Notre Dame Journal of Formal Logic
Volume 35, Number 2, Spring 1994

Extending Intuitionistic Linear Logic with Knotted Structural Rules

R. HORI, H. ONO, and H. SCHELLINX

Abstract In the present paper, extensions of the intuitionistic linear logic with knotted structural rules are discussed. Each knotted structural rule is a rule of inference in sequent calculi of the form: from $\Gamma, A, \ldots, A$ (n times) $\rightarrow C$ infer $\Gamma, A, \ldots, A$ (k times) $\rightarrow C$, which is called the $(n \rightsquigarrow k)$-rule. It is a restricted form of the weakening rule when $n < k$, and of the contraction rule when $n > k$. Our aim is to explore how they behave like (or unlike) the weakening and contraction rules, from both syntactic and semantic point of view. It is shown that when either $n = 1$ or $k = 1$, strong similarities hold between logics with the $(n \rightsquigarrow k)$ rule and logics with the weakening or the contraction rule, as for the cut elimination theorems, decidability and undecidability results and the finite model property.

1 Introduction In the present paper, we will introduce a new kind of structural rule, called *knotted structural rules*, and study syntactic and semantical properties of extensions of the intuitionistic linear logic with knotted structural rules. Each knotted structural rule is a rule of inference in sequent calculi of the form:

from $\Gamma, A, \ldots, A$ (n times) $\rightarrow C$ *infer* $\Gamma, A, \ldots, A$ (k times) $\rightarrow C$,

which is called the $(n \rightsquigarrow k)$-rule. It is a restricted form of the weakening rule when $n < k$, and of the contraction rule when $n > k$. Our aim is to explore how they behave like (or unlike) the weakening and contraction rules. It will be shown that when either $n = 1$ or $k = 1$, strong similarities hold between logics with the $(n \rightsquigarrow k)$ rule and logics with the weakening or the contraction rule. Therefore we can get the cut elimination theorems, decidability and undecidability results and the finite model property for them. On the other hand, we are faced with great difficulties in the remaining cases, which in the present paper we were not yet able to overcome.

In the next section, we will introduce our basic systems $\mathbf{FL_e}$ which is a sequent calculus for the intuitionistic linear logic as introduced by Girard [4], and

Received July 13, 1993; revised February 14, 1994

then we will introduce knotted structural rules. The logic obtained from $\mathbf{FL_e}$ by adding the $(n \rightsquigarrow k)$ rule and its implicational fragment will be called $\mathbf{FL}_{\mathbf{e_k}}^{\mathbf{n}}$ and $\mathbf{BCI}_{\mathbf{k}}^{\mathbf{n}}$, respectively. The cut elimination theorem for these logics will be discussed in Section 3. It will be shown that the cut elimination theorem holds for $\mathbf{BCI}_{\mathbf{k}}^{\mathbf{n}}$ if and only if $k = 1$. When $k = 1$ we can show moreover that the cut elimination theorem holds for the predicate logic $\mathbf{FL}_{\mathbf{e_k}}^{\mathbf{n}}$. Next, we introduce the n-mingle rule, which generalizes the mingle rule of Ohnishi and Matsumoto [12]. By replacing the $(1 \rightsquigarrow n)$ rule with the n-mingle rule, we obtain a sequent calculus equivalent to $\mathbf{FL}_{\mathbf{e_k}}^{\mathbf{1}}$, for which the cut elimination theorem holds.

We go on to derive some results on decision problems of logics with knotted structural rules by using techniques developed in Meyer [9] and Kiriyama and Ono [6]. We can extend results in [6] and show that for each $n \geq 2$ the propositional logic $\mathbf{FL}_{\mathbf{e_1}}^{\mathbf{n}}$ is decidable while its predicate extension is undecidable. On the other hand, even the predicate logic $\mathbf{FL}_{\mathbf{e_n}}^{\mathbf{1}}$ is shown to be decidable. In Section 5, the finite model property will be discussed. By extending the method developed by Meyer [10] and Meyer and Ono [11], we will show that the implicational logics $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$ and $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$ have the finite model property for each $n > 1$.

2 *Knotted structural rules*

As our basic system, we will take the sequent calculus $\mathbf{FL_e}$ for the intuitionistic linear logic introduced in Ono [14], which is called also $\mathbf{ILL_q}$ in Troelstra [19]. The implicational fragment of $\mathbf{FL_e}$ is commonly known as **BCI**, since the Hilbert-style formal system corresponding to it can be axiomatized by using axiom schemata which are *types* of combinators **B**, **C**, and **I**. For general information on *substructural logics* including extensions of $\mathbf{FL_e}$, see [4], [13], [14], and [19]. Next, we will introduce new structural rules, called *knotted structural rules*, each of which is a restricted form of either the weakening rule or the contraction rule. We will call the sequent calculus obtained from **BCI** (and $\mathbf{FL_e}$) by adding the $(n \rightsquigarrow k)$ rule, $\mathbf{BCI}_{\mathbf{k}}^{\mathbf{n}}$ (and $\mathbf{FL}_{\mathbf{e_k}}^{\mathbf{n}}$, respectively).

Following [14] we will introduce a sequent calculus $\mathbf{FL_e}$ for the intuitionistic linear predicate logic. The language $\mathcal{L}$ of $\mathbf{FL_e}$ consists of logical constants 0, 1, and $\perp$, logical connectives $\wedge$, $\vee$, $\supset$, and $*$ (*multiplicative conjunction* or *fusion*) and quantifiers $\forall$ and $\exists$. Notice that we will follow the notation for the constants 0 and $\perp$ of [14] and [19], which is different from that in [6] and [4]. Sequents in $\mathbf{FL_e}$ are defined in the same way as those in Gentzen's **LJ**. But we will adopt here the *multiset* notation so as to include the exchange rule implicitly. So each sequent is an expression of the form $\Gamma \to B$, where Γ is a finite (possibly empty) multiset of formulas and B is either a formula or empty. Following usual conventions, we will write $A_1, \ldots, A_n \to B$ when $\Gamma = \{A_1, \ldots, A_n\}$, and write also $\Gamma, \Delta \to B$ and $A, \Gamma \to B$, instead of $\Gamma \cup \Delta \to B$ and $\{A\} \cup \Gamma \to B$, respectively, where $\cup$ denotes the multiset sum.

Definition 2.1 The sequent calculus $\mathbf{FL_e}$ for the intuitionistic linear logic consists of following initial sequents from 1 to 4;

1. $A \to A$
2. $\perp, \Gamma \to C$
3. $\to 1$
4. $0 \to$,

and of the following rules of inference,

cut rule:

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}$$

rules for logical constants:

$$\frac{\Gamma \to C}{1, \Gamma \to C}\ (1w) \qquad \frac{\Gamma \to}{\Gamma \to 0}\ (0w)$$

rules for logical connectives:

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\supset R) \qquad \frac{\Gamma \to A \quad B, \Delta \to C}{A \supset B, \Gamma, \Delta \to C}\ (\supset L)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\vee R1) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\vee R2)$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\ (\vee L)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\wedge R)$$

$$\frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge L1) \qquad \frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge L2)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B}\ (*R) \qquad \frac{A, B, \Gamma \to C}{A * B, \Gamma \to C}\ (*L)$$

rules for quantifiers:

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists x A(x)}\ (\exists R) \qquad \frac{A(a), \Gamma \to C}{\exists x A(x), \Gamma \to C}\ (\exists L)$$

$$\frac{\Gamma \to A(a)}{\Gamma \to \forall x A(x)}\ (\forall R) \qquad \frac{A(t), \Gamma \to C}{\forall x A(x), \Gamma \to C}\ (\forall L).$$

Here, Γ and Δ are finite (possibly empty) multisets of formulas. Also, t is any term, and a is any variable satisfying the eigenvariable condition, that is, a does not occur in the lower sequent of $(\exists L)$ and $(\forall R)$.

It is easy to see that **LJ** is equivalent to the system which is obtained from $\mathbf{FL_e}$ by adding the following weakening and contraction rules.

$$\frac{\Gamma \to C}{A, \Gamma \to C}\ (\textit{weakening}) \qquad \frac{A, A, \Gamma \to C}{A, \Gamma \to C}\ (\textit{contraction}).$$

Sequent calculi $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$ are defined to be systems obtained from $\mathbf{FL_e}$ by adding the weakening rule and the contraction rule, respectively. The subscripts **e**, **w**, and **c** denote exchange, weakening, and contraction, respectively. (Recall that the exchange rule is implicitly included in all of our systems.)

Next, we will introduce implicational fragments of these logics. In this case, our language consists only of the implication $\supset$. Then, the sequent calculus **BCI** for the implicational linear logic is a system whose initial sequents are sequents of the form $A \to A$, and whose rules of inference consist of the cut and rules for $\supset$. Sequent calculi **BCK**, **BCIW**, and $\mathbf{LJ}_{\supset}$ are obtained from **BCI** by adding the weakening rule, the contraction rule, and both the weakening and contraction rules, respectively.

As is well known, the cut elimination theorem holds for any of **LJ**, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, and $\mathbf{FL_e}$ (see for example Ono and Komori [15] and [13]). Therefore, they are conservative extensions of $\mathbf{LJ}_{\supset}$, **BCK**, **BCIW**, and **BCI**, respectively. In the following, we will sometimes identify a given sequent calculus with the logic determined by it, i.e., the set of all sequents provable in it, when no confusions will occur.

Next we will introduce $(n \rightsquigarrow k)$ rules which are restricted forms of the weakening or the contraction rule. They are called collectively *knotted structural rules*. In the following, sometimes the multiset consisting only of n copies of a formula A is denoted by A^n and the multiset sum of n copies of the multiset Γ by Γ^n. To abbreviate parentheses in formulas, we will sometimes follow the convention that $\supset$ *associates to the right*, and moreover we will abbreviate a formula $\overbrace{A \supset A \supset \cdots A}^{n} \supset B$ to $A^n \supset B$.

Definition 2.2 Let (n,k) be any pair of natural numbers n and k such that $n \neq k$ and $k > 0$. Then the $(n \rightsquigarrow k)$ rule is the rule of inference defined as follows:

$$\frac{\overbrace{A,\ldots,A}^{n}, \Gamma \to C}{\underbrace{A,\ldots,A}_{k}, \Gamma \to C}\ (n \rightsquigarrow k).$$

It is obvious that the $(n \rightsquigarrow n)$ rule is redundant for any n. Also, the logic **BCI** with the $(n \rightsquigarrow 0)$ rule becomes odd when $n > 0$. For, the formula $(p^n \supset p) \supset p$, which is not provable even in **LK**, becomes provable in **BCI** with the $(n \rightsquigarrow 0)$ rule. Thus we have excluded these cases in the above definition.

Clearly, the $(0 \rightsquigarrow 1)$ rule and the $(2 \rightsquigarrow 1)$ rule are exactly the weakening rule and the contraction rule, respectively. Also, it is obvious that the $(n \rightsquigarrow k)$ rule can be derived from the weakening when $n < k$ and from the contraction rule when $n > k > 0$.

In this paper, we will discuss mainly extensions of **BCI** or $\mathbf{FL_e}$ obtained from them by adding some $(n \rightsquigarrow k)$ rules. $\mathbf{BCI}^{\mathrm{n}}_{\mathrm{k}}$ and $\mathbf{BCI}^{\mathrm{nk}}_{\mathrm{kn}}$ are sequent calculi obtained from **BCI** by adding the $(n \rightsquigarrow k)$ rule, and adding both $(n \rightsquigarrow k)$ and $(k \rightsquigarrow n)$ rules, respectively. $\mathbf{FL}^{\mathrm{n}}_{\mathrm{e_k}}$ and $\mathbf{FL}^{\mathrm{nk}}_{\mathrm{e_{kn}}}$ are defined likewise.

We can define also $\mathbf{BCK}^{\mathrm{n}}_{\mathrm{k}}$ and $\mathbf{FL}^{\mathrm{n}}_{\mathrm{ew_k}}$ for $n > k > 0$, and $\mathbf{BCIW}^{\mathrm{n}}_{\mathrm{k}}$ and $\mathbf{FL}^{\mathrm{n}}_{\mathrm{ec_k}}$ for $n < k$. Then it is easily shown that when $n > k > 0$ the $(n \rightsquigarrow k)$ rule is derivable from the $(k+1 \rightsquigarrow k)$ rule and vice versa in the presence of the weakening rule, and when $n < k$ it is derivable from the $(n \rightsquigarrow n+1)$ rule and vice versa in the presence of the contraction rule. Therefore, it suffices to consider only $\mathbf{BCK}^{\mathrm{k+1}}_{\mathrm{k}}$ (and $\mathbf{FL}^{\mathrm{k+1}}_{\mathrm{ew_k}}$) for each $k > 0$ and $\mathbf{BCIW}^{\mathrm{n}}_{\mathrm{n+1}}$ (and $\mathbf{FL}^{\mathrm{n}}_{\mathrm{ec_{n+1}}}$) for each n.

3 Cut elimination As we have mentioned already in the previous section, the cut elimination theorem holds for most *standard* sequent calculi for extensions of the intuitionistic linear logic like $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, and $\mathbf{LJ}$. On the other hand, we will show in this section that the cut elimination theorem does not hold for most sequent calculi with knotted structural rules introduced in the previous section. Of course, this does not mean that they cannot be formalized by sequent calculi for which the cut elimination theorem holds.

In the following, we will show that the cut elimination theorem holds for $\mathbf{BCI^n_k}$ if and only if $k = 1$. In fact, when $k = 1$ the cut elimination theorem holds even for the predicate logic $\mathbf{FL^n_{e_1}}$. From this, it follows that $\mathbf{FL^n_{e_1}}$ is a conservative extension of $\mathbf{BCI^n_1}$, for which the cut elimination theorem holds.

Though the cut elimination theorem fails for $\mathbf{FL^1_{e_n}}$, we will be able to introduce sequent calculi $\mathbf{FL^{*1}_{e_n}}$ and $\mathbf{FL^{*1n}_{e_{n1}}}$, which are equivalent to $\mathbf{FL^1_{e_n}}$ and $\mathbf{FL^{1n}_{e_{n1}}}$, respectively. That is, for any sequent S, S is provable in $\mathbf{FL^{*1}_{e_n}}$ (and $\mathbf{FL^{*1n}_{e_{n1}}}$) if and only if it is provable in $\mathbf{FL^1_{e_n}}$ (and $\mathbf{FL^{1n}_{e_{n1}}}$, respectively). Then we will show the cut elimination theorem for both $\mathbf{FL^{*1}_{e_n}}$ and $\mathbf{FL^{*1n}_{e_{n1}}}$.

First, we will show the following theorem.

Theorem 3.1 *The cut elimination theorem holds for* $\mathbf{BCI^n_k}$ *if and only if* $k = 1$.

Proof: We will give here only a proof of the only-if part of our theorem. A stronger form of the if-part will be shown in Theorem 3.4. To show the only-if part, it is enough to give a sequent which is provable in $\mathbf{BCI^n_k}$ but is not provable in $\mathbf{BCI^n_k}$ without cut. The following sequent $S(n,k)$ is a uniform counterexample, in which p, q, r, and s are distinct propositional variables;

$$r, p \supset (r \supset q), (p \supset q)^{k-1}, (p \supset q)^n \supset s \to s.$$

First we show that the above sequent $S(n,k)$ can be proved in $\mathbf{BCI^n_k}$:

$$\dfrac{\dfrac{\dfrac{p \to p \qquad \dfrac{r \to r \quad q \to q}{r, r \supset q \to q}}{p, r, p \supset (r \supset q) \to q}}{r, p \supset (r \supset q) \to p \supset q} \qquad \dfrac{\dfrac{\dfrac{p \supset q \to p \supset q \qquad \dfrac{p \supset q \to p \supset q \quad s \to s}{p \supset q, (p \supset q) \supset s \to s}}{\cdots}\,(\supset L)}{(p \supset q)^n, (p \supset q)^n \supset s \to s}}{(p \supset q)^k, (p \supset q)^n \supset s \to s}\,(n \rightsquigarrow k)}{r, p \supset (r \supset q), (p \supset q)^{k-1}, (p \supset q)^n \supset s \to s}\,(cut).$$

Next we will show that $S(n,k)$ is not provable in $\mathbf{BCI^n_k}$ without cut, when $k > 1$. Suppose that there exists a cut-free proof $\mathbf{P}$ of $S(n,k)$. Let J be the last rule applied in $\mathbf{P}$. Since $k > 1$ and no formula appears k times in $S(n,k)$, J cannot be the $(n \rightsquigarrow k)$ rule. Thus, J must be $(\supset L)$. We note here that no sequent of the form $\Sigma \to p$ is provable in $\mathbf{BCI^n_k}$ where Σ consists of formulas among $r, p \supset (r \supset q), p \supset q$ and $(p \supset q)^n \supset s$, as it is not a tautological sequent, i.e., a sequent provable in $\mathbf{LK}$. Therefore, the left upper sequent of J must be of the form $\Sigma \to p \supset q$. By a similar argument, we can show moreover that Σ is either $p \supset q$ or $r, p \supset (r \supset q)$, and hence the right upper sequent of J becomes either

$$r, p \supset (r \supset q), (p \supset q)^{k-2}, (p \supset q)^{n-1} \supset s \to s,$$

or

$$(p \supset q)^{k-1}, (p \supset q)^{n-1} \supset s \to s.$$

Next we consider the upper sequents of this sequent and repeat this. Then we can conclude that in general, either

$$r, p \supset (r \supset q), (p \supset q)^{k-m-1}, (p \supset q)^{n-m} \supset s \to s,$$

or

$$(p \supset q)^{k-m}, (p \supset q)^{n-m} \supset s \to s$$

must be provable in $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{k}}$. Finally, we have that

$$(p \supset q)^{n-k} \supset s \to s$$

must be provable in $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{k}}$ when $n > k$ and that either

$$r, p \supset (r \supset q), (p \supset q)^{k-n-1}, s \to s$$

or

$$(p \supset q)^{k-n}, s \to s$$

must be provable in $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{k}}$ when $k > n$. But neither of them is provable in $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{k}}$. Thus, we have a contradiction.

Note that the sequent $S(n,k)$ is provable without cut in $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{k}}$ when $k = 1$. In fact, by applying $(\supset L)$ n times, the sequent

$$r^n, (p \supset (r \supset q))^n, (p \supset q)^n \supset s \to s$$

is provable. Then, by the $(n \rightsquigarrow 1)$ rule, we have

$$r, p \supset (r \supset q), (p \supset q)^n \supset s \to s.$$

(In Prijatelj [17], the classical linear logic with the weakening rule and the $(n + 1 \rightsquigarrow n)$ rule is studied. It is shown that the cut elimination theorem fails for it. Došen pointed out that by modifying slightly the counterexample given there, we can generate another counterexample for $\mathbf{BCI}^{\mathbf{n+1}}_{\mathbf{n}}$ (by a personal communication).)

Next we will show that the cut elimination theorem holds for the predicate logic $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ for each $n \geq 0$. When $n = 0$, $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ is nothing but $\mathbf{FL_{ew}}$, and the cut elimination theorem for $\mathbf{FL_{ew}}$ was shown in [15]. So we assume $n \geq 2$ in the following. First we will prove the cut elimination theorem for $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{1}}$ by introducing the *multi-cut* rule instead of the *mix* rule. Then, we will prove the cut elimination theorem for $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ by modifying the multi-cut rule.

Definition 3.2 The multi-cut rule is a rule of inference of the following form:

$$\frac{\Gamma \to A \quad \Delta \to C}{\Delta^*[\Gamma/A] \to C}.$$

Here, Δ must contain at least one occurrence of A, and $\Delta^*[\Gamma/A]$ is a multiset obtained from Δ by replacing A^m by the multiset Γ^m for some $m > 0$. The formula A is called the multi-cut formula of the above multi-cut rule.

For example, let Δ be a multiset A,A,A,Π. Then, the following is an application of the multi-cut rule:

$$\frac{\Gamma \to A \quad A,A,A,\Pi \to C}{\Gamma,\Gamma,A,\Pi \to C}\ (\textit{multi-cut}).$$

Clearly, the cut rule is a special case of the multi-cut rule. Conversely, each application of the multi-cut rule can be replaced by repeated applications of the cut rule.

Theorem 3.3 *The cut elimination theorem holds for* $\mathbf{BCI}_1^n$ *for each* $n \geq 2$.

Proof: It is enough to show the following statement:

(1) If $\mathbf{P}$ is a proof figure of a sequent S containing only one multi-cut rule which occurs as the last inference of $\mathbf{P}$, then S is provable without the multi-cut rule.

We will define the grade and the rank of a given application of the multi-cut rule as follows:

1. The *grade* is the number of logical connectives occurring in the multi-cut formula.
2. The *rank* is the total number of sequents occurring in the proof figure over the lower sequent of the multi-cut rule.

The grade and the rank of a given proof figure $\mathbf{P}$ are defined by the grade and the rank of the application of the multi-cut rule which is the last inference of $\mathbf{P}$. We will prove (1) by using double induction on the grade and the rank of $\mathbf{P}$. More precisely, we assume that any application of the multi-cut rule can be eliminated if either its grade is smaller than the grade of $\mathbf{P}$, or its grade is the same as the grade of $\mathbf{P}$ but its rank is smaller than the rank of $\mathbf{P}$. It suffices to consider the following four cases according to the inference rule applied just before the application of the multi-cut rule;

1. either $\Gamma \to A$ or $\Delta \to C$ is an initial sequent,
2. either $\Gamma \to A$ or $\Delta \to C$ is a lower sequent of the $(n \leadsto 1)$ rule,
3. both $\Gamma \to A$ and $\Delta \to C$ are lower sequents of some logical rules such that principal formulas of both rules are just the multi-cut formula,
4. either $\Gamma \to A$ or $\Delta \to C$ is a lower sequent of a logical rule except Case 3.

We will give here a proof for Cases 2 and 3.

Case 2. The case where $\Delta \to C$ is a lower sequent of $(n \leadsto 1)$ rule is essential. Then it will be of the following form, where Δ is A,Π;

$$\frac{\Gamma \to A \quad \dfrac{A^n,\Pi \to C}{A,\Pi \to C}\ (n \leadsto 1)}{\Gamma,\Pi^*[\Gamma/A] \to C}\ (\textit{multi-cut}).$$

Then, this can be transformed into

$$\cfrac{\cfrac{\cfrac{\Gamma \to A \quad A^n,\Pi \to C}{\Gamma^n,\Pi^*[\Gamma/A] \to C}\,(\mathit{multi\text{-}cut})}{\cdots}\,(n \rightsquigarrow 1)}{\Gamma,\Pi^*[\Gamma/A] \to C}\quad .$$

The rank of the application of the multi-cut rule in the above is smaller than that of **P**. So, it can be eliminated by the hypothesis of induction.

Case 3. We can suppose that $\Gamma \to A$ and $\Delta \to C$ are lower sequents of $(\supset R)$ and $(\supset L)$, respectively. So, it will be of the following form:

$$\cfrac{\cfrac{A,\Sigma \to B}{\Sigma \to A \supset B}\,(\supset R) \quad \cfrac{\Pi_1 \to A \quad B,\Pi_2 \to C}{A \supset B,\Pi_1,\Pi_2 \to C}\,(\supset L)}{\Sigma,\Pi_1^*[\Sigma/A \supset B],\Pi_2^*[\Sigma/A \supset B] \to C}\,(\mathit{multi\text{-}cut}).$$

Then this can be transformed into

$$\cfrac{\cfrac{\cfrac{\Sigma \to A \supset B \quad \Pi_1 \to A}{\Pi_1^*[\Sigma/A \supset B] \to A}\,(a) \quad A,\Sigma \to B}{\Pi_1^*[\Sigma/A \supset B],\Sigma \to B}\,(b) \quad \cfrac{\Sigma \to A \supset B \quad B,\Pi_2 \to C}{B,\Pi_2^*[\Sigma/A \supset B] \to C}\,(c)}{\Sigma,\Pi_1^*[\Sigma/A \supset B],\Pi_2^*[\Sigma/A \supset B] \to C}\,(d).$$

In this proof, every rule from (a) to (d) is multi-cut, and the ranks of both (a) and (c) are smaller than that of **P**. Also, the grades of both (b) and (d) are smaller than that of **P**. So, they can be eliminated by the hypothesis of induction. Notice that when $A \supset B$ does not appear in Π_1 (and Π_2), we must omit the application (a) (and (c), respectively).

Next we will extend the above result to $\mathbf{FL}_{\mathbf{e_1}}^{\mathbf{n}}$. For the propositional logic $\mathbf{FL}_{\mathbf{e_1}}^{\mathbf{n}}$, the proof goes quite similarly to that of $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$, which we have just shown in the above. On the other hand, for the predicate logic the eigenvariable condition causes some difficulties. To overcome them, we will modify the multi-cut rule in the following form:

$$\cfrac{\Gamma \to A \quad \Delta \to C}{\Delta^*[\Gamma^\#/A] \to C}\,(\mathit{multi\text{-}cut}).$$

Here, we will give some explanations on the notation $\Delta^*[\Gamma^\#/A]$. Let $a_1,\ldots,a_k$ be variables (not necessary all the variables) which appear in Γ but do not appear in A. Suppose that A occurs at least m times in Δ. Take arbitrary m variables $b_1^i,\ldots,b_m^i$ for each $i = 1,\ldots,k$, which are not necessarily mutually distinct. For each $j \le m$, let Γ_j be a multiset obtained from Γ by replacing every free occurrence of a_i by the variable b_j^i for each $i = 1,\ldots,k$ in each formula in Γ. Then, $\Delta^*[\Gamma^\#/A]$ is the multiset obtained from Δ by replacing A^m by the multiset sum of $\Gamma_1,\ldots,\Gamma_m$.

To make our idea clearer, we will consider an example. Let Γ be a multiset $B(a),\Pi$ where the free variable a occurs neither in Π nor in a formula A, and let Δ be a multiset A^3,Σ. We will replace two of these A's in Δ, for instance. So, by taking two variables b_1 and b_2, Γ_i becomes $B(b_i),\Pi$ for $i = 1,2$. Thus, we have the following application of modified multi-cut rule:

$$\frac{B(a),\Pi \to A \qquad A^3,\Sigma \to C}{B(b_1),B(b_2),\Pi^2,A,\Sigma \to C}\ (\textit{multi-cut}).$$

It can easily be shown that modified, the multi-cut rule is a derived rule of $\mathbf{FL}^{\mathrm{n}}_{\mathrm{e}_1}$, since $\Gamma_j \to A$ is provable for each $j = 1,\ldots,m$.

Using this modified multi-cut rule, we will show (1) for the present case. Now, the following case is essential:

$$\frac{\dfrac{B(a),\Gamma \to A}{\exists xB(x),\Gamma \to A}\,(\exists L) \qquad \Delta \to C}{\Delta^*[(\exists xB(x),\Gamma)^{\#}/A] \to C}\ (\textit{multi-cut}).$$

We assume here that A^m in Δ will be replaced by the above multi-cut. So suppose that Δ is A^m,Σ and $\Delta^*[(\exists xB(x),\Gamma)^{\#}/A]$ is $\exists xB_1(x),\Gamma_1,\ldots,\exists xB_m(x),\Gamma_m,\Sigma$, where $\exists xB_i(x),\Gamma_i$ is the multiset obtained from $\exists xB(x),\Gamma$ by some substitution of variables for each i. Now take distinct, new m variables $b_1,\ldots,b_m$. Then, we will have the following:

$$\dfrac{\dfrac{\dfrac{B(a),\Gamma \to A \qquad \Delta \to C}{B_1(b_1),\Gamma_1,B_2(b_2),\Gamma_2,\ldots,B_m(b_m),\Gamma_m,\Sigma \to C}\,(\textit{multi-cut})}{\exists xB_1(x),\Gamma_1,B_2(b_2),\Gamma_2,\ldots,B_m(b_m),\Gamma_m,\Sigma \to C}\,(\exists L)}{\dfrac{\cdots}{\exists xB_1(x),\Gamma_1,\ldots,\exists xB_m(x),\Gamma_m,\Sigma \to C}}\,(\exists L) \quad .$$

Clearly, the last sequent is equal to $\Delta^*[(\exists xB(x),\Gamma)^{\#}/A] \to C$.

Thus we have the following result.

Theorem 3.4 *The cut elimination theorem holds for the predicate logic* $\mathbf{FL}^{\mathrm{n}}_{\mathrm{e}_1}$ *for each* $n \geq 2$.

As shown in the next theorem, we can obtain similar results to Theorem 3.1 for extensions of both **BCK** and **BCIW** by using the same counterexample in Theorem 3.1. Notice here that both $\mathbf{BCK}^2_1$ and $\mathbf{BCIW}^0_1$ are nothing but $\mathbf{LJ}_{\supset}$. (The second author learned in 1985 from Wroński that Došen gave a counterexample of the cut elimination theorem for $\mathbf{BCK}^{\mathrm{k}+1}_{\mathrm{k}}$ when $k > 1$. In fact, his counterexample is the same as Prijateli's in [17] (according to a recent personal communication from Došen), see also Palasiński and Wroński [16], p. 89. Question 3 of [16], which is closely related to the subjects of the present paper, seems to remain unanswered.)

Theorem 3.5 (1) *The cut elimination theorem holds for* $\mathbf{BCK}^{\mathrm{k}+1}_{\mathrm{k}}$ *if and only if* $k = 1$. (2) *The cut elimination theorem holds for* $\mathbf{BCIW}^{\mathrm{n}}_{\mathrm{n}+1}$ *if and only if* $n = 0$.

As we have shown in Theorem 3.1, the cut elimination theorem fails for $\mathbf{BCI}^1_{\mathrm{n}}$, and *a fortiori* for $\mathbf{FL}^1_{\mathrm{e}_\mathrm{n}}$. In the following, we will introduce a new rule of inference, called the *n-mingle* rule, and will show that the cut elimination theorem holds for the sequent calculus $\mathbf{FL}^{*1}_{\mathrm{e}_\mathrm{n}}$ obtained from $\mathbf{FL}_{\mathrm{e}}$ by adding the n-mingle, which is equivalent to $\mathbf{FL}^1_{\mathrm{e}_\mathrm{n}}$.

Definition 3.6 For each $n \geq 2$, the n-mingle is a rule of inference of the following form:

$$\frac{\Gamma_1, \Delta \to C \cdots \Gamma_n, \Delta \to C}{\Gamma_1, \ldots, \Gamma_n, \Delta \to C}.$$

When $n = 2$, the n-mingle rule is essentially the same as the *mingle rule* introduced by [12]. (More precisely, the original form of mingle rule in [12] is just the weak 2-mingle rule mentioned below in the proof of Theorem 3.8, see also Došen [1].) Let $\mathbf{FL}^{*1}_{\mathbf{e_n}}$ be the sequent calculus obtained from $\mathbf{FL_e}$ be adding the n-mingle. We will show the following.

Theorem 3.7 *The predicate logic* $\mathbf{FL}^{*1}_{\mathbf{e_n}}$ *is equivalent to* $\mathbf{FL}^{1}_{\mathbf{e_n}}$.

Proof: To show that the $(1 \rightsquigarrow n)$ rule is derivable in $\mathbf{FL}^{*1}_{\mathbf{e_n}}$, we assume that $A, \Delta \to C$ is provable. Then,

$$\frac{A, \Delta \to C \cdots A, \Delta \to C}{A^n, \Delta \to C}\ (n\textit{-mingle}).$$

Hence, $A^n, \Delta \to C$ is also provable. Conversely, suppose that $\Gamma_i, \Delta \to C$ is provable for each $i = 1, \ldots, n$. Suppose that Δ is the multiset $\{B_1, \ldots, B_m\}$ and D is the formula $B_1 \supset \cdots \supset B_m \supset C$. Then $\Gamma_i \to D$ is also provable. Now we have the following proof figure (in $\mathbf{FL}^{*1}_{\mathbf{e_n}}$):

$$\dfrac{\dfrac{\dfrac{\Gamma_2 \to D \qquad \dfrac{\Gamma_1 \to D \qquad \dfrac{D \to D}{D^n \to D}\,(1 \rightsquigarrow n)}{\Gamma_1, D^{n-1} \to D}}{\Gamma_1, \Gamma_2, D^{n-2} \to D}}{\dfrac{\cdots}{\Gamma_1, \ldots, \Gamma_n \to D}} \qquad D, \Delta \to C}{\Gamma_1, \ldots, \Gamma_n, \Delta \to C}.$$

Thus, $\Gamma_1, \ldots, \Gamma_n, \Delta \to C$ is provable.

Next we will show the following.

Theorem 3.8 *The cut elimination theorem holds for the predicate logic* $\mathbf{FL}^{*1}_{\mathbf{e_n}}$ *for each* $n \geq 2$.

Proof: By using the standard technique, we can show our theorem. Here we will eliminate each cut (not *mix*) in a given proof figure. In the following, we will show this only where the left side of the upper sequent in a given application of the cut rule is a consequence of the n-mingle; i.e., it is of the following form:

$$\dfrac{\dfrac{\Gamma_1, \Delta \to C \cdots \Gamma_n, \Delta \to C}{\Gamma_1, \ldots, \Gamma_n, \Delta \to C}\,(n\textit{-mingle}) \qquad C, \Sigma \to D}{\Gamma_1, \ldots, \Gamma_n, \Delta, \Sigma \to D}\,(\textit{cut}).$$

Then, this can be transformed into the following:

$$\dfrac{\dfrac{\Gamma_1, \Delta \to C \quad C, \Sigma \to D}{\Gamma_1, \Delta, \Sigma \to D}\,(\textit{cut}) \quad \cdots \quad \dfrac{\Gamma_n, \Delta \to C \quad C, \Sigma \to D}{\Gamma_n, \Delta, \Sigma \to D}\,(\textit{cut})}{\Gamma_1, \ldots, \Gamma_n, \Delta, \Sigma \to D}\,(n\textit{-mingle}).$$

The rank of every application of cut in the new proof figure is smaller than that of the former one. Therefore, these applications of cut can be eliminated.

The predicate logic $\mathbf{FI}^{\mathbf{1n}}_{\mathbf{e_{n1}}}$ can be treated in the same way. Let $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e_{n1}}}$ be the sequent calculus obtained from $\mathbf{FL}^{*\mathbf{1}}_{\mathbf{e_n}}$ by adding the $(n \rightsquigarrow 1)$ rule. In this case, we can replace n-mingle rule by the following *weak n-mingle*, due to the presence of the $(n \rightsquigarrow 1)$ rule:

$$\frac{\Gamma_1 \to C \cdots \Gamma_n \to C}{\Gamma_1, \ldots, \Gamma_n \to C} \ (\textit{weak n-mingle}).$$

In fact, from this n-mingle rule can be derived by the help of the $(n \rightsquigarrow 1)$ rule, as the following proof figure shows:

$$\cfrac{\cfrac{\cfrac{\Gamma_1, \Delta \to C \cdots \Gamma_n, \Delta \to C}{\Gamma_1, \ldots, \Gamma_n, \Delta^n \to C} \ (\textit{weak n-mingle})}{\cdots} \ (n \rightsquigarrow 1)}{\Gamma_1, \ldots, \Gamma_n, \Delta \to C} \quad .$$

Quite similarly to Theorems 3.7 and 3.8, we have the following theorem. This time we will eliminate *multi-cut* instead of cut, as in Theorem 3.4.

Theorem 3.9 (1) *The predicate logic* $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e_{n1}}}$ *is equivalent to* $\mathbf{FL}^{\mathbf{1n}}_{\mathbf{e_{n1}}}$. (2) *The cut elimination theorem holds for the predicate logic* $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e_{n1}}}$ *for each* $n \geq 2$.

4 Decision problems As corollaries of results in the previous section, we can derive some decidability and undecidability results on extensions of the intuitionistic linear logic with knotted structural rules discussed so far.

In the following, we will show that propositional logics $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ and $\mathbf{FL}^{\mathbf{1n}}_{\mathbf{e_{n1}}}$ are decidable. From results in Komori [7] and [6], we know that the existence of the contraction rule is essential in decision problems of predicate logics. In fact, it is shown that predicate logics $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$, neither of which has the contraction rule, are decidable, while $\mathbf{FL_{ec}}$ and **LJ** which have the contraction rule are undecidable. So it will be interesting to see what will happen when a predicate logic has the $(n \rightsquigarrow 1)$ rule which is a weak form of the contraction rule. We will show that the predicate logic $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ is undecidable, by using the same translation introduced in [6]. In contrast to this, we will show that the predicate logic $\mathbf{FL}^{\mathbf{1}}_{\mathbf{e_n}}$ is decidable even when our language contains function symbols.

These results can be proved essentially in the same way as in [6]. So we will assume familiarity with [6] and will give only a sketch of their proofs. (The details of proofs were described in Hori [5].)

First, we will show that the propositional logic $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ is decidable. This can be proved similarly to the decidability of the propositional logic $\mathbf{FL_{ec}}$, i.e., $\mathbf{FL}^{\mathbf{2}}_{\mathbf{e_1}}$, in [6], where a modification of Kripke's method in [8] plays an important role. (For Kripke's method, see Dunn [2]. After the publication of [6], the second author, who is also one of authors of [6], learned from Meyer that a similar result had already been obtained by him in his dissertation [9]. In fact, quite similarly to the proof given in [6], he proved that the sequent calculus **LR**, which is obtained from Gentzen's **LK** by deleting the weakening rule, is decidable.)

Similarly to $\mathbf{FL}'_{\mathbf{ec}}$ in [6], we can introduce a sequent calculus $\mathbf{FL}'^{\mathbf{n}}_{\mathbf{e_1}}$, which is equivalent to $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$, but which has neither the cut rule nor the *explicit* $(n \rightsquigarrow 1)$ rule. In $\mathbf{FL}'^{\mathbf{n}}_{\mathbf{e_1}}$, each $(n \rightsquigarrow 1)$ rule is *incorporated* into each logical rule instead.

A sequent S' is called an $(n \rightsquigarrow 1)$*-contract* of a sequent S if S' is obtained from S by some (possibly no) applications of the $(n \rightsquigarrow 1)$ rule. We say a branch in a given proof figure of $\mathbf{FL}'^{\mathbf{n}}_{\mathbf{e}_1}$ is said to be *redundant* if there exist sequents S_1 and S_2 in the branch such that (1) S_2 is below S_1 and (2) S_2 is an $(n \rightsquigarrow 1)$-contract of S_1. Then we can show that if a sequent S is provable in $\mathbf{FL}'^{\mathbf{n}}_{\mathbf{e}_1}$ then there exists a proof figure of S containing no redundant branches. In fact, this can be proved by using Curry's lemma (see for example [2]).

We say two sequents $\Gamma \rightarrow A$ and $\Delta \rightarrow A$ are *cognate* if every formula in Γ appears in Δ and vice versa, i.e., if Γ is equal to Δ as sets. Suppose that an (ordered) sequence $\mathcal{U}$ of sequents is given in which any of two sequents are cognate. Then we can show by using Kripke's lemma (see [2]) that $\mathcal{U}$ is finite whenever $\mathcal{U}$ is not redundant. Thus, any branch of a *decomposition-tree* (or a *proof-search tree*) in $\mathbf{FL}'^{\mathbf{n}}_{\mathbf{e}_1}$ having no redundant branches is finite, by König's lemma. Hence, we have the following.

Theorem 4.1 *The propositional logic* $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e}_1}$ *is decidable for each* $n \geq 2$.

Next we will show the decidability of the propositional logic $\mathbf{FL}^{\mathbf{1n}}_{\mathbf{e}_{n1}}$. This follows from Theorem 3.9 by using a method similar to Gentzen's original proof of the decidability of both **LK** and **LJ** (see [3]).

In the following we will consider proof figures in $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$. For each $k > 0$ we say that a sequent S is *k-reduced*, if each formula occurs at most k times in the antecedent of S. For each sequent S, we can obtain effectively $(n-1)$-reduced sequent S' such that S is provable in $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$ if and only if S' is provable in it, by applying the $(n \rightsquigarrow 1)$ rule or the n-mingle rule.

We will first show the following.

Lemma 4.2 *If an* $n(n-1)$*-reduced sequent* S *is provable in* $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$ *then there exists a cut-free proof of* S *in* $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$ *consisting only of* $n(n-1)$*-reduced sequents, each of which contains only subformulas of formulas in* S.

Proof: Suppose that an $n(n-1)$-reduced sequent S is provable in $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$. Then there exists a cut-free proof figure $\mathbf{P}$ of S by Theorem 3.9. It is clear that $\mathbf{P}$ satisfies the subformula property. Inserting an appropriate number of applications of the $(n \rightsquigarrow 1)$ rule, we can transform $\mathbf{P}$ into another proof figure $\mathbf{P}'$ of S in which each upper sequent of an application of rules in $\mathbf{P}'$ except the $(n \rightsquigarrow 1)$ rule is $(n-1)$-reduced. Then it can be assured that its lower sequent is $n(n-1)$-reduced. (It may happen that some formulas appear $n(n-1)$ times in the antecedent of the lower sequent by an application of the n-mingle rule.) Moreover, every initial sequent of $\mathbf{P}'$ is 1-reduced, or can be changed into an $(n-1)$-reduced one by eliminating consecutive applications of the $(n \rightsquigarrow 1)$ rule under it if it is of the form $\bot, \Gamma \rightarrow C$. Let $\mathbf{P}^*$ be the proof figure thus obtained from $\mathbf{P}'$. Clearly, it satisfies the required property in the above lemma.

Now suppose that a sequent S_0 is given. Let S be the $(n-1)$-reduced sequent obtained effectively from S_0. By Lemma 4.2, S is provable in $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$ if and only if there exists a cut-free proof figure of S in $\mathbf{FL}^{*\mathbf{1n}}_{\mathbf{e}_{n1}}$ consisting only of $n(n-1)$-reduced sequents, each of which contains only subformulas of formulas in S. So it suffices to consider the decomposition-tree $\mathcal{T}$ of S which consists only of $n(n-1)$-reduced sequents, each of which contains only subformulas of formulas in S. Clearly, $\mathcal{T}$ is finite. Therefore, we can decide whether S is provable in $\mathbf{FL}^{*1n}_{e_{n1}}$ or not.

Theorem 4.3 *The propositional logic* $\mathbf{FL}^{1n}_{e_{n1}}$ *is decidable for each* $n \geq 2$.

We will show next the undecidability of the predicate logic $\mathbf{FL}^{n}_{e_1}$. In fact, we will show a stronger undecidability result of the next theorem. In the following, we will take the language $\mathcal{L}'$ which is obtained from $\mathcal{L}$ by eliminating constants $\perp, 1$ and the logical symbol $*$. For a while we will consider predicate logics restricted to this language $\mathcal{L}'$.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be predicate logics. If every sequent provable in $\mathbf{L}_1$ is also provable in $\mathbf{L}_2$ then we will write it as $\mathbf{L}_1 \subseteq \mathbf{L}_2$.

Theorem 4.4 *Any predicate logic* $\mathbf{L}$ *such that* $\mathbf{FL}^{n}_{e_1} \subseteq \mathbf{L} \subseteq \mathbf{LJ}$ *is undecidable.*

The proof is quite similar to that of the undecidability of $\mathbf{FL}_{ec}$ in [6]. There, a sequent calculus $\mathbf{IL}$ for the intuitionistic predicate logic is introduced, which does not have the weakening rule but has initial sequents of the form instead;

1. $\Sigma, A \to A$,
2. $\Sigma, 0 \to$,

where A is an atomic formula and Σ is an arbitrary multiset of formulas.

We will take an arbitrary sequent $\Gamma \to A$ of $\mathcal{L}'$. Let $\mathcal{P}$ be the set of all predicate symbols appearing in $\Gamma \to A$ and let Φ be the set of all formulas consisting only of predicate symbols in $\mathcal{P}$. We say that a sequent $\Delta \to C$ is in Φ, whenever every formula appearing in this sequent belongs to Φ. We define a formula T as follows:

$$T = \bigwedge_{P \in \mathcal{P}} \forall x(P(x) \supset P(x)) \wedge (0 \supset 0),$$

where x is a sequence of m distinct variables when P is an m-ary predicate symbol. For any formula B in Φ, define formulas $|B|^-$ and $|B|^+$ as follows:

1. $|B|^- = B \wedge T,\ |B|^+ = B \vee 0 \qquad$ if B is atomic,
2. $|C \supset D|^- = (|C|^+ \supset |D|^-) \wedge T,\ |C \supset D|^+ = (|C|^- \supset |D|^+) \vee 0,$
3. $|C \circ D|^- = (|C|^- \circ |D|^-) \wedge T,\ |C \circ D|^+ = (|C|^+ \circ |D|^+) \vee 0$ for $\circ \in \{\vee, \wedge\}$,
4. $|QxC|^- = Qx|C|^- \wedge T,\ |QxC|^+ = Qx|C|^+ \vee 0 \qquad$ for $Q \in \{\exists, \forall\}$.

Then, the following is proved in [6].

Theorem 4.5 *For any sequent* $\Delta \to C$ *in* Φ, $\Delta \to C$ *is provable in* $\mathbf{IL}$ *if and only if* $|\Delta|^- \to |C|^+$ *is provable in* $\mathbf{FL}_{ec}$, *where* $|\Delta|^-$ *is* $|B_1|^-, \ldots, |B_m|^-$ *when* Δ *is* $B_1, \ldots, B_m$.

Next we will show the following.

Lemma 4.6 *For any sequent* $\Delta \to C$ *in* Φ, *if* $\Delta \to C$ *is provable in* $\mathbf{FL}^{n}_{e_1}$ *then* $T^m, \Delta \to C$ *is also provable in it for any* $m > 0$.

Proof: Suppose that $\mathbf{Q}$ is a proof figure of $\Delta \to C$ in $\mathbf{FL}^{n}_{e_1}$. We may assume that $\mathbf{Q}$ is cut-free by Theorem 3.4. Then we can show our lemma by induction on the length of $\mathbf{Q}$. When $\Delta \to C$ is an initial sequent, it is of the form either $P(\mathrm{x}) \to P(\mathrm{x})$ for $P \in \mathcal{P}$, or $0 \to$. In either case, we can show that $T^m, \Delta \to C$ is provable in $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ quite similarly to the proof of Lemma 3.6 of [6]. In other cases, our lemma follows immediately from the hypothesis of induction.

Using Theorem 4.5 and Lemma 4.6, we have the following.

Theorem 4.7 *For any sequent $\Delta \to C$ in Φ and any $n \geq 2$, $\Delta \to C$ is provable in* $\mathbf{IL}$ *if and only if $|\Delta|^- \to |C|^+$ is provable in* $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$.

Proof: This can be proved just as Theorem 4.5, where our claim is shown to hold for $n = 2$. The if-part is obvious since $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}} \subseteq \mathbf{FL_{ec}}$ holds. To show the converse direction, it is enough to check the case where $\Delta \to C$ is a lower sequent of the contraction rule. Thus, it is of the following form, where Δ is B, Π:

$$\frac{B, B, \Pi \to C}{B, \Pi \to C}.$$

By the hypothesis of induction, $|B|^-, |B|^-, |\Pi|^- \to C$ is provable in $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$. Notice that $|B|^-$ is of the form $B' \wedge T$ for some formula $B' \in \Phi$. By using Lemma 4.6,

$$T^{n-2}, |B|^-, |B|^-, |\Pi|^- \to |C|^+$$

is also provable. Then we have

$$\cfrac{\cfrac{T^{n-2}, |B|^-, |B|^-, |\Pi|^- \to |C|^+}{\cdots}\,(\wedge L)}{(B' \wedge T)^{n-2}, |B|^-, |B|^-, |\Pi|^- \to |C|^+}\ .$$

The last sequent is $(|B|^-)^n, |\Pi|^- \to |C|^+$. So, applying the $(n \rightsquigarrow 1)$, we have $|B|^-, |\Pi|^- \to |C|^+$.

Corollary 4.8 *Let* $\mathbf{L}$ *be any predicate logic such that* $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}} \subseteq \mathbf{L} \subseteq \mathbf{LJ}$ *for some n. Then, for any sequent $\Gamma \to A$ of $\mathcal{L}'$, $\Gamma \to A$ is provable in* $\mathbf{IL}$ *if and only if $|\Gamma|^- \to |A|^+$ is provable in* $\mathbf{L}$.

Proof: The if-part is trivial since both formulas $|B|^- \equiv B$ and $|B|^+ \equiv B$ are provable in $\mathbf{IL}$. Conversely, if $\Gamma \to A$ is provable in $\mathbf{IL}$ then $|\Gamma|^- \to |A|^+$ is provable in $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ by Theorem 4.7, and hence it is provable in $\mathbf{L}$.

From this corollary, Theorem 4.4 follows by using the undecidability of the intuitionistic predicate logic. The proof of Theorem 4.7 will suggest also the following result.

Corollary 4.9 $\mathbf{FL_e}$ *is not equivalent to the intersection of* $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ *for $n \geq 2$. More precisely, there exists a sequent which is provable in* $\mathbf{FL}^{\mathbf{n}}_{\mathbf{e_1}}$ *for every $n \geq 2$, but is not provable in* $\mathbf{FL_e}$.

Proof: Let p and q be propositional variables, and p^* be the formula $p \wedge (p \supset p)$. Then, the following sequent is an example of sequents satisfying the required property;

$$p^*, p^* \supset p^* \supset q \to q.$$

In contrast to Theorem 4.4, we can derive the following result by using Theorem 3.8. This can be proved almost in the same way as Theorem 4.5 of [6].

Theorem 4.10 *The predicate logic* $\mathbf{FL}^{\mathbf{1}}_{\mathbf{e_n}}$ *(with function symbols) is decidable for each* $n \geq 2$.

5 Finite model property In [11] Meyer and the second author proved the finite model property of the implicational logic **BCK** by using the same method as Meyer employed to prove the finite model property of **BCIW** in [10].

In this section, we will generalize the method and will show the finite model property of $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{1}}$, $\mathbf{BCI}^{\mathbf{1}}_{\mathbf{n}}$, and $\mathbf{BCI}^{\mathbf{1n}}_{\mathbf{n1}}$ for each $n \geq 2$. In the following discussions, it may be more convenient for readers to consider the system **BCI** (and $\mathbf{BCI}^{\mathbf{n}}_{\mathbf{1}}$) with n-mingle rule, instead of $\mathbf{BCI}^{\mathbf{1}}_{\mathbf{n}}$ (and $\mathbf{BCI}^{\mathbf{1n}}_{\mathbf{n1}}$, respectively). As mentioned in Section 3, the former is equivalent to the latter.

According to [11], **BCI**-*structures* are defined as follows.

Definition 5.1 A pair $\langle \mathbf{M}, \leq \rangle$ is called a **BCI**-structure, if $\mathbf{M} = \langle M, \cdot, 1 \rangle$ is a commutative monoid with unity 1, and $\leq$ is a binary relation on M satisfying that for any element $x, y, z \in M$, $x \leq y$ implies $x \cdot z \leq y \cdot z$. A valuation $\models$ on a **BCI**-structure $\langle \mathbf{M}, \leq \rangle$ is a binary relation between elements of M and propositional variables which satisfies

$x \models p$ and $x \leq y$ imply $y \models p$.

A triple $\langle \mathbf{M}, \leq, \models \rangle$ with a **BCI**-structure $\langle \mathbf{M}, \leq \rangle$ and its valuation $\models$ is called a **BCI**-model.

When the set M is finite, $\langle \mathbf{M}, \leq \rangle$ is said to be a *finite* **BCI**-structure and $\langle \mathbf{M}, \leq, \models \rangle$ a *finite* **BCI**-model. Each valuation $\models$ on $\langle \mathbf{M}, \leq \rangle$ can be extended to a relation between elements of M and formulas by

$x \models A \supset B$ if and only if for any $y \in M$, $y \models A$ implies $x \cdot y \models B$.

By using induction, we can show that for any formula A,

$x \models A$ and $x \leq y$ imply $y \models A$.

A formula A is *true* in a **BCI**-model $\langle \mathbf{M}, \leq, \models \rangle$ if $1 \models A$ holds. Also, a sequent $B_1, \ldots, B_m \to C$ is true in a **BCI**-model $\langle \mathbf{M}, \leq, \models \rangle$ if the formula $B_1 \supset \cdots \supset B_m \supset C$ is true in it. Notice here that the sequent $B_1, \ldots, B_m \to C$ is provable in **BCI** if and only if the sequent $\to B_1 \supset \cdots \supset B_m \supset C$ is provable in it.

We say a **BCI**-structure $\langle \mathbf{M}, \leq \rangle$ is a **BCK**-*structure* if it satisfies $1 \leq x$ for any $x \in M$, and is a **BCIW**-*structure* if it satisfies $x \cdot x \leq x$ for any $x \in M$. **BCK**-models and **BCIW**-models are defined similarly to **BCI**-models. In [10] and [11], the following was proved. (In this paper we will discuss mainly **BCI**-models, not **BCI**-structures, since we will put some restrictions on valuations as shown in the definition below.)

Theorem 5.2 *Both* **BCK** *and* **BCIW** *have the finite model property. That is, for any sequent* $\Gamma \to A$, $\Gamma \to A$ *is provable in* **BCK** *(and* **BCIW***) if and only if it is true in any finite* **BCK**-*model (and any finite* **BCIW**-*model, respectively).*

We will define next $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$-, $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-, and $\mathbf{BCI}_{\mathbf{n1}}^{\mathbf{1n}}$-models by modifying the definition of **BCI**-models.

Definition 5.3 Let $\langle \mathbf{M}, \leq, \models \rangle$ be a **BCI**-model. Then

1. it is a $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$-model, if $\langle \mathbf{M}, \leq \rangle$ is a $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$-structure, i.e., $x^n (= \overbrace{x \cdots x}^{n}) \leq x$ for each $x \in M$;
2. it is a $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-model, if $\models$ satisfies the following condition (2);

   (2) $x_1 \cdot u \models p, \ldots, x_n \cdot u \models p$ imply $x_1 \cdots x_n \cdot u \models p$ for any $x_1, \ldots, x_n$, $u \in M$ and any propositional variable p;
3. it is a $\mathbf{BCI}_{\mathbf{n1}}^{\mathbf{1n}}$-model, if it is a $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$-model and at the same time is a $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-model.

We remark here that in any $\mathbf{BCI}_{\mathbf{n1}}^{\mathbf{1n}}$-model the condition (2) can be replaced by the following condition (3):

(3) $x_1 \models p, \ldots, x_n \models p$ imply $x_1 \cdots x_n \models p$.

By using induction, (2) above can be extended to all formulas, i.e., in each $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-model

(4) $x_1 \cdot u \models A, \ldots, x_n \cdot u \models A$ imply $x_1 \cdots x_n \cdot u \models A$ for any $x_1, \ldots, x_n$, $u \in M$ and any formula A.

Next, we will show the completeness theorem for logics $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$ and $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$ with respect to models defined above. It is easy to see the following soundness results.

Lemma 5.4 (1) *For any sequent* $\Gamma \to A$, *if* $\Gamma \to A$ *is provable in* $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$ *then it is true in any* $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$*-model.* (2) *For any sequent* $\Gamma \to A$, *if* $\Gamma \to A$ *is provable in* $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$ *then it is true in any* $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$*-model.*

Notice that the condition $x^n \leq x$ and (4) of $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-models are necessary to validate the $(n \rightsquigarrow 1)$ rule and the n-mingle rule (or equivalently, the $(1 \rightsquigarrow n)$ rule), respectively. Like the definition of $\mathbf{BCI}_{\mathbf{1}}^{\mathbf{n}}$-models, one may have an idea of defining a $\mathbf{BCI}_{\mathbf{n}}^{\mathbf{1}}$-structure to be a **BCI**-structure satisfying $x \leq x^n$. But soundness fails for this semantics.

Similarly to Lemma 3 of [11], we can show the converse of the above lemma in a strong form. Let N_m be the set of all m-dimensional vectors, all of whose components are non-negative integers. Clearly, $\mathbf{N_m} = \langle N_m, +, \vec{0} \rangle$ forms a commutative monoid with unity $\vec{0}$ $(=\langle 0, \ldots, 0 \rangle)$, where $+$ is vector addition. We will define binary relations $|_1^n$ and $|_n^1$ on natural numbers as follows.

Definition 5.5 The binary relation $|_1^n$ on natural numbers is defined by the condition that for any natural number x and y,

$x \,|_1^n y$ if and only if (1) $x = y$, or (2) $1 \leq y \leq x$ and $x \equiv y \ (mod\, n-1)$.

The relation $|_n^1$ is the inverse of the relation $|_1^n$, i.e., $x \,|_n^1 y$ if and only if $y \,|_1^n x$.

We will extend the relations $|_1^n$ and $|_n^1$ to those on N_m by

1. for any $\vec{x} = \langle x_1, \ldots, x_m \rangle$ and $\vec{y} = \langle y_1, \ldots, y_m \rangle$ in N_m, $\vec{x}\,|_1^n\,\vec{y}$ if and only if $x_i\,|_1^n\,y_i$ for each i,
2. $\vec{x}\,|_n^1\,\vec{y}$ if and only if $\vec{y}\,|_1^n\,\vec{x}$.

Now we will show the following.

Theorem 5.6 (1) *If a sequent* $\Gamma \to A$ *is not provable in* $\mathbf{BCI_1^n}$ *then it is not true in some* $\mathbf{BCI_1^n}$*-model* $\langle \mathbf{N_m}, |_1^n, \models \rangle$. (2) *If a sequent* $\Gamma \to A$ *is not provable in* $\mathbf{BCI_n^1}$ *then it is not true in some* $\mathbf{BCI_n^1}$*-model* $\langle \mathbf{N_m}, |_n^1, \models \rangle$.

Proof: Our theorem can be shown in the same way as Lemmas 3 and 10 of [11]. So we will give here only a sketch of the proof. First we will consider the case of $\mathbf{BCI_1^n}$. Suppose that $\Gamma \to A$ is not provable in $\mathbf{BCI_1^n}$. Let $\Gamma = \{B_1, \ldots, B_m\}$ and let F be the formula $B_1 \supset \ldots \supset B_m \supset A$. Also, let $\Psi = \{D_1, \ldots, D_m\}$ be the set of all subformulas of F and K be the set of all finite multisets with elements in Ψ. Then, a multiset $\Delta \in K$ is denoted by $\Delta = \{D_1^{k_1}, \ldots, D_m^{k_m}\}$, where $k_1, \ldots, k_m$ are the multiplicity of $D_1, \ldots, D_m$ in Δ, respectively. So this Δ can be unambiguously represented by a m-dimensional vector $\vec{v} = \langle k_1, \ldots, k_m \rangle$ in N_m. In this case, we say that $\vec{v}$ represents a multiset Δ. Clearly, $\vec{0}$ represents the empty set, $\varnothing$. Now, define a binary relation $|_1^n$ on K as follows:

$\Delta\,|_1^n\,\Sigma$ if and only if for an arbitrary formula C, the sequent $\Sigma \to C$ is obtained from $\Delta \to C$ by some (possibly no) applications of the $(n \rightsquigarrow 1)$ rule.

Then it can be easily shown that if $\vec{v}$ and $\vec{w}$ represent Δ and Σ, respectively, then $\vec{v} + \vec{w}$ represents the multiset sum $\Delta \cup \Sigma$, and it holds that $\vec{v}\,|_1^n\,\vec{w}$ if and only if $\Delta\,|_1^n\,\Sigma$. We can show that $\mathbf{K} = \langle K, \cup, \varnothing \rangle$ is a commutative monoid with unity $\varnothing$. Let Δ^n be the multiset sum of n Δs. Then, it is obvious that $\Delta^n\,|_1^n\,\Delta$ holds. Thus, $\langle \mathbf{K}, |_1^n \rangle$ is a $\mathbf{BCI_1^n}$-structure which is isomorphic to $\langle \mathbf{N_m}, |_1^n \rangle$.

We will define a valuation $\models$ on $\langle \mathbf{N_m}, |_1^n \rangle$ by the condition that for any $\vec{v} \in N_m$ and any propositional variable $p \in \Psi$,

(5) $\vec{v} \models p$ if and only if $\Delta \to p$ is provable in $\mathbf{BCI_1^n}$,

where Δ is a multiset represented by $\vec{v}$. In fact, we can assume that $\models$ satisfies the condition (1) of valuations. Using induction, we can show that for any $\vec{v} \in N_m$ and any formula $B \in \Psi$,

(6) $\vec{v} \models B$ if and only if $\Delta \to B$ is provable in $\mathbf{BCI_1^n}$.

As a consequence, we have that $\vec{0} \not\models F$ since the sequent $\to F$, i.e., $\varnothing \to F$ is not provable in $\mathbf{BCI_1^n}$, by our assumption.

As for $\mathbf{BCI_n^1}$, the proof proceeds almost in the same way as the above. We will define a binary relation $|_n^1$ on K by

$\Delta\,|_n^1\,\Sigma$ if and only if $\Sigma\,|_1^n\,\Delta$.

Then we can show that $\langle \mathbf{K}, |_n^1 \rangle$ is a $\mathbf{BCI}$-structure which is isomorphic to $\langle \mathbf{N_m}, |_n^1 \rangle$. Of course, in the present case the valuation $\models$ on $\langle \mathbf{N_m}, |_n^1 \rangle$ is defined by the condition that for any $\vec{v} \in N_m$ and any propositional variable $p \in \Psi$,

(5′) $\vec{v} \models p$ if and only if $\Delta \to p$ is provable in $\mathbf{BCI_n^1}$, where Δ is a multiset represented by $\vec{v}$. This time, it is necessary to show that $\models$ also satisfies condition (2) of valuations, which will be of the following form:

(7) $\vec{v}_1 + \vec{u} \models p, \ldots, \vec{v}_n + \vec{u} \models p$ implies $\vec{v}_1 + \cdots + \vec{v}_n + \vec{u} \models p$.

Suppose that $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{u}$ represent multisets $\Delta_1, \ldots, \Delta_n$ and Σ, respectively. To show (7) it suffices to show that

(8) if $\Delta_i, \Sigma \to p$ is provable in $\mathbf{BCI^1_n}$ for each i, then $\Delta_1, \ldots, \Delta_n, \Sigma \to p$ is also provable in it.

But (8) follows immediately from the n-mingle rule. Thus we have shown our theorem.

We will make some preparations for constructing finite $\mathbf{BCI^n_1}$- and $\mathbf{BCI^1_n}$-models from models of the form $\langle \mathbf{N_m}, |^n_1 \rangle$ and $\langle \mathbf{N_m}, |^1_n \rangle$, respectively.

By $mod(a,b)$, we mean the remainder of a when divided by b. Let R and n be natural numbers such that $R > 0$ and $n > 1$. We will define an operation on natural numbers $[R, \cdot\,]_{n-1}$ by the following conditions; for any natural number c,

$$[R,c]_{n-1} = \begin{cases} c - R & \text{if } c < R \\ mod(c-R, n-1) & \text{otherwise.} \end{cases}$$

It is easy to see that the function f defined by $f(x) = R + [R,x]_{n-1}$ is a mapping from the set N of all natural numbers to the set $\{0,1,\ldots,R+n-2\}$. For instance, when $R = 7$ and $n = 4$, f takes the following values: $f(x) = x$ for $x \leq 9$, $f(10) = 7$, $f(11) = 8$, $f(12) = 9$, $f(13) = 7$, etc. In general,

(9) $R + [R,c]_{n-1} = c$ when $c \leq R + n - 2$,

since $[R,c]_{n-1} = c - R$ for such c. Moreover, we can show that for each natural number d,

(10) $R + [R,d]_{n-1} \equiv d \ (mod\, n-1)$,

(11) $d\,|^n_1 R + [R,d]_{n-1}$.

We will prove (11). If $d \leq R + n - 2$, $d = R + [R,d]_{n-1}$ by (9) and hence (11) holds. Otherwise, $d \geq R + n - 2 \geq R + [R,d]_{n-1}$. Thus, (11) holds also in this case, by using (10).

We will define also an operation $\oplus_{n-1}$ on $\{0,1,\ldots,R+n-2\}$ by

$$a \oplus_{n-1} b = R + [R, a+b]_{n-1}.$$

In the following, we will sometimes omit the subscript $n-1$ of $\oplus_{n-1}$, when no confusion will occur.

Lemma 5.7 $\langle\{0,1,\ldots,R+n-2\}, \oplus_{n-1}, 0\rangle$ *is a commutative monoid with unity* 0.

Proof: By our definition, both the commutativity of $\oplus_{n-1}$ and the neutrality of 0 follow immediately. So we will show the associativity of $\oplus_{n-1}$. By our definition $(x \oplus y) \oplus z = R + [R,(x \oplus y) + z]_{n-1}$. If $(x \oplus y) + z < R$, then $x \oplus y < R$ and hence $x \oplus y = x + y$. Thus

(12) $(x \oplus y) \oplus z = R + [R,x + y + z]_{n-1}$.

On the other hand, if $(x \oplus y) + z \geq R$, then $(x \oplus y) \oplus z = R + mod((x \oplus y) + z - R, n - 1)$. If $x + y < R$ then $x \oplus y = x + y$, and hence we have (12) in this case. So suppose that $x + y \geq R$. Then

$$\begin{aligned}(x \oplus y) \oplus z &= R + mod(R + [R,x + y]_{n-1} + z - R, n - 1)\\ &= R + mod(mod(x + y - R, n - 1) + z, n - 1)\\ &= R + mod(x + y + z - R, n - 1)\\ &= R + [R,x + y + z]_{n-1}.\end{aligned}$$

Therefore (12) holds always. By a similar argument, we can show that $x \oplus (y \oplus z) = R + [R,x + y + z]_{n-1}$. Hence we have that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

From the above argument, we can see that

(13) $x_1 \oplus \cdots \oplus x_n = R + [R,x_1 + \cdots + x_m]_{n-1}$

holds in general. Next we will show the following:

Lemma 5.8 *Let $x_1, \ldots, x_m \in \{0,1,\ldots,R + n - 2\}$. Then,* (1) $x_1 + \cdots + x_m |_1^n x_1 \oplus \cdots \oplus x_m$, *and* (2) *if* $x_1 + \cdots + x_m |_1^n k$ *and* $k \leq R$ *then* $x_1 \oplus \cdots \oplus x_m |_1^n k$.

Proof: First we will show (1). If $x_1 + \cdots + x_m \leq R + n - 2$, then $x_1 \oplus \cdots \oplus x_m = x_1 + \cdots + x_m$ by (9) and (13). On the other hand, if $x_1 + \cdots + x_m > R + n - 2$, then $x_1 + \cdots + x_m \geq x_1 \oplus \cdots \oplus x_m$ and moreover

$$\begin{aligned}x_1 \oplus \cdots \oplus x_m &= R + [R,x_1 + \cdots + x_m]\\ &\equiv x_1 + \cdots + x_m \ (mod\, n - 1)\end{aligned}$$

by (13) and (10). Thus (1) holds. Next we will show that (2) holds. Clearly (2) holds when $x_1 + \cdots + x_m \leq R + n - 2$. If $x_1 + \cdots + x_m > R + n - 2$ then $x_1 \oplus \cdots \oplus x_m \equiv x_1 + \cdots + x_m \equiv k \ (mod\, n - 1)$ by (1) and our assumption. Moreover, $k \leq R \leq x_1 \oplus \cdots \oplus x_m$. Thus, (2) holds.

For a fixed pair of R and n, let N^* be the set $\{0,1,\ldots,R + n - 2\}$ and $\mathbf{N}^*$ be the commutative monoid $\langle\{0,1,\ldots,R + n - 2\},\oplus,0\rangle$. Then we have the following.

Lemma 5.9 *Both* $\langle \mathbf{N}^*, |_1^n\rangle$ *and* $\langle \mathbf{N}^*, |_n^1\rangle$ *are finite* **BCI**-*structures. Moreover,* $\langle \mathbf{N}^*, |_1^n\rangle$ *is a* $\mathbf{BCI}_1^n$-*structure.*

Proof: We must check the monotonicity of both $|_1^n$ and $|_n^1$ with respect to $\oplus$. Suppose that x, y, and z are in the set N^* and $x |_1^n y$. When $x = y$, clearly $x \oplus z = y \oplus z$ holds. So let us assume $1 \leq y \leq x$ and $x - y \equiv 0 \ (mod\, n - 1)$. So, $x = y + k(n - 1)$ for some $k \geq 1$. If $x + z < R$, then $y + z \leq x + z < R$. So both $x \oplus z = x + z$ and $y \oplus z = y + z$ hold. Hence, $x \oplus z |_1^n y \oplus z$ holds, since

$$(x \oplus z) - (y \oplus z) = (x + z) - (y + z) = x - y \equiv 0 \ (mod\, n - 1).$$

Otherwise suppose that $x + z \geq R$. Then

$$\begin{aligned} x \oplus z &= R + mod(x + z - R, n - 1) \\ &= R + mod(y + k(n-1) + z - R, n - 1) \\ &= R + mod(y + z - R, n - 1). \end{aligned}$$

Thus, we have $x \oplus z = y \oplus z$ when $y + z \geq R$. If $y + z < R$, then $1 \leq y + z = y \oplus z \leq x \oplus z$, and

$$\begin{aligned} (x \oplus z) - (y \oplus z) &= R + mod(y + z - R, n - 1) - (y + z) \\ &= R - (y + z) - mod(R - (y + z), n - 1) \\ &\equiv 0 \ (mod\, n - 1). \end{aligned}$$

Therefore $x \oplus z \,|_1^n y \oplus z$ holds. From this, the monotonicity of $|_n^1$ with respect to $\oplus$ follows, since $|_n^1$ is the inverse of $|_1^n$.

To show that $\langle \mathbf{N}^*, |_1^n \rangle$ is a $\mathbf{BCI_1^n}$-structure, it is necessary to prove that $n(\oplus_{n-1} x) \,|_1^n x$ for each $x \in \{0,1,\ldots,R + n - 2\}$, where $n(\oplus_{n-1} x)$ denotes $\overbrace{x \oplus_{n-1} \cdots \oplus_{n-1} x}^{n}$. From (13), $n(\oplus x) = R + [R, nx]_{n-1}$. If $nx \leq R + n - 2$ then $n(\oplus x) = nx \,|_1^n x$. So suppose that $nx > R + n - 2$. In this case,

$$\begin{aligned} n(\oplus x) &= R + mod(nx - R, n - 1) \\ &= R + mod(x - R, n - 1). \end{aligned}$$

If $x \geq R$ then $R + mod(x - R, n - 1) = R + [R, x]_{n-1} = x$ since $x \leq R + n - 2$. Therefore $n(\oplus x) = x$, and thus $n(\oplus x) \,|_1^n x$. When $x < R$, clearly $x < R \leq n(\oplus x)$. Moreover, $n(\oplus x) \equiv nx \equiv x \ (mod\, n - 1)$. Therefore $n(\oplus x) \,|_1^n x$ holds also in this case.

For each $m > 0$, define $\mathbf{N_m^*}$ to be the direct product of $m\mathbf{N}^*$s. Similarly to Definition 5.5, we can extend both $|_1^n$ and $|_n^1$ to $\mathbf{N_m^*}$. Also define $\oplus_{n-1}$ on $\mathbf{N_m^*}$ by

$$\vec{v} \oplus_{n-1} \vec{w} = \langle v_1 \oplus_{n-1} w_1, \ldots, v_m \oplus_{n-1} w_m \rangle,$$

for $\vec{v} = \langle v_1, \ldots, v_m \rangle$ and $\vec{w} = \langle w_1, \ldots, w_m \rangle$. It is easy to see that (1) $\mathbf{N_m^*} = \langle N_m^*, \oplus_{n-1}, \vec{0} \rangle$ is a commutative monoid, (2) both $\langle \mathbf{N_m^*}, |_1^n \rangle$ and $\langle \mathbf{N_m^*}, |_n^1 \rangle$ form **BCI**-structures, and (3) in fact, $\langle \mathbf{N_m^*}, |_1^n \rangle$ is a $\mathbf{BCI_1^n}$-structure. The following lemma follows from Lemma 5.8.

Lemma 5.10 *Let $\vec{x}_1, \ldots, \vec{x}_m, \vec{k} \in N_m^*$ and moreover each component of $\vec{k}$ is equal to or less than R. Then, (1) $\vec{x}_1 + \cdots + \vec{x}_m \,|_1^n \vec{x}_1 \oplus \cdots \oplus \vec{x}_m$, and (2) if $\vec{x}_1 + \cdots + \vec{x}_m \,|_1^n \vec{k}$ then $\vec{x}_1 \oplus \cdots \oplus \vec{x}_m \,|_1^n \vec{k}$.*

Now we will show the finite model property of both $\mathbf{BCI_1^n}$ and $\mathbf{BCI_n^1}$. The proof can be carried out similarly to the proof in [11]. It is necessary here to add some arguments which use properties of $\oplus_{n-1}$.

Definition 5.11 Suppose that $\langle \mathbf{N_m}, \leq \rangle$ is a **BCI**-structure with a partial order $\leq$ on $\mathbf{N_m}$. Then for any valuation $\models$ on the **BCI**-structure $\langle \mathbf{N_m}, \leq \rangle$ and any formula B, an element $\vec{v}$ of $\mathbf{N_m}$ is $B^{(-)}$-critical in $\langle \mathbf{N_m}, \leq, \models \rangle$ if

1. $\vec{v} \not\models B$,
2. if $\vec{v} < \vec{w}$ (i.e., $\vec{v} \le \vec{w}$ but $\vec{v} \neq \vec{w}$) then $\vec{w} \models B$, and is $B^{(+)}$-critical in $\langle \mathbf{N_m}, \le, \models \rangle$ if
3. $\vec{v} \models B$,
4. if $\vec{w} < \vec{v}$ then $\vec{w} \not\models B$.

For each formula B and each **BCI**-model $\langle \mathbf{N_m}, \le, \models \rangle$ with a partial order $\le$, define

$$R^*(B;\le,\models) = \{\vec{v} : \vec{v} \text{ is } B^{(*)}\text{-critical in } \langle \mathbf{N_m}, \le, \models \rangle\},$$

where $*$ is either $-$ or $+$. Then it is clear that both $R^-(B;\le,\models)$ and $R^+(B;\le,\models)$ are antichains in the partially ordered set $\langle N_m, \le \rangle$, i.e., $\vec{v} < \vec{u}$ never holds for any $\vec{v}$ and $\vec{u}$ in $R^*(B;\le,\models)$. Now we can show the following.

Lemma 5.12 *There exist no infinite antichains in the partially ordered sets* $\langle N_m, |_1^n \rangle$ *and* $\langle N_m, |_n^1 \rangle$.

Proof: This lemma is an easy consequence of the following result with the fact that both $\langle N_m, |_1^n \rangle$ and $\langle N_m, |_n^1 \rangle$ are well-ordered partially ordered sets (see [11]).

Theorem 5.13 *The property of not possessing infinite antichains is preserved by any finite product of well-founded partially ordered sets.*

Now we are ready to prove the finite model property of $\mathbf{BCI_1^n}$ and $\mathbf{BCI_n^1}$. First we will consider $\mathbf{BCI_1^n}$. It is enough to show that if a formula F is not provable in $\mathbf{BCI_1^n}$, i.e., the sequent $\rightarrow F$ is not provable in it, then it is not true in some finite $\mathbf{BCI_1^n}$-model.

By Theorem 5.6, if F is not provable in $\mathbf{BCI_1^n}$ then F is not true in a $\mathbf{BCI_1^n}$-model $\langle \mathbf{N_m}, |_1^n, \models \rangle$. Let Ψ be the set of all subformulas of F, and R_F be $\bigcup_{B \in \Psi} R^-(B; |_1^n, \models)$. Since Ψ is finite and $|_1^n$ is a partial order on N_m, R_F is also finite by Lemma 5.1. Define $R = max\{1, max\{a_i : \langle a_1, \ldots, a_m \rangle \in R_F\}\}$. Let $N^* = \{0, 1, \ldots, R + n - 2\}$ for this R. Then $\langle \mathbf{N_m^*}, |_1^n \rangle$ is a $\mathbf{BCI_1^n}$-structure as shown before. Now define a valuation $\models^*$ on $\langle \mathbf{N_m^*}, |_1^n \rangle$ by the condition that for each propositional variable p in Ψ and each $\vec{v} \in N_m^*$,

(14) $\vec{v} \models^* p$ if and only if $\vec{v} \models p$.

We will show by induction that for each formula B in Ψ and each $\vec{v} \in N_m^*$,

(15) $\vec{v} \models^* B$ if and only if $\vec{v} \models B$.

Suppose that $B = C \supset D$, $\vec{v} \models C \supset D$ and $\vec{w} \models^* C$ for $\vec{v}, \vec{w} \in N_m^*$. Then, by the hypothesis of induction, $\vec{w} \models C$ and hence $\vec{v} + \vec{w} \models D$. Since $\vec{v} + \vec{w} |_1^n \vec{v} \oplus \vec{w}$, we have $\vec{v} \oplus \vec{w} \models D$. Hence $\vec{v} \oplus \vec{w} \models^* D$ by the hypothesis of induction. Thus $\vec{v} \models^* B$.

Conversely, suppose that $\vec{v} \not\models C \supset D$ for $\vec{v} \in N_m^*$. Then for some $\vec{w} \in N_m$, $\vec{w} \models C$ but $\vec{v} + \vec{w} \not\models D$. So, there must exist $\vec{u}$ such that $\vec{v} + \vec{w} |_1^n \vec{u}$ and $\vec{u}$ is $D^{(-)}$-critical in $\langle \mathbf{N^m}, |_1^n, \models \rangle$. We can find such a $D^{(-)}$-critical element $\vec{u}$, since there exist no infinite ascending chains in $\langle N^m, |_1^n \rangle$. Since each component of $\vec{u}$ is not greater than R, $\vec{v} \oplus \vec{w} |_1^n \vec{u}$ by Lemma 5.10. Therefore $\vec{v} \oplus \vec{w} \not\models D$ and hence $\vec{v} \oplus \vec{w} \not\models^* D$ by the hypothesis of induction. But this $\vec{w}$ may not belong to N_m^*, and hence we cannot conclude here that $\vec{w} \models^* C$ in general. So define $z_i = R +$

$[R,w_i]_{n-1}$ for each $i = 1,2,\ldots,m$ and $\vec{z} = \langle z_1,\ldots,z_m\rangle$, when $\vec{w} = \langle w_1,\ldots,w_m\rangle$. Then $\vec{z} \in N^*_m$ and $\vec{w}\,|^n_1\,\vec{z}$ by (11). Thus $\vec{z} \models C$ and hence $\vec{z} \models^* C$ by the hypothesis of induction. Moreover, for each i

$$\begin{aligned} v_i \oplus z_i &= R + [R,v_i + z_i]_{n-1} \\ &= R + [R,v_i + R + [R,w_i]_{n-1}]_{n-1} \\ &= R + [R,v_i + w_i]_{n-1} \\ &= v_i \oplus w_i. \end{aligned}$$

Therefore, $\vec{v} \oplus \vec{z} = \vec{v} \oplus \vec{w} \not\models^* D$. Combining this with $\vec{z} \models^* C$, we have $\vec{v} \not\models^* B$.

Theorem 5.14 *The logic* $\mathbf{BCI^n_1}$ *has the finite model property for* $n \geq 2$.

The finite model property of $\mathbf{BCI^1_n}$ can be shown almost in the same way as the above. Suppose that a formula F is not provable in $\mathbf{BCI^1_n}$. Then by Theorem 5.6, it is not true in a $\mathbf{BCI^1_n}$-model $\langle \mathbf{N_m}, |^1_n, \models\rangle$. For the set Ψ of all subformulas of F, define R_F to be the set $\bigcup_{B\in\Psi} R^+(B; |^1_n, \models)$, this time. Define R and N^* in the same as before. Then, $\langle \mathbf{N^*_m}, |^1_n\rangle$ becomes a **BCI**-structure. Define a valuation $\models^*$ on $\langle \mathbf{N^*_m}, |^1_n\rangle$ also by (14). In this case, it is necessary to check that $\models^*$ satisfies that

(16) $\vec{v}_1 \oplus \vec{u} \models^* p,\ldots,\vec{v}_n \oplus \vec{u} \models^* p$ implies $\vec{v}_1 \oplus \cdots \oplus \vec{v}_n \oplus \vec{u} \models^* p.$

Now suppose that $\vec{v}_i \oplus \vec{u} \models^* p$ for each i, or equivalently, $\vec{v}_i \oplus \vec{u} \models p$ for each i. By Lemma 5.10, $\vec{v}_i \oplus \vec{u}\,|^1_n\,\vec{v}_i + \vec{u}$ and hence $\vec{v}_i + \vec{u} \models p$. (Recall that $|^1_n$ is the inverse relation of $|^n_1$.) Since $\langle \mathbf{N^*_m}, |^1_n, \models\rangle$ is a $\mathbf{BCI^1_n}$-model, $\vec{v}_1 + \cdots + \vec{v}_n + \vec{u} \models p$ follows from them. Let $\vec{z}$ be $p^{(+)}$-critical element such that $\vec{z}\,|^1_n\,\vec{v}_1 + \cdots + \vec{v}_n + \vec{u}$. In fact, there exists such $\vec{z}$ since there exist no infinite descending chains in $\langle N_m, |^1_n\rangle$. Moreover, since each component of $\vec{z}$ is not greater than R, $\vec{z}\,|^1_n\,\vec{v}_1 \oplus \cdots \oplus \vec{v}_n \oplus \vec{u}$ by Lemma 5.10. Thus, $\vec{v}_1 \oplus \cdots \oplus \vec{v}_n \oplus \vec{u} \models p$ and hence $\vec{v}_1 \oplus \cdots \oplus \vec{v}_n \oplus \vec{u} \models^* p$. The rest of the proof can be carried out similarly to that of Theorem 5.14.

Theorem 5.15 *The logic* $\mathbf{BCI^1_n}$ *has the finite model property for* $n \geq 2$.

In [11], the finite model property for the logic $\mathbf{RMO_\to}$ is proved. $\mathbf{RMO_\to}$ is the logic obtained from **BCIW** by adding the ($1 \rightsquigarrow 2$) rule, and hence it is just $\mathbf{BCI^{12}_{21}}$ by our notation. Slightly modifying the proof, we can extend it to the proof of the finite model property of $\mathbf{BCI^{1n}_{n1}}$ for each $n > 1$, as shown in the next theorem.

Define a binary relation $\sim_n$ on the set N_m by the condition that for each $\vec{x} = \langle x_1,\ldots,x_m\rangle$ and $\vec{y} = \langle y_1,\ldots,y_m\rangle$,

> $\vec{x} \sim_n \vec{y}$ if and only if for each $i \leq m$, either (1) $x_i = y_i$ or (2) $1 \leq x_i, y_i$ and $x_i \equiv y_i\ (mod\, n-1)$.

Then, it is easy to show that $\sim_n$ is an equivalence relation on N_m which is compatible with $+$. Similarly to Theorem 5.6, we can show the following.

Theorem 5.16 *For any sequent* $\Gamma \to A$, $\Gamma \to A$ *is provable in* $\mathbf{BCI^{1n}_{n1}}$ *if and only if it is true in any* $\mathbf{BCI^{1n}_{n1}}$*-model* $\langle \mathbf{N_m}, \sim_n, \models\rangle$.

We can show that the quotient set $\mathbf{N_m}/\sim_n$ of $\mathbf{N_m}$ modulo $\sim_n$ also forms a commutative monoid, and in fact $\langle \mathbf{N_m}/\sim_n, = \rangle$ is a finite $\mathbf{BCI_1^n}$-structure with n^m elements. So, similarly to [11], we have the following.

Theorem 5.17 *The logic* $\mathbf{BCI_{n1}^{1n}}$ *has the finite model property for* $n \geq 2$. *More precisely, for any sequent* $\Gamma \to A$, $\Gamma \to A$ *is provable in* $\mathbf{BCI_{n1}^{1n}}$ *if and only if it is true in any finite* $\mathbf{BCI_{n1}^{1n}}$-model $\langle \mathbf{N_m}/\sim_n, =, \models \rangle$.

We have shown in this section that both $\mathbf{BCI_1^n}$ and $\mathbf{BCI_n^1}$ have the finite model property for each $n \geq 2$. On the other hand, we know nothing about $\mathbf{BCI_k^n}$ and $\mathbf{BCI_n^k}$ when $k > 1$. We did not even succeed in formulating completeness theorems like Theorem 5.6 for them. These difficulties seem to be strongly related to difficulties which we met in finding cut-free systems. So it will be interesting to find intrinsic relations between them.

Acknowledgments Some of our results have been announced already in preliminary reports by Schellinx [18] and Hori [5]. Discussions on them led us to a joint paper of the present form, in which most of the results in these reports are covered and extended. The authors would like to express their gratitude to K. Matsumoto for providing useful information on the mingle rule and to K. Došen and J. M. Font for their helpful comments.

REFERENCES

[1] Došen, K. "Addenda and corrigenda to 'Sequent-Systems and Groupoid Models,'" *Studia Logica*, vol. 49 (1990), p. 614.

[2] Dunn, J. M., "Relevance logic and entailment," pp. 117–224 in *Handbook of Philosophical Logic, Vol. III*, edited by D. Gabbay and F. Guenthner, D. Reidel, Dordrecht, 1986.

[3] Gentzen, G., "Untersuchungen über das logische Schliessen," *Mathematische Zeitschrift*, vol. 39 (1934–35), pp. 176–210 and 405–431.

[4] Girard, J.-Y., "Linear logic," *Theoretical Computer Science*, vol. 50 (1987), pp. 1–102.

[5] Hori, R., "Extensions of the implicational linear logic (1), (2)," Technical Report Nos. 93-C1, 93-C2, Department of Applied Mathematics, Faculty of Engineering, Hiroshima University, 1993.

[6] Kiriyama, E., and H. Ono, "The contraction rule and decision problems for logics without structural rules," *Studia Logica*, vol. 50 (1991), pp. 299–319.

[7] Komori, Y., "Predicate logics without the structure rules," *Studia Logica*, vol. 45 (1986), pp. 393–404.

[8] Kripke, S., "The problem of entailment," abstract, *Journal of Symbolic Logic*, vol. 24 (1959), p. 324.

[9] Meyer, R. K., *Topics in modal and many-valued logic*, Ph.D. dissertation, University of Pittsburgh, 1966.

[10] Meyer, R. K., *Improved decision procedures for pure relevant logic* (unpublished manuscript), 1973.

[11] Meyer, R. K., and H. Ono, "The finite model property for BCK and BCIW," *Studia Logica*, vol. 53 (1994), pp. 107–118.

[12] Ohnishi, M., and K. Matsumoto, "A system for strict implication," *Annals of the Japan Association for Philosophy of Science*, vol. 2 (1964), pp. 183–188.

[13] Ono, H., "Structural rules and a logical hierarcy," pp. 95–104 in *Mathematical Logic*, edited by P. P. Petkov, Plenum Press, 1990.

[14] Ono, H., "Semantics for substructural logics," pp. 259–291 in *Substructural Logics*, edited by K. Došen and P. Schroeder-Heister, Oxford University Press, Oxford, 1993.

[15] Ono, H., and Y. Komori, "Logics without the contraction rule," *Journal of Symbolic Logic*, vol. 50 (1985), pp. 169–201.

[16] Palasiński, M. and A. Wroński, "Eight simple questions concerning BCK-algebras," *Reports on Mathematical Logic*, vol. 20 (1986), pp. 87–91.

[17] Prijatelj, A., *Bounded contraction and many-valued semantics*, ILLC Prepublication Series for Mathematical Logic and Foundations, ML-93-04, University of Amsterdam, 1993.

[18] Schellinx, H., *How to broaden your horizon*, ILLC Prepublication Series for Mathematical Logic and Foundations ML-92-07, University of Amsterdam, 1992.

[19] Troelstra, A. S., *Lectures on Linear Logic*, CSLI Lecture Notes, No. 29, Stanford University, 1992.

Hori, Ryuichi
Matsusita Electric Information Science Laboratory of Hiroshima
12-20, Hikarimachi 1-chome, Higashi-ku
Hiroshima, 732 Japan

Ono, Hiroakira
School of Information Science
Japan Advanced Institute of Science and Technology
Tatsunokuchi, Ishikawa, 923-12 Japan

Schellinx, Harold
Faculteit Wiskunde en Informatica
Universiteit van Amsterdam
Plantage Muidergracht 24, 1018 TV
Amsterdam, The Netherlands