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ALTERNATIVE COMPLETENESS THEOREMS FOR MODAL SYSTEMS

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Since the development of semantics for the modal systems T, S4 and S5 notably in [1] there have appeared several completeness theorems for these systems. Some, e.g. [1][1] rely on the method of semantic tableaux which are shewn to give a decision procedure. A simpler proof, though one not yielding to decision procedure, follows from [3][2]. In part I of this paper we shew, by extensions of known results, some relations between the completeness of S5, S4 and T. In part II we shew how Anderson's [5] decision procedure for T can yield a relatively simple completeness proof for that system.

I. The system T (v.[6]) is a system of propositional modal logic based on the following additions to some standard axiomatic basis for the propositional calculus, (L for necessity);

LA1 $Lp \supset p$
LA2 $L(p \supset q) \supset (Lp \supset Lq)$
LR1 $\vdash \alpha \rightarrow \vdash L\alpha$

S4 is T with the addition of

LA3 $Lp \supset LLp$

S5 is T with the addition of

LA4 $\sim Lp \supset L \sim Lp$

We define validity in T, S4, and S5 in the manner of [1] as truth in all T, S4, S5 models. A T-model is an ordered triple $<VWR>$ where W is a set of objects (worlds), R a reflexive relation over W, and V a function (assignment) taking as arguments a.) wffs of T b.) members of W and as values the truth values 1 or 0, and satisfying the following:

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i) For every propositional variable p and every $x_i \varepsilon W$ $V(px_i) = 1$ or 0
ii) For any wff α and any $x_i \varepsilon W$, $V(\sim\alpha x_i) = 1$ iff $V(\alpha x_i) = 0$ otherwise 0.
iii) For any wffs α and β and any $x_i \varepsilon W$ $V((\alpha \vee \beta)x_i) = 1$ iff either $V(\alpha x_i) = 1$ or $V(\beta x_i) = 1$, otherwise 0.
iv) For any wff α and any $x_i \varepsilon W$, $V(L\alpha x_i) = 1$ iff for every $x_j R x_i$, $V(\alpha x_j) = 1$, otherwise 0.

A T-model in which R is transitive is an S4-model and in which R is an equivalence relation is an S5-model. A formula α is T, S4, S5-valid iff $V(\alpha x_i) = 1$ for every $x_i \varepsilon W$ in every T, S4, S5-model $<VWR>$. It may easily be shewn that every T, S4, S5 theorem is T, S4, S5 valid respectively by noting that the axioms are valid and the rules validity-preserving.

THEOREM I *If* T *is complete then* S5 *is complete.*

It is well known[3] that every formula α of S5 is provably equivalent in S5 to a first-degree[4] formula β. Since every provable S5 equivalence is valid in S5 then α will be valid in S5 iff β is. Hence if we can prove that every S5-valid first-degree formula is also T-valid then (by the reduction process) every S5 valid formula will be S5-equivalent to a T-valid formula. Hence if T is complete then (since the reduction process is provable in S5) S5 will be complete.

Suppose α is a first-degree formula which is not T-valid. Then for some $<VWR>$ and some $x_i \varepsilon W$, $V(\alpha x_i) = 0$. Let $<V'W'R'>$ be the model defined by; $W' = \hat{x}(xRx_i)$, $x_k R' x_j$ iff $x_k, x_j \varepsilon W'$, V' is the restriction of V to W' and R'. Clearly $<V'W'R'>$ is an S5-model. Since α is a first-degree formula it is a truth-function each of whose members is;

1.) a propositional variable
or
2.) L followed by a **PC** formula.

We shew that $V'(\alpha x_i) = 0$, and hence that α is not S5-valid. If p is a propositional variable then from the definition of V' we have $V'(px_i) = V(px_i)$. For $L\beta$ (where β is a **PC** wff) then $V'(L\beta x_i) = 1$ iff $V'(\beta x_j) = 1$ for every $x_j \varepsilon W'$ i.e. for every $x_j R x_i$. But since β is a **PC** formula then for $x_j R x_i$ $V(\beta x_j) = V'(\beta x_j)$. But $V(L\beta x_i) = 1$ iff $V(\beta x_j) = 1$ for every $x_j R x_i$, hence $V(L\beta x_i) = V'(L\beta x_i)$. Hence, since α is a truth function of such members, $V'(\alpha x_i) = 0$. Hence α is not S5 valid. Hence if α is S5-valid then α is T-valid. Hence if T is complete then S5 is complete. QED

One might have thought that by reducing S4 formulae a similar result might be obtainable but it is not clear that modal functions in S4 can be reduced to functions of a given degree.[5] One can however, given a formula α, of S4 construct a formula α' such that 1.) $\vdash_T \alpha' \rightarrow \vdash_{S4} \alpha$ and 2.) if α is S4 valid then α' is T-valid. From 1. and 2. immediately follows, given the completeness of T, that every valid S4 formula is a theorem.

Given a formula α of degree n we define α' as follows; Where β_i is a sub-formula of α and β is the conjunction of $L_n(L\beta_i \supset L_n\beta_i)$ for each β_i then α' is $\beta \supset \alpha$[6].

THEOREM II *If* T *is complete then* S4 *is complete.*

It is sufficient to prove that if α is S4-valid then $\beta \supset \alpha$ is T-valid since β is obviously an S4 theorem and hence if $\beta \supset \alpha$ is a T-theorem then α will be an S4 theorem. Given a relation R we define a R-*step* as follows; x_j is one R-step from x_k iff $x_j R x_k$. x_j is $n+1$ R-steps from x_k iff there is some x_h such that x_h is n R-steps from x_k and $x_j R x_h$. (Obviously for reflexive R any x_j less than n R-steps from x_k will also be n R-steps from x_k).

Suppose $\beta \supset \alpha$ is not T-valid. Then for some $<VWR>$ and some $x_i \in W$, $V(\alpha\, x_i) = 1$ and $V(\alpha\, x_i) = 0$. Where α is a formula of degree n we define as follows the model $<V'W'R'>$; W' is the set of all x_j which are n R-steps from x_i. R' is the restriction of R_* (the ancestral of R) to W'. V' is an S4 assignment such that for propositional variables and members of W', $V' = V$. Clearly $<V'W'R'>$ is an S4-model. Further since every $x_k R' x_j$ is n R-steps from x_j then for any formula γ if $V(L_n\gamma x_j) = 1$ then $V'(L\gamma x_j) = 1$. We shew that $V'(\alpha x_i) = 0$. Proof by induction on the construction of γ. For propositional variables $V'(p x_j) = V(p x_j)$ for every $x \in W'$ by definition of V'. Where γ is of degree h ($h \leqslant n$ since γ a sub-formula of α) it is sufficient to assume the induction hypothesis as $V'(\gamma x_j) = V(\gamma x_j)$ for every x_j, $n - h$ R-steps from x_i. In what follows x_j is understood to be $n - h$ R-steps from x_i where h is the modal degree of γ.

For truth-functions, if $V'(\gamma x_j) = V(\gamma x_j)$ for every x_j and $V'(\delta x_j) = V(\delta x_j)$ for every x_j then $V'({\sim}\gamma x_j) = V({\sim}\gamma x_j)$ for every x_j and $V'((\gamma \vee \delta)x_j) = V((\gamma \vee \delta)x_j)$ for every x_j. Hence the induction holds for truth functors. Suppose $V(\gamma x_j) = V'(\gamma x_j)$ for every x_j and suppose $V(L\gamma x_j) = 1$ for some x_j. Now for any x_k n R-steps from x_i and any wff δ if $V(L_n\, \delta\, x_i) = 1$ then $V(\delta x_k) = 1$. In particular from the assignment to the antecedant β of α' we have $V(L_n(L\gamma \supset L_n\gamma)x_i) = 1$, hence $V((L\gamma \supset L_n\gamma)x_k) = 1$. Hence if $V(L\gamma x_j) = 1$ then $V(L_n\gamma x_j) = 1$, hence $V'(L\gamma x_j) = 1$. If $V(L\gamma x_j) = 0$ then for some $x_k R x_j$ $V(\gamma x_k) = 0$. Since $L\gamma$ is of degree $h+1$ then x_j is $n - (h+1)$ R-steps from x_i. Hence x_k is $n - h$ R-steps from x_i. Hence $V'(\gamma x_k) = 0$ for some $x_k R' x_j$, hence $V(L\gamma x_j) = 0$. Hence by induction $V'(\alpha x_i) = 0$. Hence α is not S4-valid. Hence if T is complete then S4 is complete. QED.

Obviously the completeness of S5 relative to T could be proved in a similar manner. However the method of theorem I is somewhat simpler.

Not only do theorems I and II give completeness relative to T but they also give a decision procedure relative to T since in the case of S5 the reduction procedure is effective and in the case of S4 the construction of α' is effective.

II. We use a method derived from the decision procedure of [5][7] to show

THEOREM III *Every* T-*valid formula is a* T *theorem.*

Basically we shew how, given any formula of modal degree n, it will either be provable given the theoremhood of formulae of degree $<n$ or can be falsified in a finite model given the falsifiability in a finite model of formulae of degree $<n$. Clearly since formulae of degree 0 (PC formulae) are

either theorems or falsifiable in some one-world model this provides an adequate inductive proof of completeness.

Given a T-formula α written in terms of L and truth-functors only (i.e. with M, $\prec$ and $=$ eliminated by definition) we say that some wf part β of α is a *constituent*[8] of α iff;

i) β is a propositional variable.

or

ii) β has the form $L\gamma$ (γ a wff)

Where β has form ii we call it an *L-constituent* of α. We construct the *modal truth table* of α by assigning a truth value 1 or 0 to each constituent as in a **PC** truth table. Clearly this will, in every case, yield a value for α. In some cases this value will be 0. We shall call such a row of the table an *F-row*.[9] Now there will be some T theorems which have F-rows. (e.g. $Lp \supset p$ where $Lp = 1$ and $p = 0$) We formulate a set of conditions such that iff α is a T-theorem then each F-row satisfies one of these conditions. Suppose that an F-row satisfies one of the following[10] where $L\beta$, $L\gamma_1, \ldots,$ $L\gamma_n$ are constituents of α:

I $L\beta$ has 1 and β has 0[11]
II $L\beta$ has 0 where $L\gamma_1, \ldots, L\gamma_n$ all have 1 and $\vdash (\gamma_1, \ldots, \gamma_n) \supset \beta$
III $L\beta$ has 0 and $\vdash \beta$

We prove **A**: If each F-row satisfies one of I-III then—From the theory of truth functions it suffices to prove that each F-row is inconsistent in T (given one of I-III) i.e. that the conjunction δ formed from γ for every γ having 0 in the row is such that $\vdash_T \sim\delta$. If I holds then δ contains as conjuncts $(L\beta . \sim\beta)$ and hence (By LA1) is inconsistent in T. If II holds then from $\vdash (\gamma_1, \ldots, \gamma_n) \supset \beta$ we have in T $\vdash L(\gamma_1, \ldots, \gamma_n) \supset L\beta$ and hence $\vdash (L\gamma_1, \ldots, L\gamma_n) \supset L\beta$[12] and hence $\vdash \sim(L\gamma_1, \ldots, L\gamma_n . \sim L\beta)$ and the inconsistencey of the whole conjunction. If III holds then by necessitation we have $\vdash L\beta$ and hence the inconsistency of any conjunction containing $\sim L\beta$. **B**: If some F-row does not satisfy one of I-III then we may construct a finite model which falsifies α. We assume as an induction hypothesis that for any formula β of lower degree than α, β is a theorem of T iff β is true in all finite T-models. Given that all non-valid formulae of lower degree than α are falsified in some finite model they can clearly each be falsified in a set of models each of whose set of worlds contains no members in common with the worlds in any other model in the set. Let **M** be such a set of models containing for each β some model $\langle V_k W_k R_k \rangle$ in which β is false for some $x_i \in W_k$. We now take the first F-row of the modal truth table of which satisfies none of the conditions I-III (Obviously if α has no F-rows $\vdash \alpha$) Take each β_i such that $L\beta_i$ in the table (i.e. its assigned value in the row we are considering) has 0. Where $L\beta_j^1, \ldots, L\beta_j^n$ are all the L-constituents having 1 in the table form $(\beta_j^1, \ldots, \beta_j^n) \supset \beta_i$. Clearly this is of lower degree than α. Hence if it is not a theorem then it is false in some finite model (induction hypothesis). Now if it were a theorem then condition II would obtain (or condition III if there are no $L\beta_j^n$'s having 1 in the table). Hence it is false in some finite model, hence it is false in some V_k ε **M** (strictly $<V_k W_k R_k>$ ε **M**). Let the world in which it is false be called x_k. Hence $V_k(\beta_j^1 x_k), \ldots, V_k(\beta_j^n x_k) = 1$ while $V_k(\beta_i x_k) = 0$ (V_k we call the falsifying model and x_k the falsifying world of $(\beta_j^1, \ldots, \beta_j^n) \supset \beta_i$). Let **M'** be just that subset of **M** which contains a falsifying model of $(\beta_j^1, \ldots, \beta_j^n) \supset \beta_i$ for each $L\beta_i$ having 0 in the table. **M'** will be a finite set of models. We now choose some x_i not in any V_k ε **M'** and call it x_1. We shew how to construct a model V (i.e. $<VWR>$) such that $V(\alpha x_1) = 0$. Let W be the set containing x_1 and just those members of any $W_k(<V_k W_k R_k>$ ε **M'**) W will be finite. R is the set of pairs $<x_i x_j>$ such that for $x_i x_j$ ε W_k, $x_i R x_j$ iff $x_i R_k x_j$ (by the nature of **M** no x's will occur in more than one W_k) and such that for $x_i = x_1, x_j$ is the falsifying world of some model in **M'**. Let $V(p x_i) = 1$ iff for x_i ε W_k, $V_k(p x_i) = 1$, otherwise 0. Let $V(p x_1) = 1$ iff p has 1 in the table, otherwise 0. Let V be a T assignment. For x_i ε W_k we note that since any $x_j R x_i$ ε $W_k, V(\beta x_i) = V_k(\beta x_i)$. We shew that for every constituent $L\beta$ of α $V(L\beta x_1) = 1$ or 0 according as $L\beta$ has 1 or 0 in the table. (For then since α is a truth function of its constituents, and since the row in question gives 0 then $V(\alpha x_1) = 0$.). If $L\beta$ has 1 then for every falsifying model $V_k(\beta x_k) = 1$. Further, since condition I does not obtain $V(\beta x_1) = 1$. Hence by definition of V and R $V(\beta x_i) = 1$ for every $x_i R x_1$. Hence $V(L\beta x_1) = 1$. If $L\beta$ has 0 then for some falsifying model $V_k(\beta x_k) = 0$. I.e., for some $x_i R x_1$, $V(\beta x_i) = 0$. Hence $V(L\beta x_1) = 0$. Hence, since $<VWR>$ is a finite model because **M'** is a finite set of finite models, α is false in some finite model. From **A** and **B** we have that either α is a theorem (if each F-row satisfies one of the conditions) or there is a model which falsifies it, i.e. if α is valid then $\vdash \alpha$, i.e. T is complete. QED

Further the method does give a decision procedure since the set of all sub-formulae of α of lower degree than α (on which the theoremhood of α was shewn to depend) can be effectively constructed from α and this process is obviously finite. It is true though that the procedure, and especially its extensions to S4 and S5 is probably not as simple to apply in practice as some of the other methods, notably the method of semantic tableaux, and for this purpose these latter may be preferable.

NOTES

1. There are also completeness by semantic tableaux where validity is defined algebraically as in e.g. [2] and [4]. For a note on the relation between algebraic and semantic models v. [1], pp. 92-94.

2. For propositional T the device of 'C-forms' in [3] is, of course, unnecessary.

3. v.e.g. [7], pp. 117-118.

4. Following [8] we define inductively the 'modal degree' of a formula as:

   1. A propositional variable is of degree 0.
   2. If α is of degree n then $\sim\alpha$ is of degree n.

3. If α is of degree n and β is of degree m then $(\alpha \vee \beta)$ is of degree n if $n \geqslant m$, otherwise m.
4. If α is of degree n the $L\alpha$ is of degree $n + 1$.

5. Certainly 'simple' functions (v. [7] p. 94) can all be reduced in S4 to one of a number of forms as shown in [8] p. 149 but no proof has been given of a reducibility of all functions to functions of a given degree and model-theoretic considerations might lead one to suspect that no such reduction is possible.

6. $L_n\alpha$ is α preceded by n L's. Strictly this condition is a little stronger than necessary since not every $L\beta_i \supset L_n\beta_i$ need appear within the scope of n L's but since the stronger antecedant is still an S4 theorem nothing is lost and the definition is simplified.

7. Anderson actually develops the procedure for S4 and then (p. 212) mentions the simpler method for T.(M). One could develop directly a completeness proof along these lines for S4 and S5 but it would be considerably more complicated than that for T and in the light of our results in part I is here unnecessary. In [9] p. 11 Kripke considers a truth-table type of decision procedure for S5.

8. cf. [5] p. 203 though ignoring the reduction to Normal form which, as Kripke observes, ([1] p. 94) appears unnecessary.

9. [5] p. 204.

10. [5] p. 212.

11. Since $L\beta$ is a constituent then β will be a truth function of constituents and so will have a value which may be calculated from the table.

12. These are all known T theorems. v.e.g. [7] pp. 124-126 (82.1 and 83.2) and p. 71 (44.3) and p. 96 (62.41).

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