Notre Dame Journal of Formal Logic
Volume VIII, Number 4, October 1967

ON THREE RELATED EXTENSIONS OF S4

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1. This paper contains three algebraic exercises which I was led to by some recent discussions with Mr. Geach. The first is to provide a proof of decidability and a characteristic model for the system obtained by extending S4[1] with

$$CMLpCpLp\ ,$$

a system originally studied by Geach, and later by various logicians. The others arise from an enquiry of Geach's into systems which can be obtained by extending S4 with words in a single variable. Could, for example, S4.3 be so presented? It could not, but I was able to establish that its strongest fragment of this form was S4.2 + *ALCLCpLpLpLCLCLCpLpLpLp*, and that its weakest extension of this form was S4 + *CMLpALCpLpLCLCLCpLpLp*. Geach then pointed out that extending these systems with *CMLpCLCLCpLpLpLp* gave the trio S4.2 + *CMLpCLCLCpLpLpLp*, D,[2] S4 + *CMLpCpLp* (in order of increasing strength). As it seems to me that this is a more interesting trio than mine, I have adjusted my techniques to apply to it instead. My second exercise, then, is to show that S4 + *CMLpCpLp* stands to D as the weakest extension of S4 with single-variable axioms containing it; my third is to show that the closure in S4 of the single-variables theses of D is S4.2 + *CMLpCLCLCpLpLpLp*.

I use the usual machinery of closure algebras, the order closure models of [2], and the finite model property. My main concern is with the well-connected closure algebras—i.e. those were

$$\mathbf{C}a \cap \mathbf{C}b = \wedge \ \text{ iff } a = \wedge \text{ or } b = \wedge$$

which are known to characterize all (normal) extensions of S4, by Lemma 3 of [1].[3] I use the symbol - for relative complement, rather than in its normal role of complement proper. Otherwise, anything I use here will be found in [1] or [2], or in the papers referred to there.

2. The system S4 + *CMLpCpLp* can also be axiomatised, more conveniently for my purposes, as S4 + *ALNLpLCpLp*. This new axiom is

Received August 19, 1965

equivalent to *CMLpLCpLp*, which can be seen to imply the old axiom, and be derived from it via $\vdash_{S4}$ *CMLpMLCpLp*, *CMLpCpLp*, and *CMLCpLpCCpLpLCpLp*. A well-connected closure algebra verifies *ALNLpLCpLp* if and only if

$$\mathsf{C}a = \vee \text{ or } \mathsf{C}a = a$$

in it. Thus any sub-set containing $\vee$ of the elements of such an algebra is closed under C. Now suppose that we are given a non-thesis of the system. It must be rejected by some well-connected closure algebra which verifies the system; take the values of its parts in a rejecting evaluation in this algebra and form the (finite) sub-Boolean algebra on them. This is also closed under C, so it is a subalgebra of the original algebra; clearly it is a finite well-connected closure algebra which verifies the system and rejects the given non-thesis. Repeating this construction for all non-theses of the system, we have a proof that it has the finite model property.

Since the system has the finite model property it is decidable and is characterized by finite order closure models. It is easily checked that the well-connected finite order closure models which verify the system are those on quasi-ordered sets of the forms represented in Figure 1.

Figure 1.

All these order closure models can be embedded in that on the denumerable quasi-ordered set of the form represented on the left in Figure 1—a model which also verifies *ALNLpLCpLp*—so that this is a characteristic model for the system. (That the system lies strictly between D and S5 follows immediately on consideration of their models.)

3. The system D can be axiomatised as S4 + *ALCLpLqLCLqLp* + *ALNLpLCLCLCpLpLpLp* (given $\vdash_{S4.3}$ *CMLpLMLp* and $\vdash \alpha \Longrightarrow \vdash L\alpha$, the latter is equi-derivable with *CMLpCLCLCpLpLpLp*). Using this axiomatisation it can be established that

(1) $\mathsf{C}a \supseteq \mathsf{C}b$ or $\mathsf{C}a \subseteq \mathsf{C}b$.
(2) If $a \cap b = \wedge$ and $\mathsf{C}a \supseteq \mathsf{C}b$ then $\mathsf{C}a = \vee$ or $\mathsf{C}(\mathsf{C}a - \mathsf{C}b) = \mathsf{C}a$.

are necessary and sufficient conditions for a well-connected closure algebra to verify D. The necessity and sufficiency of (1) for the verification of *ALCLpLqLCLqLp* are clear. Allocating $\mathsf{C}a - b$ to p, we see the necessity of (2) for the verification of *ALNLpLCLCLCpLpLpLp*, for then:

$\mathsf{C}a \supseteq \mathsf{C}a - b \supseteq a$, by the hypothesis that $a \cap b = \wedge$;
$\therefore \mathsf{C}(\mathsf{C}a - b) = \mathsf{C}a$;
$\therefore \mathsf{C}(\mathsf{C}(\mathsf{C}a - b) - (\mathsf{C}a - b)) = \mathsf{C}(\mathsf{C}a - (\mathsf{C}a - b))$
$= \mathsf{C}(\mathsf{C}a \cap b)$
$= \mathsf{C}b$, by the hypothesis that $\mathsf{C}a \supseteq \mathsf{C}b$;
$\therefore \mathsf{C}a = \vee$ or $\mathsf{C}(\mathsf{C}a - \mathsf{C}b) = \mathsf{C}a$, by the verification of the word and well-connectedness.

Taking a to be the value of p and b to be the value of $CpLp$, the sufficiency of (2) for the verification of $ALNLpLCLCLCpLpLpLp$ is clear.

A well-connected closure algebra which satisfies (1) and (2), and has at most one closed element other than $\wedge$ and $\vee$, verifies S4 + $CMLpCpLp$, for the following reasons. If an element has closure $\wedge$ then it must be $\wedge$. If an element a has a closure which is neither $\wedge$ nor $\vee$ then $\mathsf{C}a$ must be a, for otherwise $\mathsf{C}(\mathsf{C}a - a)$ is $\mathsf{C}a$, and (2) is violated when b is taken to be $\mathsf{C}a - a$. Thus all elements satisfy $\mathsf{C}a = \vee$ or $\mathsf{C}a = a$, the condition for the algebra to verify S4 + $CMLpCpLp$. So if a system is not an extension of S4 + $CMLpCpLp$ then the supposition that it be an extension of D requires that it be verified by a well-connected algebra satisfying (1) and (2), and having more than one closed element other than $\wedge$ and $\vee$. I can now show that S4 + $CMLpCpLp$ is the weakest extension of S4 with single-variable axioms to contain D, by proving that a single-variable word which is verified by an algebra of this kind is also verified by the order closure model on the quasi-ordered set of Figure 2; for this model

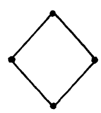

Figure 2.

does not verify D.

Suppose we are given a well-connected closure algebra satisfying conditions (1) and (2), and having elements $\mathsf{C}a$ and $\mathsf{C}b$, such that $\mathsf{C}a \supset \mathsf{C}b$ and neither is $\wedge$ or $\vee$. Then the order closure model on the quasi-ordered set of Figure 3 can

Figure 3.

be embedded in this algebra by associating the points with $\mathsf{C}b$, $\mathsf{C}a - \mathsf{C}b$, $\vee - \mathsf{C}a$, in descending order. For:

(a) These elements are a disjoint cover of the algebra.

(b) $\mathsf{C}(\mathsf{C}a - \mathsf{C}b) = \mathsf{C}a \supseteq \mathsf{C}b$, by condition (2) and hypothesis.

(c) The algebra verifies S4.3 by condition (1), so it verifies $\vdash_{S4.3} ALNLpLNLNLp$,

$\therefore$ $\mathsf{C}a = \vee$ or $\mathsf{C}(\vee - \mathsf{C}a) = \vee$, by this verification and well-connectedness;

$\therefore$ $\mathsf{C}(\vee - \mathsf{C}a) = \vee \supseteq \mathsf{C}a - \mathsf{C}b$, eliminating the other alternative by hypothesis.

(d) The converses of these containments do not hold, by hypothesis.

In turn, it may be checked that any subalgebra of the order closure model of Figure 2, generated by the element, can be embedded in the order closure model of Figure 3. Therefore every single-variable word which is verified by the given algebra is verified by the order closure model of Figure 2, as required in the previous paragraph.

4. In this section I show that S4.2 + *CMLpCLCLCpLpLpLp* contains the single-variable fragment of D. Since D is the extension of this system with *ALCLpLqLCLqLp*, it suffices to show that the system contains every word of the form $ALCL\alpha(p)L\beta(p)LCL\beta(p)L\alpha(p)$. This holds if

$$\mathrm{C}f(a) \supseteq \mathrm{C}g(a) \text{ or } \mathrm{C}f(a) \subseteq \mathrm{C}g(a)$$

in every well-connected closure algebra verifying the system. I prove this linearity by induction on the depth of C's[4] in f and g.

Given a well-connected closure algebra which verifies the system, and an element a of it, I construct sets A_n of elements of the algebra such that, with argument a, the value of every unary closure-algebraic function having a depth of C's not more then n is in A_n. These sets are defined inductively in three steps:

(1) $a_{-1} = \vee$,
$a_0 = a$,
$a_i = \mathrm{C}a_{i-1} - a_{i-1}$, for $i \geqslant 1$.

(2) $\alpha_n = \{a_n, a_{n-1}, \mathrm{C}a_{i-1} - \mathrm{C}a_i \mid 0 \leqslant i \leqslant n-1\}$.

(3) The members of A_n are $\wedge$ and the unions of the members of α_n.

Each α_n is a disjoint cover of the algebra, so each A_n is closed under the Boolean operators. A_0 contains a; it remains to show that A_{n+1} contains the closures of the members of A_n. Since the closure of a union of elements equals the union of the closures of the elements, it is sufficient to show that the closure of each member of α_n is in A_{n+1}.

(a) $$\begin{aligned}\mathrm{C}a_n &= (\mathrm{C}a_n - a_n) \cup a_n \\ &= a_{n+1} \cup a_n \\ &\varepsilon\, A_{n+1}.\end{aligned}$$

(b) $$\begin{aligned}\mathrm{C}a_{n-1} &= (\mathrm{C}a_n - a_n) \cup a_n \cup (\mathrm{C}a_{n-1} - \mathrm{C}a_n), \text{ since } \mathrm{C}a_{n-1} \supseteq \mathrm{C}a_n \\ &= a_{n+1} \cup a_n \cup (\mathrm{C}a_{n-1} - \mathrm{C}a_n) \\ &\varepsilon\, A_{n+1}.\end{aligned}$$

(c) Allocating a to p in $\vdash_{S4.2}$ *ALNLpLNLNLp* and using well-connectedness,

$$\mathrm{C}(\mathrm{C}a_{-1} - \mathrm{C}a_0) = \wedge \text{ or } \mathrm{C}(\mathrm{C}a_{-1} - \mathrm{C}a_0) = \vee$$

and $\wedge$ and $\vee$ are in A_{n+1}.

(d) Allocating a_{i-1} to p in *ALNLpLCLCLCpLpLpLp* (this thesis can be derived from *CMLpCLCLCpLpLpLp* with $\vdash_{S4.2}$ *CMLpLMLp* and $\vdash \alpha \Longrightarrow \vdash L\alpha$), noting that *CpLp* takes value a_i, and using well-connectedness,

$$\mathrm{C}a_{i-1} = \vee \text{ or } \mathrm{C}(\mathrm{C}a_{i-1} - \mathrm{C}a_i) = \mathrm{C}a_{i-1}, \text{ for } 1 \leqslant i \leqslant n-1.$$

If the first alternative holds, for some $i = m$, the construction may be terminated at $n = m - 1$, since then

$$\mathsf{C}a_n = \vee \, \varepsilon \, A_n$$

and consideration of $\mathsf{C}(\mathsf{C}a_{m-1} - \mathsf{C}a_m)$ never arises. Therefore we are left with the second alternative:

$$\begin{aligned}\mathsf{C}(\mathsf{C}a_{i-1} - \mathsf{C}a_i) &= \mathsf{C}a_{i-1} \\ &= a_{n+1} \cup a_n \cup (\mathsf{C}a_{n-1} - \mathsf{C}a_n) \cup \ldots \cup (\mathsf{C}a_{i-1} - \mathsf{C}a_i) \\ &\varepsilon \, A_{n+1}.\end{aligned}$$

Note that the closures of the members of $a_n - \vee, \mathsf{C}a_0, \mathsf{C}a_1, \ldots, \mathsf{C}a_n, \wedge$ are linearly ordered under containment, by the definition of the a_i's. It follows that the closures of the members of A_n, being their unions, are also linearly ordered under containment. The result required in the first paragraph of this section follows immediately, taking n large enough for A_n to contain $f(a)$ and $g(a)$.

NOTES

1. Throughout this paper I take S4, and extensions of it, to be given with $\vdash\alpha \Longrightarrow \vdash L\alpha$ as a derivation rule. (So all systems are automatically normal.)

2. For a brief account of this system of Prior's, and its identification as S4.3 + *CMLpCLCLCpLpLpLp*, see [1]. I gather that Mr. D. C. McKinson has produced a paper on single-variable words in D; I have not seen it.

   Added 21-8-65: I learn that this trio has been discussed in Sobociński's [3], as S4.2.1, S4.3.1, S4.4, and the point made that they are proper extensions of each other.

3. The work of sections 2 and 3 could be done more quickly by using the result, given in a forthcoming paper of mine, that all (normal) extensions of S4.3 are characterized by finite order closure models; however it seems to me that these nuts should be spared the sledge-hammer.

4. The identity function has depth 0 of C's; a Boolean function of functions has the maximum of the depths of C's of those functions for its depth of C's; if f has depth n of C's then $\mathsf{C}f$ has depth $n+1$ of C's.

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