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REGRESSIVE FUNCTIONS AND COMBINATORIAL FUNCTIONS

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1. *Introduction.** Let ε denote the set of all non-negative integers and let ε^* denote the set of all integers. Every function $f(n)$ from ε into ε uniquely determines a function c_i from ε into ε^* such that

$$(1) \qquad f(n) = \sum_{i=1}^{n} c_i \binom{n}{i}, \quad \text{for } n \in \varepsilon .$$

The function $f(n)$ is called *combinatorial* if the function c_i related to $f(n)$ by (1) assumes no negative values. The function c_i is called the *associated function* of $f(n)$. The function c_i can be explicitly expressed in terms of the function $f(n)$ by the formula:

$$(2) \qquad c_n = \sum_{i=1}^{n} (-1)^i \binom{n}{i} f(n-i).$$

Combinatorial functions were introduced by Myhill in a set-theoretic manner in [3] and play a fundamental role in the theory of recursive equivalence types; however, in what follows we need only the number-theoretic definition of a combinatorial function given above.

We note that if c_i is an effectively computable function (or formally, a recursive function), so is $f(n)$. For given n we can effectively calculate $c_0, \ldots, c_n$ and hence $f(n)$ by (1). Conversely, if $f(n)$ is a recursive combinatorial function, we can, given n, compute $f(0), \ldots, f(n)$, and hence c_n by (2). Thus c_i is a recursive function if $f(n)$ is. We conclude that for a combinatorial function $f(n)$,

$$f(n) \textit{ is recursive} \Longleftrightarrow c_i \textit{ is recursive.}$$

A function t_n from ε into ε is *regressive*, if it is one-to-one (1-1) and there exists a partial recursive function $p(x)$ such that

$$(3) \qquad \rho t \subset \delta p ,$$

$$(4) \qquad (\forall n)[p(t_n) = t_{n \dot{-} 1}] .$$

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Intuitively, t_n is regressive if given t_{n+1}, we can effectively find t_n. Since the notion of a regressive function is a generalization of that of a recursive function, a natural question arises: "Does there exist any correlation between the regressiveness of a combinatorial function and the regressiveness of its associated function?" In view of the fact that every regressive function is 1-1, we restrict our attention to the case where both $f(n)$ and its associated function c_i are 1-1. A priori, there are four possibilities:

(i) neither $f(n)$ nor its associated function c_i is regressive;
(ii) $f(n)$ is not regressive, but its associated function c_i is;
(iii) $f(n)$ is regressive, but its associated function c_i is not;
(iv) $f(n)$ and its associated function c_i are regressive.

The purpose of this paper is to show that all four possibilities do in fact exist.

2. *Preliminaries.* It is assumed that the reader is familiar with some of the terminology and theorems concerning partial recursive, recursive, and regressive functions. The following propositions are stated without proof.

Proposition 1. *Let t_n be a regressive function. Then there exists a partial recursive function $p(x)$ which in addition to* (3) *and* (4) *satisfies*

$$\rho p \subset \delta p, \tag{5}$$

$$(\forall x)[x \in \delta p \Longrightarrow (\exists k)[p^{k+1}(x) = p^k(x)]] . \tag{6}$$

Definition. Let t_n be a regressive function. If a partial recursive function $p(x)$ satisfies conditions (3), (4), (5), and (6), we call $p(x)$ a *regressing* function of t_n. We say $p(x)$ *regresses* t_n.

Definition. Let a_n, b_n be functions from ε to ε; then $a_n \leq^* b_n$ if there exists a partial recursive function $p(x)$ such that

$$\rho a \subset \delta p, \quad (\forall n)[p(a_n) = b_n] .$$

Also, $a_n \simeq b_n$ if a_n and b_n are 1-1 and there exists a 1-1 function $p(x)$ such that

$$\rho a \subset \delta p, \quad (\forall n)[p(a_n) = b_n] .$$

Proposition 2. *Let a_n, b_n, c_n be functions from ε into ε. Then*

(i) *$a_n \leq^* b_n$ and $b_n \leq^* c_n \Longrightarrow a_n \leq^* c_n$,*
(ii) *let a_n, b_n be 1-1; then $a_n \leq^* b_n$ and $b_n \leq^* a_n \Longrightarrow a_n \simeq b_n$.*

Proposition 3. *Let $a_n \simeq b_n$. Then a_n regressive $\Longleftrightarrow$ b_n regressive.*

Propositions 1 through 3 are discussed in [2]. It is known that there are exactly $\mathfrak{c}$ regressive functions, where $\mathfrak{c}$ denotes the cardinality of the continuum.

3. *Theorems.*

Theorem 1. *There exist exactly $\mathfrak{c}$ increasing combinatorial functions $f(n)$ such that neither $f(n)$ nor its associated function c_i is a regressive function.*

Proof. We shall first prove a lemma.

Lemma. *Let $g(x)$ be a function from a subset of ε into ε with an infinite range. Let c, d, p, q be four positive constants such that $p \neq q$. Then*

(1.1) $(\exists x)[x \geqslant c$ *and* $g(x) \neq p$ *and* $g(x+d) \neq q]$.

Proof of Lemma. If in the following, a set is defined by enumerating its elements, then any element of the form $g(x)$, with $x \notin \delta g$, has to be ignored. Put

$$\gamma^* = (g(0), \ldots, g(d-1));$$
$$\gamma_i = (g(d+i), g(2d+i), \ldots), \text{ for } 0 \leqslant i \leqslant d-1 \text{ ; then}$$

(1.2) $\rho g = \gamma^* + \gamma_0 + \ldots + \gamma_{d-1}$.

We first prove (1.1) for $c = 0$. Assume that (1.1) is false in this case; then

(1.3) $(\forall x)\, [g(x) = p$ or $g(x+d) = q]$.

We note that γ_0 consists of all numbers that occur at least once in the sequence

(1.4) $g(d), g(2d), \ldots$.

If all elements of (1.4) are equal to p then $\gamma_0 = (p)$ and γ_0 is finite. Now assume that not all the members of (1.4) are equal to p. Let $g(md)$ where $m \geqslant 1$ be the first element in (1.4) which does not equal p. Relation (1.3) implies $g((m+1)d) = q$ since $g(md) \neq p$. But then $g((m+1)d) \neq p$, hence $g((m+2)d) = q$. Using induction we see that $g(id) = q$ for $i > m$. Thus (1.4) contains only finitely many distinct members and γ_0 is again finite. Similarly, $\gamma_1, \ldots, \gamma_{d-1}$ are finite. Also, γ^* is finite in view of the definition. It now follows from (1.2) that ρg is finite contrary to the hypothesis. Thus (1.1) holds for $c = 0$. Now assume $c > 0$; we then put $\bar{g}(x) = g(x+c)$. Then $\bar{g}(x)$ has an infinite range; applying the case $c = 0$ of (1.1) to $\bar{g}(x)$ we obtain (1.1) itself for $g(x)$.

We now prove the theorem. There are exactly denumerably many functions which regress regressive functions from ε into ε, and all these functions have an infinite range. Hence there exists a sequence $g_0(x)$, $g_1(x), \ldots$ of partial recursive functions such that

(i) for every $i \in \varepsilon$, $g_i(x)$ has an infinite range;
(ii) every partial recursive function which regresses at least one regressive function occurs at least once in $\{g_i(x)\}$;
(iii) $g_0(1) \neq 1$, $g_1(3) \neq 1$, $g_1(2) \neq 1$.

We now define two functions $f(n)$ and c_i from ε into ε such that none of the functions $g_0(x), g_1(x), \ldots$ regresses $f(n)$ or c_i.

Basis.

(1.5) $f(0) = 1,\ c_0 = 1,\ f(1) = 3,\ c_1 = 2.$

Inductive Step. Assume as inductive hypothesis, for $k \geqslant 1$, the numbers $f(0), \ldots, f(k)$, $c_0, \ldots, c_k$ have been defined and that $c_k > 0$ and $c_k \neq f(k)$. Then let

(1.6) $c_{k+1} = (\mu x)[x \geqslant c_k+1$ and $g_{k+1}(x) \neq c_k$

and

$$g_{k+1}\left(x + \sum_{i=1}^{k} c_i \binom{k+1}{i}\right) \neq f(k)],$$

(1.7) $f(k+1) = \sum_{i=1}^{k} c_i \binom{k+1}{i} + c_{k+1}.$

Note that $f(n)$ and c_i are defined for $n \leqslant 1$ by (1.5). Also $c_1 > 0$ and $c_1 \neq f(1)$. Under the induction hypothesis c_{k+1} exists in view of the lemma, hence $c_{k\ 1}$ and $f(k+1)$ are well defined. It readily follows from (1.5 - 1.7) that $f(n)$ is a strictly increasing combinatorial function and that c_n is 1-1.

We shall now show that neither $f(n)$ nor c_i is regressive. The function $g_0(x)$ does not regress $f(n)$ nor c_i, since $f(0) = 1$, $c_0 = 1$, $g_0(1) \neq 1$, while 1 is the smallest value assumed by $f(n)$ or c_i. Similarly, the function $g_1(x)$ does not regress $f(n)$ or c_i. Finally, for each number $k \geqslant 1$, the function $g_{k+1}(x)$ does not regress $f(n)$ or c_i, in view of (1.6). Since none of the functions $g_0(x), g_1(x), \ldots$ regress $f(n)$ or c_i, neither $f(n)$ nor c_i is a regressive function.

A minor modification of c_i will enable us to prove that there are $\mathfrak{c}$ functions $f(n)$. Let $\mathcal{B}$ denote the family of all functions b_n from ε into $\{0,1\}$ such that $b_0 = 0$, $b_1 = 0$. We associate with every $b_n \in \mathcal{B}$ the functions c_n and $f(n)$ in the following manner:

(1.5') As above.

(1.6') If $b_{k+1} = 0$, c_{k+1} is defined as above. If $b_{k+1} = 1$ let

$$\bar{c}_{k+1} = (\mu x)\left[x \geqslant c_k + 1 \text{ and } g_{k+1}(x) = c_k \text{ and } g\left(x + \sum_{i=0}^{k} c_i \binom{k+1}{i}\right) \neq f(k)\right].$$

Put

$$c_{k+1} = (\mu x)\left[x \geqslant \bar{c}_k + 1 \text{ and } g_{k+1}(x) \neq c_k \text{ and } g_{k+1}\left(x + \sum_{i=1}^{k} c_i \binom{k+1}{i}\right) \neq f(k)\right].$$

(1.7') As above.

Note that if $b_{k+1} = 1$, both $\bar{c}_{k+1}$ and c_k exist in view of the lemma. It is readily seen that different choices of b_n yield different functions c_n and hence different functions $f(n)$. Since the family $\mathcal{B}$ has cardinality $\mathfrak{c}$, we conclude that the family of all combinatorial functions such that neither it nor its associated function is regressive has at least, hence, exactly, cardinality $\mathfrak{c}$. The following propositions will be used in the proofs of theorems 2 - 4.

Proposition 4. *There exists a family $\mathcal{A}$ of strictly increasing functions from ε into ε such that $\mathcal{A}$ has cardinality $\mathfrak{c}$ and for every $a_n \in \mathcal{A}$,*

(1) *the function $g_n = a_{2n+1}$ is regressive;*
(2) *the function $h_n = a_{2n}$ is regressive;*
(3) *the function a_n is not regressive;*
(4) $a_n \leqslant^* n$.

Let $\mathcal{D}$ denote the family of all functions from ε into $\{1, \ldots, 9\}$. We associate with every function $d_n \in \mathcal{D}$ a function $g_n = \phi_1 d_n$ in the following manner:

$$g_0 = 10\, d_0 \qquad = \overline{d_0 0,}$$

$$\vdots$$

$$g_{n+1} = 100\, g_n + 10\, d_{n+1} = \overline{d_0 0\, d_1 0 \ldots 0\, d_n 0 d_{n+1} 0}\,.$$

Let $\mathcal{G} = \phi_1 \mathcal{D}$. We note

(i) g_n is a strictly increasing function from ε into ε,

(ii) g_n is a regressive function. For let $p(x)$ be the recursive function defined by

$$p(x) = \begin{cases} x, & \text{if } x < 100, \\ \left[\dfrac{x}{100}\right], & \text{if } x \geqslant 100\,. \end{cases}$$

Then $p(x)$ is a regressing function of g_n.

(iii) the family $\mathcal{G}$ has cardinality $\mathfrak{c}$; for $\mathcal{D}$ has cardinality $\mathfrak{c}$ and ϕ_1 is 1-1.

We also associate with each $d_n \in \mathcal{D}$ a function $h_n = \phi_2 d_n$ in the following manner:

$$h_0 = d_0 \qquad = \overline{d_0}\,.$$

$$\vdots$$

$$h_{n+1} = 100\, h_n + d_{n+1} = \overline{d_0 0 d_1 0 \ldots 0 d_n 0\, d_{n+1}}\,.$$

Let $\mathcal{H} = \phi_2 \mathcal{D}$. We note that $\mathcal{H}$ has the same properties listed for the family $\mathcal{G}$. We claim:

$$g_n \in \mathcal{G} \implies (\exists h_n)\,[h_n \in \mathcal{H} \text{ and } \sim(g_n \leqslant^* h_n)]\,.$$

For assume $g_n \in \mathcal{G}$. Clearly

$$g_n \leqslant^* t_n \implies (\exists p)[p(g_n) = t_n \text{ and } p \in \mathcal{M}_1]\,,$$

where $\mathcal{M}_1$ denotes the family of all partial recursive functions of one variable. Since $\mathcal{M}_1$ is denumerable, there exists at most a countable number of functions t_n such that $g_n \leqslant^* t_n$. On the other hand, $\mathcal{H}$ has cardinality $\mathfrak{c}$. Thus there exists a function h_n such that the relation $g_n \leqslant^* h_n$ is false. We now define the function $a = \phi_3 g_n h_n$ in the following manner: for $g_n \in \mathcal{G}$, let

$$a_{2n+1} = g_n,\ a_2 = h_n, \text{ where } \sim[g_n \leqslant^* h_n]\,.$$

Let $\mathcal{A} = \phi_3 \mathcal{G}\mathcal{H}$. We note that $\mathcal{A}$ is a family of $\mathfrak{c}$ strictly increasing functions; also, the functions $g_n = a_{2n+1}$ and $h_n = a_{2n}$ are regressive. However, a_n is not regressive; for if it were, we would have $a_{2n+1} \leqslant^* a_{2n}$, i.e., $g \leqslant^* h$, contrary to our choice of h. Since $10^{2n} \leqslant a_{2n} = h_n < 10^{2n+1}$ and $10^{2n+1} \leqslant a_{2n+1} = g_n < 10^{2n+2}$, we have $n = \max\{y | 10^y \leq a_n\}$; thus n can be effectively computed from a_n; i.e. $a_n \leq^* n$. Hence each of the functions in $\mathcal{A}$ satisfy (1)-(4).

Proposition 5. *Let s_n be a strictly increasing function such that $s_0 > 0$. Then*

(1) $s_k + s_{k+1}! \leq s_k!$, *for* $k \geq 2$,
(2) $s_{k-1} + s_k! \leq s_{k-1}!$, *for* $k \geq 1$.

The proof is left to the reader.

Theorem 2. *There exist exactly* $\mathfrak{c}$ *combinatorial functions* $f(n)$ *such that:*

(a) $f(n)$ *is a strictly increasing regressive function;*
(b) *the associated function* c_i *of* $f(n)$ *is strictly increasing but not regressive.*

Let $\mathcal{A}$ be the family of functions with the properties listed in the statement of proposition 4. With every function $a_n \in \mathcal{A}$ we associate the function $f(n) = \psi_1 a_n$ by

$$(2.1)\quad f(n) = \sum_{i=0}^{n} 2^{a_i!}\binom{n}{i}.$$

Let c_i be the associated function of $f(n)$. Then

$$(2.2)\quad c_i = 2^{a_i!}.$$

It is readily seen that the family $\psi_1\mathcal{A}$ consists of $\mathfrak{c}$ strictly increasing combinatorial functions $f(n)$ whose associated function c_i is also strictly increasing. We claim:

(1) $c_n \simeq a_n$;
(2) $f(n) \leq^* c_n$;
(3) $f(n) \leq^* c_{n \dot- 1}$;
(4) $f(n)$ *is regressive;*
(5) c_n *is not regressive.*

If we let

$$t(x) = \begin{cases} 2^{x!}, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$$

then $t(x)$ is a one-to-one recursive function which maps a_n onto c_n. Hence $c_n \simeq a_n$.

Re(2). We shall use the relation

$$(2.3)\quad \sum_{i=0}^{k} c_i \binom{k+1}{i} < c_{k+1}, \quad \text{for } k \geq 0,$$

which will be proved later. We claim:

$$(2.4)\quad a_n! = \max\{y \in \varepsilon \mid 2^y \leq f(n)\}, \quad \text{for } n \in \varepsilon.$$

To prove (2.4) we first observe that $f(0) = c_0 = 2^{a_0!}$. Hence, (2.4) holds for $n = 0$. Now assume $n > 0$, say $n = k + 1$. Clearly,

$$(2.5)\quad f(k+1) = \sum_{i=0}^{k} c_i \binom{k+1}{i} + c_{k+1}\,.$$

Taking into account that $c_i > 0$ for $c_i \in \mathcal{E}$, we see that $c_{k+1} < f(k+1)$. In view of (2.3) and (2.5),

$$2^{a_{k+1}!} = c_{k+1} < f(k+1) < 2c_{k+1} = 2^{a_k!+1}\,,$$

from which (2.4) follows. Relation (2.4) implies

$$f(n) \leqslant^* 2^{a_n!} = c_n\,.$$

It remains to prove (2.3). First of all, (2.3) holds for $k = 0$, for $c_0 < c_1$. Now assume $k > 0$, then

$$(2.6)\quad \sum_{i=1}^{k} c_i \binom{k+1}{i} < c_k \sum_{i=1}^{k} \binom{k+1}{i} < 2^{k+1} c_k = 2^{k+1} + a_k!\,;$$

for c_n is a strictly increasing function. Since $a_0 > 0$ and a_n is strictly increasing we see that $k+1 > a_{k+1}$; thus, it follows from proposition 5 that

$$k + a_k! < a_{k+1} + a_k! \leqslant a_{k+1}!\,.$$

Combining this last relation with (2.6), we obtain (2.3).

Re(3). Let $f(n)$ be given. From (2) and (1) we can compute c_n and a_n respectively. In view of the definition of a_n, we can compute n. If $n = 0$, $c_{n \dot{-} 1} = c_0$; if $n = 1$, $c_{n \dot{-} 1} = c_0 = f(1) - c_1$. Now assume $n \geqslant 2$, say $n = k+1$, where $k \geqslant 1$. We wish to prove that $c_{n \dot{-} 1} = c_k$ can be effectively computed from $f(n) = f(k+1)$. We assume

$$(2.7)\quad \sum_{i=1}^{k-1} c_i \binom{k+1}{i} < c_k, \quad \text{for } k \leqslant 1\,,$$

Whose proof is similar to that of (2.3). Clearly,

$$(2.8)\quad f(k+1) - c_{k+1} = \sum_{i=0}^{k-1} c_i \binom{k+1}{i} + (k+1)c_k\,.$$

Using (2.7) and (2.8), we conclude that

$$(k+1)2^{a_k!} = (k+1)c_k < f(k+1) - c_{k+1} < (k+2)c_k < (k+1)\,2^{a_k!} + 1\,,$$

$$(2.9)\quad a_k! = \max\,\{y \mid (k+1)2^y < f(k+1) - c_{k+1}\}\,.$$

Since $f(k+1)$ is given, $k+1$ and c_{k+1} can be computed. Hence $a_k!$, and therefore c_k, can be computed from (2.9).

Re(4). Let the number $f(k+1)$ be given. Consider the two sequences

$$\text{(i)}\quad a_{k+1}, a_{k-1}, \ldots, a_{i+2}, a_i\,,$$
$$\text{(ii)}\quad a_k, a_{k-2}, \ldots, a_{j+2}, a_j\,,$$

where $i = 0, j = 1$ in the case $k+1$ is even and $i = 1, j = 0$ in case $k+1$ is odd.

If $f(k+1)$ is given, we can compute c_{k+1} and c_k by (2) and (3), hence, a_{k+1} and a_k by (1). In view of the fact that a_{2n+1} and a_{2n} are regressive functions of n, we can effectively find the sequences (i) and (ii), thus also

$$f(k) = \sum_{i=0}^{k} 2^{a_i!}\binom{k}{i} ,$$

Re(5). Since $a_n \simeq c_n$ and a_n is not regressive, by proposition 3, we conclude that c_n is not regressive.

Theorem 3. *There exists exactly* $\mathfrak{c}$ *combinatorial functions* $f(n)$ *such that:*

(a) $f(n)$ *is strictly increasing but not regressive*;
(b) *the associated function* c_i *of* $f(n)$ *is a strictly increasing regressive function.*

Proof: Let $\mathcal{A}$ be a family of functions with the properties listed in the statement of proposition 4. We also assume $a_0 = 2$ so that in particular the relation

(3.1) $\quad a_k + a_{k-1}! < a_k!$

of proposition 5 holds for $k = 1$. With every function $a_n \in \mathcal{A}$ we associate a function $f(n) = \psi_2 a_n$ by

(3.2) $\quad f(n) = 2^{a_n!}$, for $n \in \varepsilon$.

We now define

(3.3) $\quad c_i$ = the associate function of $f(n)$.

It readily follows that the family $\psi_2\mathcal{A}$ of strictly increasing functions has cardinality $\mathfrak{c}$. Also, for $a_n \in \mathcal{A}$, $a_n \simeq f(n)$. We shall prove the following:

(1) c_i *is a strictly increasing function from* ε *into* ε;
(2) $c_n \leqslant^* f(n)$;
(3) $c_n \leqslant^* f(n \dot{-} 1)$;
(4) c_i *is regressive*;
(5) $f(n)$ *is not regressive.*

Re(1) *and* (2). If $n = 0$, $f(0) = c_0 = 2^{a_0!} = 4 > 0$. Let us assume that $n > 0$, say $n = k$. It follows from the definition of c_i that

$$(3.4) \quad c_k = f(k) + \sum_{i=1}^{k} (-1)^i \binom{k}{i} f(k-i) .$$

We shall use the relation

$$(3.5) \quad \sum_{i=1}^{k} (-1)^i \binom{k}{i} f(k-i) > -2^{a_k!-1},\ k \geqslant 1,$$

which will be proved later. Combining (3.4) and (3.5), we obtain the inequality

(3.6) $\quad c_k > f(k) - 2^{a_k!-1} = 2^{a_k!} - 2^{a_k!-1} = 2^{a_k!-1} > 0.$

From (3.6) and the fact that $c_0 > 0$, we see that $f(n)$ is combinatorial. We therefore have

(3.7) $c_k < c_0 + c_k \leqslant f(k)$, for $k \geqslant 1$.

Combining (3.6) and (3.7) we have

(3.8) $2^{a_k! - 1} < c_k < f(k) = 2^{a_k!}$, for $k \geqslant 1$.

We conclude from (3.8) that c_i is strictly increasing and that

$$a_k! = (\mu y)[2^y > c_k], \text{ for } k \geqslant 1, \; f(k) = 2^{a_k!}.$$

It follows that $k = 0 \Longleftrightarrow c_k = 4$. Hence if we are given a_k, where $c_k \neq 4$, we can effectively find $f(k)$ by the last two relations. Thus $c_k \leqslant^* f(k)$. It remains to prove (3.5). Let $k > 0$, then since $f(n)$ is strictly increasing,

$$(3.9) \quad \sum_{i=1}^{k} (-1)^i \binom{k}{i} f(k-i) \geqslant -f(k-1) \sum_{i=1}^{k} \binom{k}{i} > -f(k-1)\,2^k = -2^{k+a_{k-1}!}.$$

Since $a_0 > 1$ and a_n is strictly increasing, we see that $k < a_k$. In view of (3.1),

$$k + a_{k-1}! < a_k + a_{k-1}! \leqslant a_k!, \text{ i.e., } k + a_{k-1}! \leqslant a_k! - 1.$$

Combining this last relation with (3.9), we obtain (3.5).

Let c_n be given; then $f(n)$, a_n, and hence n can be computed. If $n = 0$, $f(n \dot{-} 1) = f(0) = c_0$. If $n = 1$, then $f(0) = f(1) - c_1$. We now assume $n > 1$, say $n = k + 1$. We wish to prove that $f(n \dot{-} 1) = f(k)$ can be effectively computed from $c_n = c_{k+1}$. We assume

$$(3.10) \quad \sum_{i=2}^{k+1} (-1)^{i+1} \binom{k+1}{i} f(k+1-i) > -(k+1)\,2^{a_k! - 1}, \text{ for } k \geqslant 1;$$

whose proof is similar to that of (3.5). In view of (3.3),

$$(3.11) \quad f(k+1) - c_{k+1} = (k+1) f(k) + \sum_{i=2}^{k+1} (-1)^{i+1} f(k+1-i).$$

Combining (3.10) and (3.11), we have

$$(3.12) \quad f(k+1) - c_{k+1} > (k+1) f(k) - (k+1)\,2^{a_k! - 1} = (k+1)\,2^{a_k! - 1}.$$

Since $\binom{k+1}{i} = \binom{k+1}{k+1-i} \binom{k}{i} \leqslant (k+1)\binom{k}{i}$, for $0 \leqslant i \leqslant k$, $k \geqslant 0$, we obtain

$$(3.13) \quad f(k+1) - c_{k+1} = \sum_{i=0}^{k} c_i \binom{k+1}{i} \leqslant (k+1) \sum_{i=0}^{k} c_i \binom{k}{i} (k+1) f(k), \; k \geqslant 1.$$

Combining (3.12) and (3.13) we obtain

$$(k+1)\,2^{a_k! - 1} < f(k+1) - c_k \leqslant (k+1) f(k) = (k+1)\,2^{a_k!},$$
$$a_k! = (\mu y)\,[(k+1)\,2^y \geqslant f(k+1) - c_{k+1}].$$

Since c_{k+1} is given, the number k+1 and $f(k+1)$ can be computed; by the last relation, we can also compute $f(k) = 2^{a_k!}$.

Re(4). In a proof similar to (3) of theorem 2, if we are given c_{k+1}, then we can compute $f(k+1)$ and $f(k)$, and hence $f(k-1), \ldots, f(0)$. Thus

$$c_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(k-i)$$

can be computed from c_{k+1}; i.e., c_k is regressive.

Since $a_n \simeq f(n)$ and a_n is not regressive, we conclude that $f(n)$ is not regressive.

Theorem 4. *There exist exactly* $\mathfrak{c}$ *combinatorial functions* $f(n)$ *such that*

(a) $f(n)$ *is a strictly increasing regressive function,*
(b) *the associated function* c_i *of* $f(n)$ *is a strictly increasing regressive function.*

Proof: Let $\mathcal{K}$ be a family of $\mathfrak{c}$ strictly increasing regressive functions such that for every $k_n \in \mathcal{K}$, $k_0 = 2$. Then in particular (1), (2), and (4) of proposition 4 hold. Define for every $k_n \in \mathcal{K}$, the function $f(n) = \psi_3 k_n$ by

$$f(n) = 2^{k_n!}\ .$$

Note that $f(n) \simeq k_n$, and the family $\psi_3\mathcal{K}$ has cardinality $\mathfrak{c}$. From the definition of k_n, relations (1) through (4) of theorem 3 hold, i.e., we can show that $f(n)$ is a strictly increasing combinatorial function and that its associated function is strictly increasing and regressive. Since $f(n) \simeq k_n$ and k_n is regressive, $f(n)$ is regressive. We note that if c_n is regressive then $c_n \leqslant^* f(n)$; for given c_n we can compute $c_n, c_{n-1}, \ldots, c_0$ and hence $f(n)$. Similarly, if $f(n)$ is regressive, then $f(n) \leqslant^* c_n$. If $f(n)$ and c_n are both 1-1 and regressive, it follows from the above and proposition 2 that

$$f(n) \simeq c_n .$$

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