Notre Dame Journal of Formal Logic
Volume 45, Number 2, 2004

Maximality and Refutability

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Abstract In this paper we study *symmetric inference systems* (that is, pairs of inference systems) as refutation systems characterizing maximal logics with certain properties. In particular, the method is applied to paraconsistent logics, which are natural examples of such logics.

1 Introduction

The aim of this paper is to introduce the concept of a symmetric inference system and to illustrate some of its applications that seem interesting.

A propositional logic $\mathcal{T}$ is usually characterized by an axiom system $\mathcal{P}$ or by a semantic (or algebraic) system $\mathbf{M}$ or by both. The systems $\mathcal{P}$ and $\mathbf{M}$ are quite different. $\mathcal{P}$ is on the propositional level, while $\mathbf{M}$ must be on a higher one. However, it is possible to characterize $\mathcal{T}$ by two propositional inference systems: $\mathcal{POS}$ and $\mathcal{NEG}$. $\mathcal{POS}$ consists of a set $\mathcal{POS}_0 \subseteq \mathcal{T}$ and a rule $\mathcal{POS}_1$ generating $\mathcal{T}$. And $\mathcal{NEG}$ consists of a set $\mathcal{NEG}_0 \subseteq -\mathcal{T}$ and a rule $\mathcal{NEG}_1$ generating $-\mathcal{T}$. Such a pair $\mathbf{S}$ of inference systems will be called a symmetric inference system. We find this concept attractive because of its economy and possible applications. For example, it provides new proofs of old results concerning decidability and refined semantic completeness (see Skura [10]).

Moreover, for any symmetric inference system $\mathbf{S}$ we can consider the class of logics (or more generally, sets of formulas) $\mathcal{T}$ that are $\mathbf{S}$-closed (that is, $\mathcal{T}$ is $\mathcal{POS}$-closed and $-\mathcal{T}$ is $\mathcal{NEG}$-closed). Further, we can view a system $\mathbf{S}$ as a refutation device generating the set $\mathcal{RF}(\mathbf{S})$ of $\mathbf{S}$-refutable formulas (by using $\mathcal{POS}$ derivations as well as $\mathcal{NEG}$ derivations). If $\mathcal{RF}(\mathbf{S}) = -\mathcal{T}$ then $\mathcal{T}$ is maximal in the class of $\mathbf{S}$-closed sets. This aspect of symmetric systems is described in Section 3, where we give a couple of general theorems as well as a few concrete examples.

Received June 16, 2003; accepted September 16, 2003; printed September 1, 2004
2000 Mathematics Subject Classification: Primary, 03B20, 03B22, 03B53, 03B55, 03B99
Keywords: maximal propositional logics, inference systems, syntactic refutability, disjunction property, paraconsistency

In Section 4 we apply this method to paraconsistent logics, which are defined by both positive conditions and negative ones and in which maximality is an important property. We give an interesting example of a paraconsistent logic that is determined by a symmetric system. Starting with certain simple conditions and employing our method, we arrive at a unique maximal set with the desired property.

Finally, for any system $\mathbf{S}$ we give an abstract characterization of the class of maximal $\mathbf{S}$-closed sets by syntactic refutability using the elegant format of sequent consequence relations.

2 Symmetric Inference Systems

Let FOR be the set of all formulas generated from a set $\mathsf{VAR} = \{p, q, r, \ldots\}$ of propositional variables by standard connectives. Logics are subsets of FOR satisfying certain conditions. They may be positive (that some formulas are in $\mathcal{T}$), for example,

1. $p \to p \in \mathcal{T}$,
2. $B \in \mathcal{T}$ whenever $A,\ A \to B \in \mathcal{T}$.

And they may be negative (that some formulas are not in $\mathcal{T}$), for example,

1. $p \to (\neg p \to q) \notin \mathcal{T}$,
2. $A \vee B \notin T$ whenever $A, B \notin \mathcal{T}$.

Such properties can be described by *rules*, which are sets of *inferences* $\mathcal{X}/A$, where $A \in \mathsf{FOR}$ and $\mathcal{X}$ is a finite set of formulas. By an *inference system* we mean a pair $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$, where $\mathcal{P}_1$ is a rule and $\mathcal{P}_0 \subseteq \mathsf{FOR}$. We say that a set $\mathcal{T} \subseteq \mathsf{FOR}$ is $\mathcal{P}$-closed, if $\mathcal{P}_0 \subseteq \mathcal{T}$ and for every $\mathcal{X}/A \in \mathcal{P}_1$, if $\mathcal{X} \subseteq \mathcal{T}$ then $A \in \mathcal{T}$. By a *symmetric inference system* we shall mean a pair $\mathbf{S} = (\mathcal{POS}, \mathcal{NEG})$ of inference systems. We say that a set $\mathcal{T} \subseteq \mathsf{FOR}$ is $\mathbf{S}$-closed if $\mathcal{T}$ is $\mathcal{POS}$-closed and the complement of $\mathcal{T}$ (that is, the set $-\mathcal{T} = \mathsf{FOR} - \mathcal{T}$) is $\mathcal{NEG}$-closed. And we say that a set $\mathcal{T}$ is a *SYM* set, if $\mathcal{T}$ is $\mathbf{S}$-closed for some symmetric inference system $\mathbf{S}$.

Intuitively speaking, $\mathcal{POS}_0$ is a set of valid formulas (axioms), $\mathcal{POS}_1$ is a rule generating new valid formulas, $\mathcal{NEG}_0$ is a set of invalid formulas, and $\mathcal{NEG}_1$ is a refutation rule generating new invalid formulas.

Example 2.1 (Paraconsistent logics) Leaving the question of positive conditions for these logics aside, we could say that they are $\mathbf{S}$-closed sets with $\mathcal{NEG}_0 = \{p \to (\neg p \to q)\}$ and $\mathcal{NEG}_1 = \varnothing$.

Example 2.2 (Intermediate logics) Here $\mathcal{POS}_0$ is the set $\mathcal{I}_0$ of the axioms for the intuitionistic logic and $\mathcal{POS}_1$ is the set $\mathcal{I}_1$ of all inferences determined by the rules: substitution and modus ponens. And $\mathcal{NEG}_0 = -\mathcal{C}$, $\mathcal{NEG}_1 = \varnothing$, where $\mathcal{C}$ is the set of all laws of Classical Logic.

Example 2.3 (Intermediate logics with the disjunction property) They are $\mathbf{S}$-closed sets with $\mathcal{POS} = (\mathcal{I}_0, \mathcal{I}_1)$ and $\mathcal{NEG} = (-\mathcal{C}, \mathcal{D})$, where $\mathcal{D} = \{A, B/A \vee B : A, B \in \mathsf{FOR}\}$.

3 Maximal *SYM* Sets

For a given symmetric inference system $\mathbf{S}$ there are usually plenty of $\mathbf{S}$-closed sets, and we are often interested in maximal ones. Here we say that an $\mathbf{S}$-closed set $\mathcal{T}$ is *maximal*, if there is no $\mathbf{S}$-closed set $\mathcal{T}'$ such that $\mathcal{T}$ is a proper subset of $\mathcal{T}'$. By

Zorn's lemma we know that such sets exist, although we do not know how to find them. It is thus worth exploring methods for characterizing them. We are going to present a method of generating maximal *SYM* sets by syntactic refutability. The idea is to regard a symmetric inference system as a refutation device for refuting formulas by derivations, and it is based on Łukasiewicz's method of refutation rules introduced in [6]. (For more recent information on the method see [10].)

Every inference system $\mathcal{P}$ determines a consequence relation $\vdash_{\mathcal{P}}$ in a natural way. We say that a formula A is $\mathcal{P}$-derivable from a finite set $\mathcal{X} \subseteq \mathsf{FOR}$ (in symbols $\mathcal{X} \vdash_{\mathcal{P}} A$), if there is a sequence $A_1, \ldots, A_n$ of formulas such that $A_n = A$ and each A_i is in $\mathcal{X} \cup \mathcal{P}_0$ or is obtained from some preceding formulas by $\mathcal{P}_1$. We also write $\vdash_{\mathcal{P}} A$ for $\varnothing \vdash_{\mathcal{P}} A$. Note that such a relation $\vdash$ has the following properties.

1. $A \vdash A$.
2. If $\mathcal{X} \vdash A$ then $\mathcal{X}' \vdash A$, where $\mathcal{X} \subseteq \mathcal{X}'$.
3. If $\mathcal{X}, A \vdash B$ and $\mathcal{X} \vdash A$, then $\mathcal{X} \vdash B$.

Since both $\mathcal{POS}$ and $\mathcal{NEG}$ are inference systems, a symmetric inference system can be used for generating both valid formulas and invalid ones. Of course, $\mathcal{NEG}$ derivations are refutations. But $\mathcal{POS}$ can also be used for refuting formulas, namely, if C is refutable and $B \vdash_{\mathcal{POS}} C$, then so is B. More formally, for any symmetric inference system $\mathbf{S} = (\mathcal{POS}, \mathcal{NEG})$, we say that a formula A is $\mathbf{S}$-*refutable*, if

$$\varnothing \vdash_{\mathcal{N}(\mathbf{S})} A$$

where $\mathcal{N}(\mathbf{S}) = (\mathcal{NEG}_0, \mathcal{NEG}_1 \cup \mathcal{N}_{\mathcal{P}})$ and $\mathcal{N}_{\mathcal{P}} = \{C/B : B \vdash_{\mathcal{POS}} C\}$. The set of all $\mathbf{S}$-refutable formulas will be denoted by the symbol $\mathcal{RF}(\mathbf{S})$, and its complement $\mathsf{FOR} - \mathcal{RF}(\mathbf{S})$ by $\mathcal{F}(\mathbf{S})$.

Proposition 3.1 *If $\mathcal{L}$ is $\mathbf{S}$-closed, then $\mathcal{L} \subseteq \mathcal{F}(\mathbf{S})$.*

Proof By induction on the length n of an $\mathcal{N}(\mathbf{S})$ derivation $A_1, \ldots, A_n$ we show that $A_n \notin \mathcal{L}$.

$(n = 1)$ Then $A_1 \in \mathcal{NEG}_0$, so $A_1 \notin \mathcal{L}$ because $\mathcal{NEG}_0 \subseteq -\mathcal{L}$.

$(n \geq 2)$

Case 1 A_n is obtained from say A_1 by $\mathcal{NEG}_1$. Also $A_1 \notin \mathcal{L}$ by the induction hypothesis. Since $-\mathcal{L}$ is $\mathcal{NEG}_1$-closed, we get $A_n \notin \mathcal{L}$.

Case 2 A_n is obtained from say $A_1 \notin \mathcal{L}$ by $\mathcal{N}_{\mathcal{P}}$. Then $A_n \vdash_{\mathcal{POS}} A_1$. Now $\mathcal{L}$ is $\mathcal{POS}$-closed, so if $B \in \mathcal{L}$ and $B \vdash_{\mathcal{POS}} C$ then $C \in \mathcal{L}$. Hence $A_n \notin \mathcal{L}$. □

Corollary 3.2 *Let $\mathcal{T}$ be an $\mathbf{S}$-closed set such that $\mathcal{F}(\mathbf{S}) \subseteq \mathcal{T}$. Then $\mathcal{T} = \mathcal{F}(\mathbf{S})$ and $\mathcal{T}$ is the greatest $\mathbf{S}$-closed set.*

Example 3.3 (Intuitionistic Logic—$\mathcal{INT}$) Does the disjunction property uniquely characterize $\mathcal{INT}$? By Corollary 3.2, it suffices to prove that every $A \notin \mathcal{INT}$ is $\mathbf{S}$-refutable, where $\mathbf{S}$ is the system defined in Example 2.3 (which was, in fact, conjectured by Łukasiewicz). However, this is impossible for there are intermediate logics with the disjunction property that are proper extensions of $\mathcal{INT}$ (see, e.g., Gabbay [5]). A property of this kind (called the generalized disjunction property) that does characterize $\mathcal{INT}$ is given in Skura [8] where it is shown that $\mathcal{INT}$ is $\mathbf{S}'$-closed and every $A \notin \mathcal{INT}$ is $\mathbf{S}'$-refutable. The system $\mathbf{S}'$ results from $\mathbf{S}$ by replacing $\mathcal{D}$ by $\mathcal{GD} =$

$\{A \to A_1, \ldots, A \to A_n/A \to A_1 \vee \cdots \vee A_n : A = (A_1 \to B_1) \wedge \cdots \wedge (A_n \to B_n)\}$.

Let $\mathcal{AX} \subseteq \mathsf{FOR}$. By the axiomatic strengthening of $\mathcal{POS}$ by $\mathcal{AX}$ we mean the system $\mathcal{POS}^{\mathcal{AX}} = (\mathcal{AX} \cup \mathcal{POS}_0, \mathcal{POS}_1)$. We also define $\mathbf{S}^{\mathcal{AX}} = (\mathcal{POS}^{\mathcal{AX}}, \mathcal{NEG})$.

Proposition 3.4 *Let $\mathcal{T}$ be an $\mathbf{S}^{\mathcal{AX}}$-closed set such that $\mathcal{F}(\mathbf{S}^{\mathcal{AX}}) \subseteq \mathcal{T}$. Then $\mathcal{T}$ is a maximal $\mathbf{S}$-closed set.*

Proof Suppose that $\mathcal{T}$ is $\mathbf{S}^{\mathcal{AX}}$-closed and $\mathcal{F}(\mathbf{S}^{\mathcal{AX}}) \subseteq \mathcal{T}$, but $\mathcal{T}$ is not a maximal $\mathbf{S}$-closed set. Then there is an $\mathbf{S}$-closed set $\mathcal{T}'$ such that $\mathcal{T}$ is a proper subset of $\mathcal{T}'$, so there is a formula $B \in \mathcal{T}' - \mathcal{T}$. Since $B \notin \mathcal{T}$, $B \notin \mathcal{F}(\mathbf{S}^{\mathcal{AX}})$. Now $\mathcal{T}$ is $\mathbf{S}^{\mathcal{AX}}$-closed, so $\mathcal{AX} \subseteq \mathcal{T} \subseteq \mathcal{T}'$, and so $\mathcal{T}'$ is $\mathbf{S}^{\mathcal{AX}}$-closed. Hence $\mathcal{T}' \subseteq \mathcal{F}(\mathbf{S}^{\mathcal{AX}})$ by Proposition 3.1. Therefore $B \notin \mathcal{T}'$, which is a contradiction. □

Example 3.5 (Medvedev's logic $\mathcal{M}$ of finite problems) Let $\mathbf{S}$ be the system defined in Example 2.3, and let $\mathbf{S}' = ((\mathcal{I}_0', \mathcal{I}_1), \mathcal{NEG})$ with

$$\mathcal{I}_0' = \mathcal{I}_0 \cup \{(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)\}.$$

It is known that $\mathcal{M}$ is $\mathbf{S}'$-closed, and it can be shown that $\mathcal{F}(\mathbf{S}') \subseteq \mathcal{M}$ (see Skura [9]). By Corollary 3.2, $\mathcal{M} = \mathcal{F}(\mathbf{S}')$ and $\mathcal{M}$ is the greatest $\mathbf{S}'$-closed set. And by Proposition 3.4, $\mathcal{M}$ is a maximal $\mathbf{S}$-closed set.

It is an interesting question whether other maximal intemediate logics with the disjunction property can be generated in this way.

Remark 3.6 Let us say that a system $\mathbf{S}$ is *consistent*, if for no formula A we have both $\vdash_{\mathcal{POS}} A$ and $\vdash_{\mathcal{NEG}} A$. Clearly, if $\mathbf{S}$ is consistent then there is an $\mathbf{S}$-closed set $\mathcal{T}$ (for example, the set $\{A :\vdash_{\mathcal{POS}} A\}$). On the other hand, if $\mathcal{T}$ is $\mathbf{S}$-closed and for some formula A we have $\vdash_{\mathcal{POS}} A$ and $\vdash_{\mathcal{NEG}} A$, then $A \in \mathcal{T}$ and $A \in -\mathcal{T}$, which is impossible. Therefore a system $\mathbf{S}$ is consistent if and only if there is a set $\mathcal{T}$ that is $\mathbf{S}$-closed.

Remark 3.7 The set $\mathcal{RF}(\mathbf{S})$ is $\mathcal{NEG}$-closed for any system $\mathbf{S}$. However, $\mathcal{F}(\mathbf{S})$ need not be $\mathcal{POS}$-closed. Indeed, let $\mathbf{S}$ be the system defined in Example 2.3. If $\mathcal{F}(\mathbf{S})$ is $\mathcal{POS}$-closed, then $\mathcal{F}(\mathbf{S})$ is an $\mathbf{S}$-closed set, so (by Corollary 3.2) it is the greatest $\mathbf{S}$-closed set, which is impossible (see Chagrov and Zakharyashchev [1]).

Remark 3.8 It is natural to define $\mathbf{S}$-provability as follows. A formula A is $\mathbf{S}$-provable, if $\varnothing \vdash_{\mathcal{POS}} A$. By symmetry, one could also use refutations in $\mathbf{S}$ proofs (just as proofs are used in $\mathbf{S}$ refutations). But I do not know whether this idea has any interesting applications.

4 Paraconsistency

Paraconsistent logics are natural examples of *SYM* sets. They are defined by both positive conditions and negative ones. Further, a paraconsistent logic should contain as many classical theorems as possible (see da Costa [2]), which can be expressed by means of maximality. In order to define $\mathbf{S}$, let us reject the classical explosive law $E = p \to (\neg p \to q)$, which we do not want. It is also natural to reject all formulas that are not classical theorems. On the positive side there are various possibilities. What do we want to have? It seems reasonable to adopt substitution and modus ponens together with the positive axioms for the intuitionistic logic as well as the formula $N = (p \to q) \to (\neg q \to \neg p)$. This guarantees the replacement of equivalent formulas by the law $(A \equiv B) \to (\ldots A \ldots \equiv \ldots B \ldots)$.

More precisely, we assume that FOR is the set of formulas generated from VAR by the connectives $\wedge, \vee, \to, \neg$. And our system $\mathbf{S} = (\mathcal{POS}, \mathcal{NEG})$, where

$$
\begin{aligned}
\mathcal{NEG}_0 = \quad & \{p \to (\neg p \to q)\} \cup (-\mathcal{C}) \qquad \mathcal{NEG}_1 = \varnothing \\
\mathcal{POS}_0 = \quad & \{p \to (q \to p), \quad (p \to (q \to r)) \to ((p \to q) \to (p \to r)), \\
& p \wedge q \to p, \quad p \wedge q \to q, \quad p \to (q \to p \wedge q), \\
& p \to p \vee q, \quad q \to p \vee q, \quad (p \to r) \to ((q \to r) \to (p \vee q \to r)), \\
& (p \to q) \to (\neg q \to \neg p)\} \\
\mathcal{POS}_1 = \quad & \{A/s(A) : A \in \mathsf{FOR},\ s \text{ is a substitution}\} \cup \\
& \{A, A \to B/B : A, B \in \mathsf{FOR}\}.
\end{aligned}
$$

For any $A, B \in$ FOR we say that A is *equivalent* to B if $\vdash_{\mathcal{POS}} A \equiv B$. (Here $A \equiv B = (A \to B) \wedge (B \to A)$.) If $\mathcal{X} = \{A_1, \ldots, A_n\}$ with each $A_i \in$ FOR, we write $\mathcal{X} \to A$ instead of $A_1 \to (A_2 \to \cdots \to (A_n \to A) \ldots)$. Note that $\mathcal{X} \to A$ is equivalent to $\bigwedge \mathcal{X} \to A$, where $\bigwedge \mathcal{X} = A_1 \wedge \cdots \wedge A_n$.

How many maximal $\mathbf{S}$-closed sets are there? Only one! The system $\mathbf{S}$ determines a unique maximal $\mathbf{S}$-closed set, namely, $\mathcal{F}(\mathbf{S})$. We are going to show that $\mathcal{F}(\mathbf{S}) = \mathcal{CPC}$, where $\mathcal{CPC} = \mathcal{C} \cap \mathcal{PC}$ and $\mathcal{PC}$ is the set of formulas valid in the algebra $\mathbf{2}'$ resulting from the 2-element Boolean algebra $\mathbf{2} = (\{0, 1\}, \wedge, \vee, \to, \neg)$ by replacing $\neg$ with the operation $-$ defined thus: $-0 = -1 = 1$. More formally, $A \in \mathcal{PC}$ if $v(A) = 1$ for every valuation v assigning either 0 or 1 to each propositional variable and extended for all the formulas by using the $\mathbf{2}'$ operations in the familiar way. ($v(\neg B) = -vB$ and so on.)

Theorem 4.1 *$\mathcal{CPC} = \mathcal{F}(\mathbf{S})$ and $\mathcal{CPC}$ is the greatest* $\mathbf{S}$*-closed set.*

Proof (1) We first observe that both $\mathcal{C}$ and $\mathcal{PC}$ are $\mathcal{POS}$-closed. (We remark that if $\mathcal{X} \vdash_{\mathcal{POS}} A$ and $\mathcal{X} \subseteq \mathcal{PC}$, then $A \in \mathcal{PC}$.) Moreover, if $vp = 1$, $vq = 0$, then $vE = 0$, and so $E \notin \mathcal{CPC}$. Also $-\mathcal{C} \subseteq -\mathcal{CPC}$. Hence $\mathcal{CPC}$ is $\mathbf{S}$-closed, so by Proposition 3.1, $\mathcal{CPC} \subseteq \mathcal{F}(\mathbf{S})$.

(2) In order to show that $\mathcal{F}(\mathbf{S}) \subseteq \mathcal{CPC}$, we introduce a technique that is quite general and useful in syntactic procedures. (The technique was known to Wajsberg [11].)

For any formula A we construct a normal form A' as follows. First of all, with every compound subformula B of A we associate a new propositional variable p_B and we put $p_a = a$ for every variable a occurring in A. Then we define the following set of formulas.

$$
\begin{aligned}
\Gamma_A = \quad & \{(\neg p_B) \equiv p_{\neg B} : \neg B \text{ is a subformula of } A\} \cup \\
& \{(p_B \otimes p_C) \equiv p_{B \otimes C} : B \otimes C \text{ is a subformula of } A, \otimes \in \{\wedge, \vee, \to\}\}.
\end{aligned}
$$

Finally, we define $A' = \Gamma_A \to p_A$. It is not difficult to prove that

1. $A' \vdash_{\mathcal{POS}} A$ (Substitute B for p_B.)
2. $\vdash_{\mathcal{POS}} A \to A'$ (By using *replacement* show that $\vdash_{\mathcal{POS}} \Gamma_A \to (B \equiv p_B)$ for every subformula B of A.)

Let us now assume that $A \notin \mathcal{CPC}$. Then $A \notin \mathcal{C}$ or $A \notin \mathcal{PC}$. If $A \notin \mathcal{C}$, A is $\mathbf{S}$-refutable, so $A \notin \mathcal{F}(\mathbf{S})$, and so we may assume that $A \notin \mathcal{PC}$. Then $A' \notin \mathcal{PC}$, so there is a valuation v such that $v(A') = 0$. Hence $v(p_A) = 0$ and $vF = 1$ for every $F \in \Gamma_A$, so that $-v(p_B) = v(p_{\neg B})$ and so on. Let s be a substitution such that $sa = q$ if $va = 0$ and $sa = \top = p \to p$ otherwise ($a \in \mathsf{VAR}$). Note that the $\neg$-free formulas in $s(\Gamma_A)$ (like $(\top \to q) \equiv q$) are equivalent to $\top$. Thus we may concentrate on the formulas of the kind $s(\neg p_B \equiv p_{\neg B})$. Now $v(p_{\neg B}) = 1$ (because $-v(p_B) = 1$), so $s(p_{\neg B}) = \top$, and so each of these formulas is equivalent to $\neg s(p_B)$, where $s(p_B)$ is either $\top$ or q. Therefore $s(A')$ is equivalent to $G \to q$, where $G = \neg a_1 \wedge \cdots \wedge \neg a_n$ with each $a_i \in \{\top, q\}$. By using N and $p \to (q \to p)$, one easily shows that $\vdash_{\mathcal{POS}} (p \wedge \neg p) \to \neg r$. Hence $\vdash_{\mathcal{POS}} (p \wedge \neg p) \to G$, so $\vdash_{\mathcal{POS}} s(A') \to E$, and so $A \vdash_{\mathcal{POS}} E$. This means that A is **S**-refutable, so that $A \notin \mathcal{F}(\mathbf{S})$, as required.

(3) Since $\mathcal{CPC}$ is an **S**-closed set such that $\mathcal{F}(\mathbf{S}) \subseteq \mathcal{CPC}$, we obtain the result by Corollary 3.2. □

The logic $\mathcal{CPC}$ is known in the literature and it is known to be maximal (see da Costa and Béziau [3] and Nowak [7]). But the fact that it is the greatest **S**-closed set seems new. The simple conditions defining **S** are quite natural, so the fact that $\mathcal{CPC}$ is the unique maximal **S**-closed set provides a natural justification for $\mathcal{CPC}$.

5 Maximality and Refutability

It is possible to give a general characterization of maximal *SYM* sets by syntactic refutability. Because of its generality, the characterization will be an abstract one. As axioms we take all formulas in a set $\mathcal{T}$. What is more, we transform a symmetric inference system **S** into one abstract rule by using sequents. Here by a *sequent* we mean a pair $\mathcal{X}/\mathcal{Y}$, where $\mathcal{X}$, $\mathcal{Y}$ are finite sets of formulas. And by a *sequent rule* we mean a set Σ of sequents. We say that a set $\mathcal{T} \subseteq \mathsf{FOR}$ is Σ-closed, if for every $\mathcal{X}/\mathcal{Y} \in \Sigma$ we have $\mathcal{Y} \cap \mathcal{T} \neq \varnothing$ whenever $\mathcal{X} \subseteq \mathcal{T}$. For any symmetric inference system $\mathbf{S} = (\mathcal{POS}, \mathcal{NEG})$ we define the sequent rule

$$\begin{aligned}\Sigma(\mathbf{S}) = \quad & \{\varnothing/A : A \in \mathcal{POS}_0\} \cup \mathcal{POS}_1 \cup \\ & \{A/\varnothing : A \in \mathcal{NEG}_0\} \cup \{A/\mathcal{X} : \mathcal{X}/A \in \mathcal{NEG}_1\}.\end{aligned}$$

And if $\mathcal{AX} \subseteq \mathsf{FOR}$, we define $\Sigma^{\mathcal{AX}} = \Sigma(\mathbf{S}^{\mathcal{AX}})$ $(= \Sigma(\mathbf{S}) \cup \{\varnothing/A : A \in \mathcal{AX}\})$.

Every sequent rule Σ determines a sequent consequence relation $\vdash_\Sigma$ between finite sets of formulas defined as follows. $\mathcal{X} \vdash_\Sigma \mathcal{Y}$ if there is a finite sequence $\alpha_1, \ldots, \alpha_n$ of sequents such that $\alpha_n = \mathcal{X}/\mathcal{Y}$ and each α_i is in Σ or is obtained from preceding sequents by one of the following metarules.

(*refl*) $\quad \dfrac{\varnothing}{\mathcal{X}/\mathcal{X}} \quad$ where $\mathcal{X} \neq \varnothing$

(*mon*) $\quad \dfrac{\mathcal{X}/\mathcal{Y}}{\mathcal{X}'/\mathcal{Y}'} \quad$ where $\mathcal{X} \subseteq \mathcal{X}'$, $\mathcal{Y} \subseteq \mathcal{Y}'$

(*cut*) $\quad \dfrac{\mathcal{X}, A/\mathcal{Y} \quad \mathcal{X}/A, \mathcal{Y}}{\mathcal{X}/\mathcal{Y}}$

For any sets $\mathcal{T}, \mathcal{U} \subseteq \mathsf{FOR}$ we say $\mathcal{T} \vdash_\Sigma \mathcal{U}$, if there are finite sets $\mathcal{X} \subseteq \mathcal{T}$, $\mathcal{Y} \subseteq \mathcal{U}$ such that $\mathcal{X} \vdash_\Sigma \mathcal{Y}$.

Let $A \in \mathsf{FOR}$. We say that A is Σ-refutable (in symbols, $A \in \mathcal{RF}(\Sigma)$), if

$$A \vdash_\Sigma \varnothing.$$

Theorem 5.1 *An **S**-closed set $\mathcal{T}$ is maximal if and only if every $A \notin \mathcal{T}$ is $\Sigma^{\mathcal{T}}$-refutable.*

Proof ($\leftarrow$) First we prove that

($*$) If $\mathcal{L}$ is **S**-closed, then $\mathcal{L} \subseteq \mathcal{F}(\Sigma) = \mathsf{FOR} - \mathcal{RF}(\Sigma)$, where $\Sigma = \Sigma(\mathbf{S})$.

Indeed, let $\alpha_1, \dots, \alpha_n$ (where $\alpha_i = \mathcal{X}_i/\mathcal{Y}_i$) be a Σ derivation. By induction on n we show that if $\mathcal{X}_n \subseteq \mathcal{L}$ then $\mathcal{Y}_n \cap \mathcal{L} \neq \varnothing$. (Then $B \notin \mathcal{L}$ whenever $B \vdash_\Sigma \varnothing$.)

($n = 1$) Then $\alpha_1 \in \Sigma$ (or it is obtained by *refl*), and this is true because $\mathcal{L}$ is Σ-closed.

($n \geq 2$) We only consider the case where $\alpha_n = \mathcal{X}/\mathcal{Y}$ is obtained by *cut* from $\mathcal{X}, B/\mathcal{Y}$ and $\mathcal{X}/B, \mathcal{Y}$. Assume $\mathcal{X} \subseteq \mathcal{L}$. By the induction hypothesis $\mathcal{Y}' \cap \mathcal{L} \neq \varnothing$, where $\mathcal{Y}' = \{B\} \cup \mathcal{Y}$, so $B \in \mathcal{L}$ or $\mathcal{Y} \cap \mathcal{L} \neq \varnothing$. If $B \in \mathcal{L}$ then $\mathcal{X} \cup \{B\} \subseteq \mathcal{L}$, so (by the induction hypothesis) $\mathcal{Y} \cap \mathcal{L} \neq \varnothing$, which gives ($*$).

Finally, suppose that $\mathcal{T}$ is **S**-closed and $\mathcal{F}(\Sigma^{\mathcal{T}}) \subseteq \mathcal{T}$, but $\mathcal{T}$ is not a maximal **S**-closed set. Then there is an **S**-closed set $\mathcal{T}' \supset \mathcal{T}$, and so some formula $B \in \mathcal{T}' - \mathcal{T}$. Thus $B \notin \mathcal{F}(\Sigma^{\mathcal{T}})$. Since $\mathcal{T}'$ is $\mathbf{S}^{\mathcal{T}}$-closed, $\mathcal{T}' \subseteq \mathcal{F}(\Sigma^{\mathcal{T}})$ by ($*$). Hence $B \notin \mathcal{T}'$. This is a contradiction.

($\rightarrow$) Let us suppose that $\mathcal{T}$ is a maximal **S**-closed set but some $A \notin \mathcal{T}$ is not $\Sigma^{\mathcal{T}}$-refutable. Then $A \nvdash_{\Sigma^{\mathcal{T}}} \varnothing$, so $\mathcal{T}, A \nvdash_{\Sigma^{\mathcal{T}}} \varnothing$.

It can be shown that there is a set $\mathcal{T}'$ such that $\mathcal{T} \cup \{A\} \subseteq \mathcal{T}'$ and $\mathcal{T}' \nvdash_{\Sigma^{\mathcal{T}}} \mathcal{U}$, where $\mathcal{U} = \mathsf{FOR} - \mathcal{T}'$. To this end enumerate all formulas $A_1, A_2, \dots$ and define a sequence $(\mathcal{T}_0, \mathcal{U}_0), (\mathcal{T}_1, \mathcal{U}_1), \dots$ of pairs of sets of formulas in such a way that $\mathcal{T}_0 = \mathcal{T} \cup \{A\}$, $\mathcal{U}_0 = \varnothing$ and $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n\}$, $\mathcal{U}_{n+1} = \mathcal{U}_n$ if $\mathcal{T}_n, A_n \nvdash_{\Sigma^{\mathcal{T}}} \mathcal{U}_n$ and $\mathcal{T}_{n+1} = \mathcal{T}_n$, $\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{A_n\}$ otherwise. Then check that the set $\mathcal{T}' = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots$ has the required property (for more details see [5], p. 11).

It follows that $\mathcal{T}'$ is **S**-closed. Indeed, suppose that $\mathcal{X}/\mathcal{Y} \in \Sigma(\mathbf{S})$ and $\mathcal{X} \subseteq \mathcal{T}'$, but $\mathcal{Y} \cap \mathcal{T}' = \varnothing$. Then $\mathcal{X} \vdash_{\Sigma^{\mathcal{T}}} \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{U}$, so $\mathcal{T}' \vdash_{\Sigma^{\mathcal{T}}} \mathcal{U}$, which is impossible. Therefore $\mathcal{T}'$ is an **S**-closed set such that $\mathcal{T} \subset \mathcal{T}'$, which contradicts the assumption that $\mathcal{T}$ is maximal. □

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