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NON-PARADOXICAL PARADOXES?

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1. It is commonly said that, whatever the exact nature of Lewis' paradoxes of strict implication, the only sense in which strict implication is paradoxical is the sense in which 'implies' means 'entails'. Critics of the *identity-thesis*, the thesis, namely that strict implication and entailment are one and the same, have not wanted to deny that Lewis' four puzzle-theorems are true of, or hold for, *material* and *strict* implication. On this view, if you interpret therein the main connective, '$\prec$', as 'materially implies' or 'strictly implies' (*as opposed to* 'entails') the air of paradox vanishes (Von Wright [8], p. 172). What they resist is construing the main connective of these theorems to be the entailment-connective. Fundamentally, they take the occurrence of the paradoxes conclusively to show that strict implication (for which the paradoxes are true) is not the same relation as entailment (for which the paradoxes are false).

Here is one version of what we might call the *no-conflict hypothesis*. On the present version of it, we are entitled to say both that the paradoxes are true (and their proofs sound) and that our intuitions (on the basis of which we reject the paradoxes) about entailment are true; the tension between paradox and intuition is only apparent. The paradoxes reveal facts about strict implication; whereas our intuitions reflect truths about entailment. It is only when the facts about strict implication revealed by the paradoxes are thought to be facts about entailment, and when the truths about entailment reflected by our intuitions are thought to be truths about strict implication, that the illusion of incompatibility is created. But once it is recognized that we have in strict implication and entailment two distinct, albeit similar, relations, and that what holds for the one does not, in all respects, hold for the other, the illusion of conflict evaporates. We may therefore, without further anxiety, continue to hold that it is a law "of any reasonable modal logic that an impossible proposition strictly implies any proposition whatever, and that a necessary proposition is strictly implied by any proposition whatever."[1]

1. See also E. J. Nelson ([7], p. 270). "I do not mean that the *systems* of material and of strict implication are as such absurd: I mean only that I am convinced of the falsity of the view that either strict or material implication is the true analysis of implication."

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We may do so because the paradoxes of strict implication "are no paradoxes in themselves, but become paradoxical if... strict implication is equated with entailment." (Von Wright [8], p. 175).

2. This ingenuous move begs all the important questions. If one defines a concept K, and then proceeds to derive therefrom, by appeal to unquestionably correct inference rules, what one *intuitively* takes to be patent falsehoods, one has a compelling, but not necessarily conclusive, reason for doubting the adequacy of the definition of K. However, if in addition, one manages to derive those very consequences of the definition of K by techniques of proof, apparently cogent and logically independent of the definition, one has a compelling, though not necessarily conclusive, reason for doubting, *not* the definition of K, but rather our original intuitions that the derivations from K were false. If one perseveres in thinking the derivations false, it would seem either that one is obliged to fault their independent proofs, or that one must show that the proofs are somehow irrelevant.

This version of the no-conflict hypothesis denies the incompatibility of the paradoxes with our intuitions about entailment by appeal to an alleged reference-split. ("What the paradoxes are about and what our intuitions pertain to are not one and the same.") It is evident, I trust, that the hypothesis requires justification; yet what may not be evident is that it is to be doubted whether an adequate defense of it is possible. Consider one such imaginary defense which, for the sake of greater clarity, we render schematically, without actual reference to entailment, strict implication and the paradoxes. In this way it is hoped better to reveal its formal contours and those of the hypothesis for which it is a defense.

3. Suppose: there exist the transitive relations R and R^*, under which some class of propositions is closed, about which it is the case

(1) that the relation R is so defined that certain specific consequences, $a_1R\, b_1, \ldots, a_nR\, b_n$, can be derived therefrom in strict accordance with sound rules of inference. Let **C** be the class of just those consequences;

(2) that it is believed by some that R and R^* are one and the same relation. If $R = R^*$, then every occurrence of 'R' in the statements in **C** can be replaced *salva veritate* by an occurrence of 'R^*'. If substitution within **C** breaks down, it follows that $R \neq R^*$.

Our concern here is with the narrower question whether 'R^*' is interchangeable with 'R' within **C**, whether as we shall say, 'R^*' **C**-*interchanges* 'R'. **C**-interchange is non-symmetrical.

To continue, suppose

(3) that the **C**-interchange of 'R^*' for 'R' yields results which, by reference to our intuitions about the nature of R^*, seem patently false. Let the class of such intuitions be **In**.

It would appear, then, either that $\sim$('R^*' **C**-interchanges 'R') or that the statements in **C** are, thanks to **In**, false.

(4) Suppose further that independent conditional proofs are available for the truth of the statements in **C**. The independence of these proofs is important. It means that the proofs neither imply nor presuppose either the truth or falsity of the definition of R, or any fact about the identity or non-identity of R and R^*. Thus, it would seem to follow that $\sim$('R^*' **C**-interchanges 'R'), and hence that $R = R^*$.

Thus far the no-conflict hypothesis seems plausible enough; but it owes its plausibility to the fact that an important feature about the nature of valid proof has been overlooked. Let it be the case

(5) that there exists, with respect to the proofs of the statements in **C** a deduction theorem such that for any valid rule of inference, $a \therefore b$, there is a true statement asserting that a bears R to b. That is, the rules of inference invoked by the proofs for the statements in **C** are, as we shall say, *R-generating*. To say that a rule of inference, $a \therefore b$, is R-generating is to say that if the rule is valid, then $a\ R\ b$. Similarly, to say that a proof is R-generating is to say that, if valid, its first step bears R to its last step. It is evident that, if valid, the proofs of the statements in **C** are R-generating. In those proofs, every step bears to its successor the R-relation; and because R is transitive, the first step of each proof stands in the R-relation to its last step.

The proofs are valid and R-generating, which is to say that the statements of **C** are true. But those very statements, interchanging 'R^*' for 'R', yield results which, by **In**, are false. It must be the case, then, that the **C**-interchange is illegitimate, and that $R \neq R^*$.

But suppose, further,

(6) that not only is it the case that the inference rules appealed to by the proofs of the statements in **C** are R-generating, but R^*-generating as well. On this assumption the first step of each proof would bear R^* to its last step, from which it follows, given the transitivity of R^*, the results of the **C**-interchange of 'R^*' for 'R' would be true. It would follow from the fact that the proofs were valid, that they are R-generating, which is just to say that **C**-interchange does not break down.

By appeal to **In** the results of the **C**-interchange of 'R^*' for 'R' are false; by appeal to the R^*-generating character of valid proof, those results are true. But this is absurd.

4. How is the absurdity to be avoided? Consider a possible escape.

(a) If it is the case that 'R^*' **C**-interchanges 'R', that the proof of the statements in **C** are valid, and that our intuitions, **In**, about R^* are sound, then the statements of **C** are both true and false.

(b) The consequent of (a) is impossible.

(c) Therefore, either (i) $\sim$('R^*' **C**-interchanges 'R'),
or (ii) the proofs are invalid,
or (iii) the results of **C**-interchanging 'R^*' for R are not false.

(d) But, by the no-conflict hypothesis,
(i) the proofs are valid,
and (ii) the results of C-interchange of 'R*' for 'R' are false.

(e) Therefore, ~('R*' C-interchanges 'R').

It is just here that the no-conflict hypothesis would like to rest. If 'R*' does not C-interchange 'R', then R is not the same relation as R*. The statements in C involve R alone; our intuitive doubts about the correctness of the results of C-interchange concern R* alone.
There is no conflict.

If one ignores (6) above, or fails to appreciate its force, one is likely to remain blind to the fatal weakness of the no-conflict hypothesis. If (6) is true, if that is, the relevant inference rules are R*-generating, the no-conflict argument quickly reduces to absurdity:

(f) The proofs involve R*-generating inference rules; the proofs, if valid, are R*-generating. (By hypothesis).

(g) Therefore, given (di), 'R*' C-interchanges 'R'.

(h) From (g), (dii) and (c) it follows that the proofs are invalid.

But (h) contradicts (di); and (f) contradicts (g).

An advantage of this schematized presentation is that it yields a relatively uncluttered view of the terrain. It is manifestly clear that our no-conflict theorist is put in a position where he must defeat step (f) above, or relinquish the right to be taken seriously. If he persists in thinking that the C-interchange of 'R*' for 'R' yields false results, he is obliged to show that the proofs, while R-generating, are not R*-generating. Unless he shows this, his argument explodes.

5. We may now appreciate how the no-conflict hypothesis, as it touches upon the substantive issue of the identity-thesis, encounters parallel difficulties. This can be seen by filling in our schematic map as follows:

(i) for 'R', depending on the context, read 'strictly implies' or 'strict implication';

(ii) for 'R*', depending on the context, read 'entails' or 'entailment';

(iii) let the statements of C be Lewis' paradoxes:

(iv) let the members of **In** be those of our intuitions about entailment which allegedly falsify the paradoxes;

(v) let the proofs of the statements of C be Lewis' independent proofs of the paradoxes.

On this interpretation, (5), above, asserts the Lewis' proofs of the paradoxes are *strict implication-generating*; this means that if valid, the first step of each strictly implies its last step. Thus the paradoxes are true when their main connective is interpreted as being the strict implication-connective.

But (6), above, asserts the stronger claim that Lewis' proofs are entailment-generating, which is to say that (if valid) the first step of each entails its last step. From which it follows, that it cannot be the case both that the proofs are valid and that the results of interchanging '*N*' for '$\prec$' within the paradoxes are false. (Read '*N*' as 'entails'). Lewis obviously did think that the relevant inference rules are entailment-generating, and obviously did think that a valid proof is entailment-generating. (Lewis [4], p. 252, and [5], p. 531. Of his system Lewis wrote that, "its primary advantage over any present system lies in the fact that its meaning is precisely that of ordinary inference and proof"). That he thought so is evident from the fact that he rendered the main connective, both in his rules of inference and in the conclusions of his conditional proofs of the paradoxes, by the symbol '$\prec$', which, for Lewis was interpreted as we interpret 'entails', viz. as did Moore ([6], p. 291). Lewis issued to his critics a challenge to refute his proofs of the paradoxes, which the proponent of the present version of the no-conflict hypothesis has ignored; and for just this reason that hypothesis remains unjustified. From the outset it begs the question against the identity-thesis.

Lewis' challenge is expressable in two ways: A critic who appeals to the paradoxes as a sufficient reason for denying the identity-thesis must either

(a) refute the proofs of the paradoxes; or, otherwise put,
(b) show that the proofs are valid, but not entailment-generating.

The formulation of the no-conflict hypothesis which we have been considering accepts the 'validity' of the proofs, but denies their entailment-generating character. So it is formulation (b) of Lewis' challenge that he needs to focus upon.

6. How is it that challenge (b) has not been met? How is it that it seems scarcely to have been recognized? This curious fact might plausibly be explained as follows. (For other possible explanations see Woods [9], pp. 405-21). There follow, directly from Lewis' definition of strict implication, four consequences, which, if strict implication is identified with entailment, seem obviously false. They outrage our preanalytic appreciation of the nature of entailment. It is not plausible to question the accuracy of the derivation of the paradoxes from the definition of strict implication. In this respect, everything seems in order. But it *is* plausible to think that 'strictly implies' is wrongly interpreted to mean 'entails'.

The suggestion that the derivability of the paradoxes refutes the identity-thesis is challenged by Lewis' independent proofs. But the challenge seems to have been emasculated by the fact that each of the rules of inference and each of the conclusions of the proofs has as its main connective the symbol for *strict implication*. But the very strong intuitive evidence is that strict implication is weaker than entailment. All that Lewis seems to have proved is that an impossible proposition *strictly implies* any proposition, that a necessary proposition is *strictly implied* by any proposition, and so on. Similarly, all that he has appealed to are rules of inference in which it is asserted that certain propositions *strictly imply* certain others. These proofs are perfectly in order, given what they actually assert. What they assert is not, however, sufficient to defeat the claim that the paradoxes are false of *entailment*. A glance at the proof of the first paradox makes this obvious.

Assumption:	$p \cdot \sim p$	(1)
(1)	$\prec p$	(2)
(1)	$\prec \sim p$	(3)
(2)	$\prec p \vee q$	(4)
(3), (4)	$\prec q$	(5)
	$\therefore (p \cdot \sim p) \prec q$	

Here all the rules of inference are framed in terms of strict implication; hence so, too, is the conclusion. Yet given our initial very strong doubts about the truth of $(p \cdot \sim p)\, N\, q$, it is otiose and silly to suppose that those doubts are legitimately assuaged by Lewis' proof that $(p \,.\, \sim p) \prec q$; for this is a "weaker" conclusion which Lewis' critics never wanted to deny. Still, the proof is certainly valid, at least in the sense that each step follows in accordance with a principle of inference framed in terms of strict implication. There may be other senses of 'valid' according to which the argument is not valid. The proof one might say, as Belnap ([2], p. 31) did say, is *strictly* valid, where a *strictly* valid proof is strict implication-generating, but not entailment-generating. Accordingly, there seems to be a clear sense in which we can say that the proofs are valid and still consistently deny the truth of the paradoxes construed as entailment-statements.

7. This may explain, but it does not justify. Earlier I remarked the importance of realizing that Lewis himself interpreted the fish-hook sign '$\prec$' as the sign for '*entails*'. The importance of this is obvious. Lewis would be quite prepared to substitute in his system, S2, for every occurrence of '$\prec$' the *entailment*-connective 'N'; for every occurrence of 'strictly implies', 'entails'; and for every occurrence of 'strict implication', 'entailment'. Having completely abandoned the *terminology* of strict implication, in favour of that of entailment, one could go through those parts of S2 presently under suspicion, and systematically 'strengthen' its claims by making explicit the fact that they are and were meant to be entailment-statements.

It could no longer be supposed that the rules of inference of the system, in their rewritten form, are valid but not entailment-generating; nor that, in their re-written form, the proofs of the paradoxes are valid, but not entailment-generating. For Lewis, non-entailment generating proofs and inference-rules are *invalid* and useless for his purposes; for Lewis, "strict validity" is *not* a species of validity.

Thus, a proponent of the present version of the no-conflict theory, one who holds both that the paradoxes are false of entailment and that the proofs are valid, cannot be attributing validity in the entailment-generating sense. Until he shows that the proofs are, in that sense, *in*valid he cannot legitimately deny the truth of the paradoxes (as re-written in terms of 'entails'). What this comes down to, of course, is that the no-conflict theorist must invalidate the proofs as re-written in terms of entailment. The claim that the proofs are not entailment-generating just is the claim that the proofs in the relevant sense are not valid. Our no-conflict theorist is now charged to refute Lewis' proofs. The question is, *can he do this*? (Anderson and Belnap [1], pp. 9-24; Hockney and Wilson [3] pp. 211-20; and Woods [10].)

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