $P_\alpha \in M_{\alpha+1}$, and since $|P_\alpha| < \lambda$ and $M_{\alpha+1} \cap \lambda$ is an initial segment of λ, it follows that $P_\alpha \subseteq M_{\alpha+1}$ as well. Thus, if $\delta \in S \cap E$ and $M_\delta \cap \lambda = \delta$, then every initial segment of c (from the definition of $I[\lambda]$) is in the model M_δ and $S \in I$. □

This is not the place for a recounting of the importance of the ideal $I[\lambda]$ in combinatorial set theory. We recommend Kojman [4] for an overview of how $I[\lambda]$ is used in PCF theory, or the forthcoming Eisworth [2] for a more comprehensive treatment. Cummings also has an excellent survey [1] elsewhere in this volume. We note that the abbreviation 'AP(μ)' (due to Foreman and Magidor) has become a standard way to denote the statement '$\mu^+ \in I[\mu^+]$'. For our next example, we take a look at an ideal associated with the very weak square principle of Foreman and Magidor [3].

Definition 2.6 Let $\lambda = \mu^+$ where μ is singular. A set $S \subseteq \lambda$ is in $I^{\mathrm{VWS}}[\lambda]$ if there is an $x \in H(\chi)$ such that for every λ-approximating sequence $\langle M_\alpha : \alpha < \lambda \rangle$ over x there is a closed unbounded $E \subseteq \lambda$ such that for every $\delta \in S \cap E$,

1. $\lambda \cap M_\delta = \delta$ and
2. if $\mathrm{cf}(\delta) > \aleph_0$, then there is an unbounded $A \subseteq \delta$ of order-type $< \delta$ such that every countable subset of A is in M_δ.

Definition 2.7

1. (Foreman and Magidor [3]) A sequence $\langle C_\alpha : \alpha < \lambda \rangle$ is a *very weak square sequence* if and only if for a closed unbounded set of α,
   (a) C_α is unbounded in α with order-type $\mathrm{cf}(\alpha)$ and
   (b) for all bounded $x \in [C_\alpha]^{\aleph_0}$ there is $\beta < \alpha$ with $x = C_\beta$.
2. A set $S \subseteq \lambda$ has a very weak square if there is a sequence $\langle C_\alpha : \alpha < \lambda \rangle$ such that $C_\alpha \subseteq \alpha$ and for some closed unbounded $E \subseteq \lambda$, if $\delta \in E \cap S$ then
   (a) C_δ is cofinal in δ of order-type $\mathrm{cf}(\delta)$ and
   (b) if $\mathrm{cf}(\delta) > \aleph_0$, then $[C_\delta]^{\aleph_0} \subseteq \{C_\alpha : \alpha < \delta\}$.

The following claim is quite straightforward; it clarifies the connection between very weak squares and the ideal $I^{\mathrm{VWS}}[\lambda]$.

Claim 2.8 *$S \in I^{\mathrm{VWS}}[\lambda]$ if and only if S has a very weak square.*

Proof The implication $\Longleftarrow$ is easy—if S has a very weak square $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$ then we set $x = \bar{C}$ and the rest follows. For the other implication, assume $S \in I^{\mathrm{VWS}}[\lambda]$ and let $\mathfrak{M}$ and E be as in Definition 2.6. For each $i < \lambda$ we let $\delta_i = M_i \cap \lambda$. Given $i < \lambda$, let F_i be a one-to-one function from $M_i \cap [i]^{\aleph_0}$ to the successor ordinals between δ_i and δ_{i+1}. (Note that this is possible as $||M_i|| \in M_{i+1}$, hence $||M_i|| \le |\delta_{i+1} \setminus \delta_i|$.) We now define a very weak square sequence $\langle C_\alpha : \alpha < \lambda\rangle$ for S.

Case 1: α a successor If $\alpha < \delta_0$, then we set $C_\alpha = \emptyset$. Otherwise, there is a unique i such that $\delta_i < \alpha < \delta_{i+1}$, and we define

$$C_\alpha = F_i^{-1}(\alpha). \tag{1}$$

Case 2: $\alpha \in E \cap S$ and $\mathrm{cf}(\alpha) > \aleph_0$ In this case, we let $C_\alpha = A_\alpha$ where A_α is as in the definition of '$S \in I^{\mathrm{VWS}}[\lambda]$'.

Case 3: Neither of the first two cases We let C_α be an arbitrary cofinal subset of α of order-type $\mathrm{cf}(\alpha)$.

Now given $\delta \in E \cap S$, we know $M_\delta \cap \lambda = \delta$ and $M_\delta = \bigcup_{\beta<\delta} M_\delta$. We are guaranteed that $[C_\delta]^{\aleph_0} \subseteq M_\delta$. Thus if $A \in [C_\delta]^{\aleph_0}$, there is an $\alpha < \delta$ with $A \in M_\alpha \cap [\alpha]^{\aleph_0}$ and hence $A = C_\beta$ for some β between δ_α and $\delta_{\alpha+1}$. Therefore $[C_\delta]^{\aleph_0} \subseteq \{C_\beta : \beta < \delta\}$, as required. $\square$

Our next result is important for our purposes because it demonstrates the existence of nonobvious relationships between ideals of the type we are considering. The argument is a simple modification of an unpublished result of Shelah that very weak square at $\aleph_\omega$ is equivalent to $\mathrm{AP}(\aleph_\omega)$ if $2^{\aleph_0} < \aleph_\omega$.[2]

Theorem 2.9 *Let $\lambda = \mu^+$ for μ strong limit of cofinality $\aleph_0$. Let $\kappa < \mu$ be an $\aleph_0$-closed regular cardinal, that is,*

$$\theta < \kappa \Longrightarrow \theta^{\aleph_0} < \kappa. \tag{2}$$

Then $I[\lambda] \restriction S^\lambda_\kappa = I^{\mathrm{VWS}}[\lambda] \restriction S^\lambda_\kappa$.

Proof One inclusion holds trivially, so assume we are given $S \subseteq S^\lambda_\kappa$ in $I^{\mathrm{VWS}}[\lambda]$; we must show $S \in I[\lambda]$ as well. By the preceding claim, we know that S carries a very weak square, so fix a sequence $\langle C_\alpha : \alpha < \lambda\rangle$ and closed unbounded $E \subseteq \lambda$ witnessing this. If $\delta \in E \cap S^\lambda_\kappa$, we may assume that

$$\alpha \in C_\delta \Longrightarrow [C_\delta \cap \alpha]^{\aleph_0} \subseteq \{A_\gamma : \gamma < \alpha\}. \tag{3}$$

We can achieve this—given our assumption on κ—by simply thinning out C_δ if necessary.

Let $\langle \mu_n : n < \omega\rangle$ be an increasing sequence of regular cardinals cofinal in μ. By induction on $\alpha < \lambda$, we can define $\langle b_{\alpha,n} : n < \omega\rangle$ satisfying

1. $b_{\alpha,n} \subseteq \alpha$,
2. $|b_{\alpha,n}| \le \mu_n$,
3. $b_{\alpha,n} \subseteq b_{\alpha,n+1}$,
4. $\beta \in b_{\alpha,n} \Longrightarrow b_{\beta,n} \subseteq b_{\alpha,n}$,
5. $\alpha = \bigcup_{n<\omega} b_{\alpha,n}$, and
6. if $|A_\alpha| = \aleph_0$ then $A_\alpha \subseteq b_{\alpha,0}$.

Let $x = \{\langle C_\alpha : \alpha < \lambda\rangle, S, E, \langle b_{\alpha,n} : n < \omega, \alpha < \lambda\rangle\}$, and let $\langle M_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence over x. Suppose that $\delta \in S$ with $M_\delta \cap \lambda = \delta$. It should be clear that $\delta \in E$; we claim

$$\alpha \in C_\delta \Longrightarrow C_\delta \cap \alpha \subseteq b_{\alpha,n} \tag{4}$$

for some $n < \omega$. By way of contradiction, assume that (4) fails. Then we can choose

$$\beta_i \in C_\delta \setminus b_{\alpha,i} \tag{5}$$

for each i. Since $\delta \in E$, by (5) there is $\gamma < \alpha$ such that $\{\beta_i : i < \omega\} = A_\gamma$. Choose n such that $\gamma \in b_{\alpha,n}$. Then, by construction, $b_{\gamma,n} \subseteq b_{\alpha,n}$. But

$$\{\beta_i : i < \omega\} = A_\gamma \subseteq b_{\gamma,0} \subseteq b_{\gamma,n} \subseteq b_{\alpha,n}, \tag{6}$$

and this contradicts (5) for $i = n$ and, therefore, (4) is established.

Suppose now that $\alpha \in C_\delta$. There is an n such that $C_\delta \cap \alpha \subseteq b_{\alpha,n}$. Since $b_{\alpha,n} \in M_\delta$, $|b_{\alpha,n}| \leq \mu_n$, and$2^{\mu_n} < \mu$, every subset of $b_{\alpha,n}$ is in M_δ. In particular, $C_\delta \cap \alpha \in M_\delta$. This tells us that $S \in I[\lambda]$, as required. □

The last ideal that we explicitly consider in this note is related to the "not so very weak square" of [3].

Definition 2.10 A sequence $\langle C_\alpha : \alpha < \lambda\rangle$ is a *not-so-very-weak square sequence* if and only if for a closed unbounded set of α,

1. C_α is closed and unbounded in α with order-type $\mathrm{cf}(\alpha)$ and
2. for all bounded $x \in [C_\alpha]^{\aleph_0}$ there is $\beta < \alpha$ with $x = C_\beta$.

The difference between very weak square and not-so-very-weak square is that the latter requires almost all of the C_α to be closed. For our purposes, note that the obvious modification to the definition of $I^{\mathrm{VWS}}[\lambda]$ yields an ideal associated to the not-so-very-weak square. Results given in [3] show us that the not-so-very-weak square ideal is a proper ideal in the case where $\lambda = \mu^+$, where μ is a limit of supercompact cardinals and $\mathrm{cf}(\mu) = \aleph_0$; yet consistently $I^{\mathrm{VWS}}[\lambda]$ is not a proper ideal. We refer the reader to [3] for a discussion of this phenomenon; we will use this ideal only as motivation for some questions.

3 Open Questions

Question 3.1 *Can we classify when these ideals coincide?*

As demonstrated by Theorem 2.9, this is not necessarily an easy question—there are nonobvious relationships between the ideals.

Question 3.2 *If two of these ideals are consistently distinct, then where is the first place that they can differ?*

Under GCH, the ideals $I[\aleph_{\omega+1}]$ and $I^{\mathrm{VWS}}[\aleph_{\omega+1}]$ coincide by Theorem 2.9. Shelah [5] has outlined a proof that it is consistent with GCH that these two ideals differ at $\aleph_{\omega+\omega+1}$. What about the not-so-very-weak-square ideal? Where is the first place that this ideal can consistently be different from $I^{\mathrm{VWS}}[\lambda]$?

Question 3.3 *What influence do large cardinals have on the structure of these ideals?*

This can be considered a variant of the question "which squarelike principles can consistently hold above a supercompact cardinal?" For a specific question, consider the following.

Let κ be a supercompact cardinal and set $\mu = \kappa^{+\omega}$ and $\lambda = \mu^+$. Let $\theta < \kappa$ be regular. Assume GCH and suppose that the very weak square holds at μ (this is consistent by [3]). Theorem 2.9 tells us that if $I[\lambda] \restriction S^\lambda_\theta \neq I^{\mathrm{VWS}}[\lambda] \restriction S^\lambda_\theta$, then θ must be the successor of a singular cardinal of cofinality $\aleph_0$.

Modify the definition of $I[\lambda]$ to demand that if $\mathrm{cf}(\delta)$ is the successor of a singular cardinal τ of cofinality $\aleph_0$, then there is a cofinal $A_\delta \subseteq \delta$ such that $[A_\delta]^{<\tau} \subseteq M_\delta$ (as opposed to having every initial segment of A_δ in M_δ). Is it the case that λ can consistently be in this ideal? The usual proof that $\lambda \notin I[\lambda]$ (for our specific λ) does not seem to cover this new ideal.

Question 3.4 *When do these ideals have nice representations?*

For example, if μ is a strong limit singular (and $\lambda = \mu^+$) then it is known that $I[\lambda]$ is generated over the nonstationary ideal by the addition of a single set. Under what circumstances do our ideals behave in this way? Does a given ideal admit a description as in Claim 2.8?

Obviously there are more questions to be asked but these four provide a nice starting point for a general investigation.

Notes

1. This is really no requirement at all.

2. Note that if $2^{\aleph_0} > \aleph_\omega$, then very weak square fails for what might be termed "silly" reasons. One can modify the definition of 'very weak square' by requiring only that every countable subset of C_δ is covered by some earlier C_β and, therefore, get a more robust theorem.

References

[1] Cummings, J., "Notes on singular cardinal combinatorics," *Notre Dame Journal of Formal Logic*, vol. 46 (2005), pp. 251–82. 303

[2] Eisworth, T., "Successors of singular cardinals," forthcoming chapter in the *Handbook of Set Theory*. 303

[3] Foreman, M., and M. Magidor, "A very weak square principle," *The Journal of Symbolic Logic*, vol. 62 (1997), pp. 175–96. Zbl 0880.03022. MR 98i:03062. 303, 305, 306

[4] Kojman, M., "The ABC of PCF," unpublished manuscript. 303

[5] Shelah, S., "Note on $I[\lambda]$," email correspondence. 305

[6] Shelah, S., "On successors of singular cardinals," pp. 357–80 in *Logic Colloquium '78*, vol. 97 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1979. Zbl 0449.03045. MR 82d:03079. 303

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