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# A Minimal Predicative Set Theory

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**Abstract** The central idea of this paper is to perform Nelson's program starting from an extremely weak set theory instead of Robinson's Q. Our theory, which is called N after Nelson, has two non-logical axioms; one asserts the existence of an empty set, the other one asserts that, given two sets $x$ and $y$, we can form the union of $x$ and the singleton of $y$. A strictly finitistic proof of the Herbrand consistency of N is given. Moreover, it is shown that Q, and therefore Nelson's Q*, is interpretable in N. Thus Q* is proved by strictly finitistic means to be consistent relative to N.

***0 Introduction*** It is well known that traditional mathematics can be done by means of set theory. However, the usual set theories, like ZF or GB, can be criticized from many points of view: constructivists can complain that many axioms guarantee the existence of sets for which no construction is produced; finitists disagree on the existence of infinite sets, and formalists can doubt about consistency, because of Gödel's Second Incompleteness Theorem. In this paper we present a theory, suggested to us by Edward Nelson, and called N after him, that is probably the weakest theory in which some mathematics can be done; this theory is so weak that its axioms should be accepted without problems by any logician, but it is also so weak that mathematics that can be done directly inside it is very poor. N has only two nonlogical axioms: the first one guarantees the existence of a set without elements, and the second one allows us to add to an already existing set $x$ the singleton of another already existing set $y$. In N, we can construct all hereditarily finite sets, but we cannot prove, e.g., that $x \cup \{y\} = (x \cup \{y\}) \cup \{y\}$. The theory N has been studied in a different context by other authors (cf. Bellé and Parlamento [1] and Omodeo, Parlamento, and Policriti [3]), and is called NW by them.

Even if N is so weak, we can try to realize Nelson's Program starting from this theory; in other words, we can try to produce a finitistic consistency proof for N, and then to interpret in it a mathematically strong theory; this is reasonable from the point of view of Hilbert's Program, because if a theory T is interpretable in N, then there is a finitistic proof of the consistency of T relative to the consistency of N. This kind of work has already been done by Nelson in [2],

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but starting from Robinson's theory Q. Nelson interprets in Q a rather strong theory, called Q* by him; moreover, Nelson gives a proof of restricted consistency of Q that is not only finitistic according to all traditional criteria, but can be also formalized in predicative arithmetic, and is therefore finitistic according to Nelson's point of view. The advantage of the theory N is that first it is even simpler than Q, and second it is a set theory, and therefore it is perhaps more natural for the foundations of mathematics. However, we have not been able to develop Nelson's Program in N in an independent way; therefore, we first interpret Q in N, and then we borrow from Nelson's results, getting an interpretation of Q* in N. In his [5], Tarski shows that N plus extensionality is interpretable in Q; however, N plus extensionality is not only stronger, but also slightly more problematic from the point of view of the consistency, because it is not an open theory. (Of course, it can be transformed into an open theory by means of some Skolem functions, but this is completely unnatural.) We will not use Tarski's result here, because it is not easy to interpret extensionality in N (it is possible a posteriori by our result, because it is easily seen that N plus extensionality is interpretable in Q). Last, we shall prove that if we drop any of the two nonlogical axioms of N, or if we weaken the second one, we get a theory that is still undecidable, but not essentially indecidable (therefore Q is not interpretable in it). So, N is in a sense a minimal set theory in which Q is interpretable.

As regards to the consistency problem, we shall prove the Herbrand consistency of N; this proof is, we believe, finitistic also in Nelson's sense, and can be formalized in predicative arithmetic; when doing it, we produce an interesting model of N, which we call "canonical", because all atomic formulas valid in it are provable in N. In order to get a full consistency proof for N, we need Hilbert-Ackermann's Theorem. However, this theorem seems to be more problematic from the point of view of Nelson's Program: its proof requires the totality of superexponential; therefore, its proof cannot be formalized in predicative arithmetic (cf. Remark 1.3).

Some months after finishing this paper, we learned that interpretability of N in Q was proved first by Jan Krajicek. Since Krajicek's proof is unpublished and our proof is completely independent from his, we present it here, but we acknowledge Krajicek's priority about this result.

***1 The theory N and a consistency proof for it*** The language of N consists of one constant symbol, $\phi$, for the empty set, two binary predicate symbols, $\in$ (for membership) and $=$ (for equality), and a binary function symbol $\cup$ representing the function that associates to $x$ and $y$ the union of $x$ and the singleton of $y$. In the following, we write $y \cup \{z\}$ for $U(x,y)$. The words "the empty set" and "the union of $x$ and the singleton of $y$" have been used somehow improperly, because, in absence of the axiom of extensionality, we cannot prove that there is only one empty set, or that the union of $x$ and the singleton of $y$ is uniquely determined by $x$ and $y$. The axioms of N are logical axioms (including identity and equality axioms) plus the following ones:

1. $\forall x(\neg x \in \phi)$
2. $\forall x \forall y \forall z(x \in y \cup \{z\} \leftrightarrow x \in y \vee x = z)$.

In this section we present a finitistic restricted consistency proof for N.

**Definition 1** A finitely axiomatized theory T is said to be *Herbrand consistent* iff no disjunction of substitution instances of Herbrand matrices of negations of axioms of T (including equality and identity axioms) is a tautology.

By Herbrand's Theorem, a finitely axiomatized theory is Herbrand consistent iff it is consistent. If a theory is axiomatized by universal formulas (as in the case of N), then the Herbrand matrix of the negation of any axiom of T coincides with its matrix (i.e., the quantifier free part of it that follows the stack of existential quantifiers in the prenex normal form of the negation of the axiom taken into consideration). Moreover, we do not need the full Herbrand Theorem in order to prove that Herbrand consistency of T is equivalent to consistency of T (Hilbert Ackermann's Theorem, cf. Schönfield [4], is enough). So, in order to prove Herbrand consistency of N, it suffices to prove that there is no tautology which is a disjunction of negations of substitution instances of the following formulas: $x = x$; $x = y \to y = x$; $x = y \wedge y = z \to x = z$; $(x = y \wedge u = v) \to (x \in u \to y \in v)$; $(x = y \wedge u = v) \to x \cup \{u\} = y \cup \{v\}$; $\neg x \in \phi$; $(x \in y \cup \{z\} \leftrightarrow x \in y \vee x = z)$.

To this purpose, we first define a map $\Phi$ from atomic formulas of N into $\{0,1\}$, as follows: let $t, t'$ be arbitrary terms; we define $\Phi(t = t')$ to be 1 if $t$ and $t'$ are syntactically equal, and 0 otherwise; note that, if $\Phi(t = t')$ is 1, then $N \vdash t = t'$. In order to define $\Phi(t \in t')$, we start from the following observation: for any term $t$, there is a unique sequence $t_0, t_1, \ldots, t_{2n}$ such that $t_{2n}$ is $t$, $t_0$ is either $\phi$ or a variable, and either $n = 0$ or, for all $i < n$, $t_{2i+2}$ is $t_{2i} \cup \{t_{2i+1}\}$. We call this sequence "the constructing sequence of $t$". Now, let $t_0, t_1, \ldots, t_{2n}$ be the constructing sequence of $t'$; we define $\Phi(t \in t')$ to be 1 if $n \neq 0$ and there is an $i < n$ such that $t$ is syntactically equal to $t_{2i+1}$. Note that, if $\Phi(t \in t')$ is 1, then $\mathrm{N} \vdash t \in t'$. Thus, our interpretation has the property that each atomic formula that is true according to it is provable in N. Now, we extend $\Phi$ to all Boolean combinations of atomic formulas according to truth tables for classical propositional logic (thus, e.g., $\Phi(A \wedge B)$ is 1 iff both $\Phi(A)$ and $\Phi(B)$ are 1, $\Phi(\neg A)$ is 1 iff $\Phi(A)$ is 0 and so on). It is clear that each theorem of propositional calculus receives value 1; we prove that, if $\varphi$ is a substitution instance of any of the formulas $x = x$; $x = y \to y = x$; $x = y \wedge y = z \to x = z$; $(x = y \wedge u = v) \to (x \in u \to y \in v)$, $(x = y \wedge u = v) \to x \cup \{u\} = y \cup \{v\}$, $\neg x \in \phi$, $s \in t \cup \{u\} \leftrightarrow s \in t \vee s = u$, then $\Phi(\varphi)$ is 1; the claim is immediate if $\varphi$ is a substitution instance of any of the formulas $x = x$; $x = y \to y = x$; $x = y \wedge y = z \to x = z$; $(x = y \wedge u = v) \to (x \in u \to y \in v)$, $(x = y \wedge u = v) \to x \cup \{u\} = y \cup \{v\}$; let $\varphi$ be a substitution instance of $\neg x \in \phi$; since the constructing sequence of $\phi$ consists of only one element, $\Phi(t \in \phi)$ is 0 for any term $t$, therefore, $\Phi(\varphi)$ is 1; last, for any triple $s, t, u$ of terms $\Phi(s \in t \cup \{u\} \leftrightarrow s \in t \vee s = u)$ is 1; to see this, it is enough to prove that $\Phi(s \in t \cup \{u\})$ is 1 iff either $\Phi(s \in t)$ is 1 or $s$ is syntactically equal to $u$; to do this, simply note that the constructing sequence of $t \cup \{u\}$ consists of the constructing sequence of $t$ followed by $u$ and $t \cup \{u\}$; thus, the terms of the constructing sequence of $t \cup \{u\}$ having an odd index are those of the constructing sequence of $t$ having an odd index plus $u$, and the claim follows from the definition of $\Phi(s \in t \cup \{u\})$. Note that incidentally we have proved that our interpretation constitutes a model of N in which only provable atomic formulas are true. Our proof of Herbrand consistency of N follows from the simple observation that $\Phi$ maps every instance $\varphi$ of an axiom into 1, and that the property "being mapped into 1 by $\Phi$" is preserved under propositional deduction. Thus, we have shown:

**Theorem 1.2** *N is Herbrand consistent.*

**Remark 1.3** Even if we did not check details, it should not be difficult to formalize our Herbrand consistency proof in predicative arithmetic; the map $\Phi$ we were speaking about is $\Delta_0$ definable (here, $\Delta_0$ refers to a language with sum, product, 0, 1, $\leq$, and the smash function as primitive symbols), the proof that each substitution instance of an axiom receives value 1 can be done by $\Delta_0$ induction on the size of the terms occurring in the formula; that any disjunction of negations of substitution instances of axioms of N is mapped into 0 by $\Phi$ is proved by $\Delta_0$ induction on the number of disjuncts; moreover, we do not need the totality of rapidly growing functions in our proof. However, the reduction of consistency to Herbrand consistency in the usual sense requires the use of Hilbert Ackermann's Theorem, whose proof requires in general the totality of the superexponential function; thus, the above argument does not show that a full consistency proof for N can be formalized in predicative arithmetic. In fact, full consistency of N cannot be proved even in $I\Delta_0 + \text{Exp}$, because otherwise, by our interpretation of Q in N, $I\Delta_0 + \text{Exp}$ would prove the consistency of Q, which is impossible (cf. Wilkie and Paris [6]).

***2 An interpretation of Q into N*** Our plan in order to interpret Q in N is the following: we first restrict our universe to a subuniverse in which one can perform some very basic set theoretical operations; then, we construct the class of natural numbers (i.e., we produce a formula that is satisfied by sets having most of the properties of natural numbers). Since our theory is very weak, we cannot define sum and product directly in our natural numbers; however, we can restrict the class of natural numbers to subclasses in which sum and product are well-defined total operations and satisfy the axioms of Q.

**Notation** (a) $x \approx y : \forall v(v \in x \leftrightarrow v \in y)$ and (b) $z \approx x \cup y : \forall v(v \in z \leftrightarrow v \in x \vee v \in y)$.

First of all, we restrict the universe $V$ of N to a subuniverse $V_1$ closed under the "operation" $\cup$. The name "operation" is between " ", since there can be many $z$ such that $z \approx x \cup y$ (because of the absence of the axiom of extensionality). Since if $x \approx x'$, $y \approx y'$, $z \approx x \cup y$, and $z' \approx x' \cup y'$ then $z \approx z'$, we can consider $\cup$ as a partial function modulo $\approx$; we write, e.g., $x \cup y \approx x' \cup y'$ instead of $\exists z \exists z'[z \approx x \cup y \wedge z' \approx x \cup y \wedge z \approx z']$, as well as $w \approx (x \cup y) \cup z$ instead of $\exists u[u \approx x \cup y \wedge w \approx u \cup z]$ and finally $w \in (x \cup y) \cup z$ instead of $\exists u \exists u'[u \approx x \cup y \wedge u' \approx u \cup z \wedge w \in u']$; note that even if there can be many sets $w$ such that $w \approx x \cup y$ or $w \approx (x \cup y) \cup z$, all such sets have the same extension.

**Definition 2.1** In the following, $V_1(x)$ denotes the formula $\forall y \exists z[z \approx x \cup y]$.

In the sequel, we write $x \in V_1$ instead of $V_1(x)$ because we like to think of $V_1(x)$ as of the universe of all $x$ such that $V_1(x)$. Of course, this does not mean that such universe is a set. According to an old tradition in set theory, we shall adopt this abuse of notation also in the following, and we shall prove several theorems of N in the metalanguage. We start from the following properties:

**Proposition 2.1** $(V_1 1)\ \phi \in V_1$.

*Proof:* Obvious, since $y \approx \phi \cup y$.

**Proposition 2.2** $(V_1 2)\ x, y \in V_1 \rightarrow \exists v \in V_1[v \approx x \cup y]$.

*Proof:* Since $x \in V_1$, there is a $v$ such that $v = x \cup y$. We prove that $v \in V_1$. Let $h$ be any set; since $y \in V_1$, there is a $z$ such that $z \approx y \cup h$; since $x \in V_1$, there is a $z'$ such that $z' \approx x \cup z$. So $z' \approx x \cup (y \cup h)$, $z' \approx (x \cup y) \cup h \approx v \cup h$. By the arbitrariness of $h$, we conclude $v \in V_1$.

**Proposition 2.3** $(V_1 3)\ x, y \in V_1 \rightarrow x \cup \{y\} \in V_1$.

*Proof:* Let $h$ be any set; since $x \in V_1$, there is a $z'$ such that $z' \approx x \cup h$; now, consider $z' \cup \{y\}$. It is easily seen that $z' \cup \{y\} \approx (x \cup \{y\}) \cup h$.

**Notation** (a) $z \approx x \cap y : \forall v(v \in z \leftrightarrow v \in x \wedge v \in y)$ and (b) $v \in x \cap y : v \in x \wedge v \in y$.

We restrict again our universe $V_1$ to a subuniverse $V_2$ of it which is closed also under $\cap$.

**Definition 2.4** $V_2 = \{x \in V_1 : \forall y \in V_1\ \exists z \in V_1[z \approx x \cap y]\}$.

We prove the following properties:

**Proposition 2.5** $(V_2 1)\ \phi \in V_2$.

*Proof:* Obvious, since, for all $y \in V_2$, $\phi \approx \phi \cap y$, and $\phi \in V_1$.

**Proposition 2.6** $(V_2 2)\ x, y \in V_2 \rightarrow \exists v \in V_2[v \approx x \cup y]$.

*Proof:* Since $x, y \in V_2$, there is a $v \in V_1$ such that $v \approx x \cup y$. We prove that any such $v$ belongs to $V_2$. Let $u \in V_1$; since $x \in V_2$, there is a $z' \in V_1$ such that $z' \approx x \cap u$, and since $y \in V_2$, there is a $z'' \in V_1$ such that $z'' \approx y \cap u$. Furthermore, from $z' \in V_1$ and $z'' \in V_1$ it follows that there is a $z''' \in V_1$ such that $z''' \approx z' \cup z''$; so $z''' \approx (x \cap u) \cup (y \cap u) \approx (x \cup y) \cap u \approx v \cap u$.

**Proposition 2.7** $(V_2 3)\ x, y \in V_2 \rightarrow \exists v \in V_2[v \approx x \cap y]$.

*Proof:* Since $x \in V_2$, there is a $v \in V_1$ such that $v \approx x \cap y$. We prove that $v \in V_2$. Let $h \in V_1$; since $y \in V_2$, there is a $z' \in V_1$ such that $z' \approx y \cap h$; since $x \in V_2$, there is a $z \in V_1$ such that $z \approx x \cap z'$. So $z \in V_1$ and $z \approx x \cap (y \cap h) \approx (x \cap y) \cap h \approx v \cap h$, therefore $v \in V_2$.

**Proposition 2.8** $(V_2 4)\ x, y \in V_2 \rightarrow x \cup \{y\} \in V_2$.

*Proof:* We have already seen that $x \cup \{y\} \in V_1$. We prove that $x \cup \{y\} \in V_2$. Let $h \in V_1$; since $x \in V_2$, there is a $z' \in V_1$ such that $z' \approx x \cap h$; if $y \notin h$, it is easily seen that $z' \approx (x \cup \{y\}) \cap h$; if $y \in h$, it is easily seen that $z' \cup \{y\} \approx (x \cup \{y\}) \cap h$; moreover, $z' \cup \{y\} \in V_1$, by Theorem $V_1 3$).

**Notation** (a) $z \approx x - y : \forall v(v \in z \leftrightarrow v \in x \wedge v \notin y)$ and (b) $y \subset x : \forall v(v \in y \rightarrow v \in x)$.

**Definition 2.9** $V_3 = \{x \in V_2 : \forall y \in V_2[y \subset x \rightarrow \exists z \in V_2[z \approx x - y]\}$.

We prove the following properties:

**Proposition 2.10** $(V_3\,1)\ \phi \in V_3$.

*Proof:* Of course, $\phi \in V_2$. For all $y \in V_2$, if $y \subset x$, then $\phi \approx \phi - y$ and $\phi \in V_2$; so $\phi \in V_3$.

**Proposition 2.11** $(V_3\,2)\ x, y \in V_3 \rightarrow \exists z \in V_3[z \approx x \cup y]$.

*Proof:* We have already seen that there is a $z \in V_2$ such that $z \approx x \cup y$. We prove that $z \in V_3$. Let $h \in V_2$ be such that $h \subset x \cup y$; let $h', h'' \in V_2$ such that $h' \approx x \cap h$ and $h'' \approx y \cap h$; since $h' \subset x$ and $h'' \subset y$, there are $z', z'' \in V_2$ such that $z' \approx x - h'$ and $z'' \approx y - h''$, so there is a $z \in V_2$ such that $z \approx z' \cup z''$. It is easily seen that $z \approx (x \cup y) - h$.

**Proposition 2.12** $(V_3\,3)\ x, y \in V_3 \rightarrow x \cup \{y\} \in V_3$.

*Proof:* We have already seen that $x \cup \{y\} \in V_2$. We prove that $x \cup \{y\} \in V_3$. Let $h \in V_2$, $h \subset x \cup \{y\}$; if $y \notin h$, then $h \subset x$, and by our assumption there is a $z' \in V_2$ such that $z' \approx x - h$. So, $z' \cup \{y\} \in V_2$ and is easily seen that $z' \cup \{y\} \approx x \cup \{y\} - h$; if $y \in h$ then, by our assumption on $x$, there is a $z' \in V_2$ such that $z' \approx x - h$. But then $z' \in V_2$, $z' \approx x - h \approx (x \cup \{y\}) - h$. So, $x \cup \{y\} \in V_3$.

**Proposition 2.13** $(V_3\,4)\ x, y \in V_3 \wedge y \subset x \rightarrow \exists z \in V_3[z \approx x - y]$.

*Proof:* Since $x \in V_3$ and $y \in V_2$, there is a $z \in V_2$ such that $z \approx x - y$. We prove that $z \in V_3$: if $h \in V_2$ and $h \subset z$, since $h \subset x$, there are a $z' \in V_2$ such that $z' \approx x - h$ and a $z'' \in V_2$ such that $z'' \approx z \cap z'$. It is easily seen that $z'' \approx (x - y) \cap (x - h) \approx (x - y) - h \approx z - h$; so $z \in V_3$.

**Proposition 2.14** $(V_3\,5)\ x, y \in V_3 \rightarrow \exists z \in V_3[z \approx x \cap y]$.

*Proof:* Since $x \in V_2$ and $y \in V_2$, there is a $z \in V_2$ such that $z \approx x \cap y$. We prove that $z \in V_3$: if $h \in V_2$ and $h \subset x \cap y$, then $h \subset x$ and $h \subset y$, hence there are $z, z' \in V_2$ such that $z \approx x - h$ and $z' \approx y - h$. From the properties of $V_2$ it follows that there is a $v \in V_2$ such that $v \approx z \cap z'$; it is easily seen that $v \approx z \cap z' \approx (x \cap y) - h \approx z - h$; so, $z \in V_3$.

Now, we try to define the class N of natural numbers in N.

**Notation**

a. Trans$(x)$ denotes the following formula:

$$\forall u, v \in V_3(u \in v \wedge v \in x \rightarrow u \in x).$$

b. "$a$ is well ordered by $\in$" denotes the conjunction of the following formulas: (i) $\forall x, y \in a(y \in x \vee x \in y \vee x = y)$; (ii) $\forall v \in V_3[(v \subset a \wedge \exists u(u \in v)) \rightarrow \exists v' \in v(\forall v'' \in v \neg(v'' \in v'))]$.

**Definition 2.15** $N = \{x \in V_3$: (i) Trans$(x)$; (ii) $\forall u \in x\,(\text{Trans}(u) \wedge u \in V_3)$; (iii) "$x$ is well ordered by $\in$"; (iv) $\forall u(u = x \vee u \in x \rightarrow (u = \phi \vee \exists v[u = v \cup \{v\}]))\}$.

Thus, N is the class of all sets that are hereditarily transitive and in $V_3$, are well ordered by $\in$, and which are either empty or the "successor" of some set; note that from Definition 2.15 it follows that the unique element of N without elements is $\phi$. Note also that, if $x \in N$ and $y \in z \in u \in x$, then $\mathrm{Trans}(u)$ (by Definition 2.15), therefore $y \in u$. So $\in$ is a transitive relation on $x$; from Definition 2.15 it follows also that $\in$ is total, antireflexive, antisymmetric (the last two claims follow from the condition (ii) in the Definition of "$a$ is well ordered by $\in$", cf. the proof of Proposition 2.16 and the subsequent Remark).

**Notation** In the following we use $\{x\}$ and $\{x,y\}$ as abbreviations for $\phi \cup \{x\}$ and $(\phi \cup \{x\}) \cup \{y\}$, respectively.

We prove the following properties:

**Proposition 2.16** $(N1)\ x \in N \rightarrow x \notin x$.

*Proof:* If $x \in x$, then $\{x\} \in V_3$, $\{x\} \subset x$ and for all $u \in \{x\}$ there is a $w \in \{x\}$ such that $w \in u$ (indeed the unique element of $\{x\}$ is $x$, therefore, if $u, w \in \{x\}$, then $w = x$, $u = x$, and $w \in u$), against our assumption that $x$ is well ordered by $\in$.

**Remark 2.17** From Proposition 2.16 and from the transitivity of $\in$ on the elements of $N$, it follows also that there are no $x, y \in N$ such that $x \in y \in x$, otherwise by transitivity we would obtain $x \in x$, against Proposition 2.16.

**Proposition 2.18** $(N2)\ \phi \in N$.

*Proof:* Obvious from Definition 2.15, since $\phi$ has no elements.

**Proposition 2.19** $(N3)\ x \in N \rightarrow x \cup \{x\} \in N$.

*Proof:* First, $x \cup \{x\} \in V_3$ by the properties of $V_3$; i) $x \cup \{x\}$ is transitive: indeed if $u \in y \in x \cup \{x\}$, then either $y \in x$ or $y = x$; in the former case $u \in x$, by the transitivity of $x$; in the latter, $u \in y = x$ and again $u \in x$; in any case, $u \in x$, therefore $u \in x \cup \{x\}$; ii) if $v \in x \cup \{x\}$, $v$ is transitive and $v \in V_3$: indeed, if $v \in x \cup \{x\}$, then either $v \in x$ or $v = x$; in both cases the claim follows from condition (ii) in the Definition 2.15; (iii) $x \cup \{x\}$ is well ordered by $\in$: let us verify the condition (i) in the Definition of well ordering: if $a, b \in x \cup \{x\}$ and $a \neq b$, then either $a, b \in x$ in which case, from $x \in N$ it follows $a \in b$ or $b \in a$, or else $a \in x$, $b = x$, in which case $a \in b$, or $a = x$, $b \in x$ in which case $b \in a$; let us verify condition (ii) in the definition of well ordering: suppose $y \in V_3$, $y$ not empty, $y \subset x \cup \{x\}$; if $x \notin y$, then $y \subset x$, therefore the claim follows from the assumption that $x$ is well ordered by $\in$; if $x \in y$, then either the unique element of $y$ is $x$ and the claim follows from the fact that $x \notin x$, or there is a $u \in y$ such that $u \neq x$, in which case there is a $u \in y$ such that $u \in y \cap x$. Note that there is a $z \in V_3$ such that $z \approx y \cap x$, by the properties of $V_3$. Clearly, $z \subset x$ and $z$ has at least one element. So, there is $v \in z$ such that for all $v' \in z$, $v' \notin v$. If $v' \in y$, then either $v' \in z$, and then $v' \notin v$, or $v' = x$ and again $v' \notin v$ otherwise $x = v' \in v \in z \approx x \cap y$, $x \in v \in x$ and by transitivity, $x \in x$, against Proposition 2.16; iv) $x \cup \{x\}$ and any element $v \neq \phi$ of $x \cup \{x\}$ are of the form $u \cup \{u\}$ for some $u \in v$: this is obvious for $x \cup \{x\}$; if $v \in x \cup \{x\}$, no matter that $v \in x$ or $v = x$, from $x \in N$ and from condition (iv) in Definition 2.15 it follows that $v$ has the required form.

**Notation** In the following, if $x \in N$ we write $Sx$ instead of $x \cup \{x\}$; by Theorem 2.19, $S$ is a total map from $N$ to $N$.

**Proposition 2.20** $(N4)$ $x \in N \to Sx \neq \phi$.

*Proof:* $x \in Sx$.

**Proposition 2.21** $(N5)$ $\forall x, y \in N(Sx \approx Sy \to x = y)$.

*Proof:* Since $x \in Sx$, if $Sx \approx Sy$ then $x \in Sy$, therefore either $x \in y$ or $x = y$. Similarly one can prove that under the same assumption, either $y \in x$ or $y = x$. From $x \in y$ and $y \in x$, it would follows $x \in x$ (by the transitivity of $x$) against Proposition 2.16. So, $x = y$.

From Proposition 2.21 it follows:

**Proposition 2.22** $(N5')$: $\forall x, y \in N(Sx = Sy \to x = y)$.

*Proof:* Obvious.

**Proposition 2.23** $(N6)$ $\forall x, y \in N(x \approx y \to x = y)$. (In other words, $N$ is extensional.)

*Proof:* Since $x, y \in N$, $x = \phi \vee \exists u[x = Su]$, and $y = \phi \vee \exists u[y = Su]$. If $x \approx y$, then either $x = \phi = y$, or $x = Su \approx Sv = y$ from which, by Proposition 2.21, $u = v$ and, finally, $x = y$.

**Proposition 2.24** $(N7)$ *$N$ is transitive, i.e.,* $x \in y \in N \to x \in N$.

*Proof:* Suppose $x \in y \in N$; then $x \in V_3$ by condition (ii) in Definition 2.15; furthermore: i) $x$ is transitive, since the elements of an element of $N$ are transitive (cf. part (i) in Definition 2.15; ii) since trans$(y)$, each element of $x$ is an element of $y$ and is, therefore, transitive and in $V_3$, by condition (ii) in Definition 2.15; iii) $x$ is well ordered by $\in$: first of all, the property (i) in the definition of well order follows from $x \subset y$ and $y \in N$; let us verify condition (ii) in the definition above: if $z \subset x$, $z \in V_3$ and $z$ has at least one element, then $z \subset y$, since $x \subset y$; so, there is a $v' \in z$ such that if $v \in z$, $v \notin v'$; iv) if $v \in x$ or $v = x$, then from $x \in y$ and trans$(y)$ it follows $v \in y$. By condition (iv) in Definition 2.15, it follows that either $v = \phi$ or $v = Su$ for some $u \in v$.

**Remark 2.25** If $y \in N$ and $y \neq \phi$, there is a $u$ such that $y = Su$; such $u$ belongs to $N$, as $N$ is transitive, and is uniquely determined, by Proposition 2.22; we express this saying that the predecessor of a natural number different from $\phi$ is a natural number.

**Notation** If $y \in N$, $Py$ denotes $\phi$ if $y = \phi$ and the unique $z \in N$ such that $Sz = y$ otherwise. Note that, if $y \in N$, then $PSy = y$ and if $y \neq \phi$, then $SPy = y$.

**Proposition 2.26** $(N7')$ $\forall x \in N(x \neq \phi \to \phi \in x)$.

*Proof:* Since $x$ is well ordered and $x \neq \phi$, there is a $z \in x$ such that, if $v \in x$, then $v \notin z$. Such $z$ is an element of $N$, as $N$ is transitive; so, either $z = \phi$, in which case $\phi \in x$, or $z = Sv$ for some $v$. But in this last case we would obtain $v \in z \in x$ and $v \in x$, a contradiction.

**Proposition 2.27** $(N8)$ *If $x, y \in N$ and $x \in y$, then either $Sx \in y$ or $Sx = y$. (So $Sx$ is the smallest natural number $y$ (with respect to $\in$) such that $x \in y$.)*

*Proof:* $Sx \subset y$, since $x \subset y$ ($y$ is transitive) and $x \in y$; if $Sx \neq y$, $x$ and $y$ have not the same extension, therefore, since $Sx \subset y$, there is a $v \in y - Sx$ such that, if $v' \in v$, then $v' \notin y - Sx$, therefore, $v' \in y \cap Sx \approx Sx$; it follows $v \subset Sx$; from $v \in y - Sx$ it follows also $v \neq x$, $v \notin x$, therefore, since $x, v \in y$ and $\in$ is a total order on $y$, we obtain $x \in v$; by the previous argument, $Sx \subset v$; so $Sx \approx v$; by the transitivity of $N$, $v \in N$ and by the extensionality of $N$, $Sx = v \in y$.

**Proposition 2.28** ($N9$) $\in$ *is a total* (*strict*) *order on* $N$, *i.e.*, (a) $\forall x, y, z \in N(x \in y \in z \to x \in z)$ *and* (b) $\forall x, y \in N(x \in y \vee y \in x \vee x = y)$.

*Proof:* (a) is obvious, since, if $z \in N$, then $z$ is transitive; (b) if $x, y \in N$, then either $x = y$, or $x = \phi$, $y \neq \phi$, in which case $x \in y$, or $x \neq \phi$, $y = \phi$ in which case $y \in x$, or $x \neq \phi$, $y \neq \phi$. In this last case, one cannot have $x \approx y$ (otherwise $x = y$, by Proposition 2.23); so there is a $u$ such that $(u \in x \wedge u \notin y)$ or $(u \notin x \wedge u \in y)$. Suppose w.l.o.g. $u \in x \wedge u \notin y$; consider a $v \in V_3$ such that $v \approx x - x \cap y$; since $v \subset x$ and $x$ is well ordered, there is $m \in v$ such that for all $v' \in m$, $v' \notin v$; certainly $m \neq \phi$ since $\phi \in x \cap y$ and $m \in v \approx x - x \cap y$. But then there is $w \in m$ such that $m = Sw$; since $w \in m$, $w \notin v \approx x - x \cap y$; so, $w \in x \cap y$, $w \in y \in N$ and then (by Proposition 2.28) either $Sw = y$ or $Sw \in y$, i.e., either $m = y$ or $m \in y$. Moreover, from $m \in x - x \cap y$ we obtain that $m \notin x \cap y$ and $m \in x$, therefore, $m \notin y$. So $m = y$, and then, from $m \in x$ we get $y \in x$. Similarly, if $u \notin x \wedge u \in y$, we can prove that $x \in y$. Concluding, $\forall x, y \in N(x \in y \vee y \in x \vee x = y)$.

Now, we try to define the sum in $N$; we shall define it first as a partial function, and then we shall restrict ourselves to a subuniverse of $N$ in which this operation is total.

**Notation** In the following, we write $\langle x, y \rangle$ as an abbreviation for $\{\{x\}, \{x, y\}\}$. Note that $\langle x, y \rangle = \langle u, v \rangle$ iff $x = u$ and $y = v$. We also write "$F$ is a function from $x$ to $y$" as an abbreviation for:

$$\forall u \in F \exists v \in x \exists w \in y[u = \langle v, w \rangle] \wedge \forall v \in x \exists w \in y(\langle v, w \rangle \in F)$$
$$\wedge\ \forall v \in x \forall w \forall w'(\langle v, w \rangle \in F \wedge \langle v, w' \rangle \in F \to w = w').$$

**Definition 2.29** Constr + $(F, x, y, z)$ denotes the conjunction of the following formulas: (i) $F \in V_3 \wedge x, y, z \in N$; (ii) $F$ is a function from $Sy$ to $Sz$; (iii) $\langle \phi, x \rangle \in F$; (iv) $\forall u \forall v(u \in y \wedge \langle u, v \rangle \in F \to \langle Su, Sv \rangle \in F)$; (v) $\langle y, z \rangle \in F$; (vi) if $\langle u, v \rangle \in F$, $\langle u', v' \rangle \in F$ and $u \in u'$, then $v \in v'$.

In the following, Add$(x, y, z)$ denotes the following formula: $\exists F$[Constr + $(F, x, y, z) \wedge \forall F' \forall z'$(Constr + $(F', x, y, z') \to F \approx F')$].

We prove the following properties:

**Proposition 2.30** (Add1) Add$(x, y, z) \wedge$ Constr + $(E, x, y, t) \to z = t$.

*Proof:* From Add$(x, y, z) \exists F$[Constr + $(F, x, y, z) \wedge \forall F' \forall z'$(Constr + $(F', x, y, z') \to F \approx F')$] follows. Thus, if Constr + $(E, x, y, t)$, then Constr + $(F, x, y, z) \wedge F \approx E$, $\langle y, z \rangle \in F$ and $\langle y, t \rangle \in E$; since $F$ and $E$ are functions and $F \approx E$, we conclude $z = t$.

**Proposition 2.31** (Add2) Add$(x, y, z) \wedge$ Add$(x, y, t) \to z = t$.

*Proof:* Obvious, by Proposition 2.30.

**Notation** By Proposition 2.31, the predicate Add defines a partial binary map on $N$; we denote such a map by $+$ and we adopt the terminology of partial maps when using $+$; e.g., we write $x+y=z$ for $\text{Add}(x,y,z)$, $x+y \in V$ ($V$ any definable class) for $\exists z[\text{Add}(x,y,z) \wedge z \in V]$, $x+(y+z)=v$ for $\exists w[\text{Add}(y,z,w) \wedge \text{Add}(x,w,v)]$, etc.

**Proposition 2.32** (Add3) $\forall x \in N(x+\phi=x)$.

*Proof:* Let $F=\{\langle\phi,x\rangle\}$; one has: $F \in V_3$, since $\phi, x \in V_3$, $V_3$ is closed under the operations $\langle\ ,\ \rangle$ and $\{\ \}$; $\phi, x \in N$; $F$ is a function from $S\phi$ to $Sx$; $\langle\phi,x\rangle \in F$, therefore (iii) and (v) in Definition 2.29 are satisfied; the other conditions in Definition 2.29 follow from the fact that $F$ has only one elements. Thus, Constr $+$ $(F,x,\phi,x)$. Furthermore, if Constr $+$ $(F',x,\phi,t)$, then, since $\phi$ is the unique element of $S\phi$, the unique element of $F'$ is $\{\langle\phi,x\rangle\}$; so, $F \approx F'$.

**Proposition 2.33** (Add4) Constr $+$ $(H,x,Sy,v) \to \exists z \in v[v=Sz]$.

*Proof:* From $v=\phi$ and Constr $+$ $(H,x,Sy,v)$, we would deduce Constr $+$ $(H,x,Sy,\phi)$; by Definition 2.29, we would have $\langle\phi,x\rangle \in H$; since $\phi \in Sy$, by part (vi) in Definition 2.29, we would obtain $x \in \phi$, a contradiction. Thus, $v \neq \phi$, and since, again by Definition 2.29, $v \in N$, we conclude that there is a $z \in v$ such that $v=Sz$.

**Lemma 2.34**
(a) [Constr $+$ $(F,x,y,v) \wedge H=F \cup \{\langle Sy,Sv\rangle\}] \to$ Constr $+$ $(H,x,Sy,Sv)$.
(b) [Constr $+$ $(H',x,Sy,v) \wedge \forall \in V_3 \wedge A \approx H'-\{\langle Sy,v\rangle\}] \to$ Constr $+$ $(A,x,y,Pv)$.

*Proof:* (a) (i) From Constr $+$ $(F,x,y,v)$, it follows $F \in V_3$, $H=F \cup \{\langle Sy,Sv\rangle \in V_3$, and $x, Sy, Sv \in N$; (ii) since $Sy \notin Sy$ and $F$ is a function from $Sy$ to $Sv$, $H=F \cup \{\langle Sy,Sv\rangle\}$ is a function from $SSy$ to $SSv$; (iii) $\langle\phi,x\rangle \in H$ since $\langle\phi,x\rangle \in F$; (iv) if $u \in Sy$ and $\langle u,z\rangle \in H$, then, since $u \neq Sy$, one has $\langle u,z\rangle \in F$. If $u \in y$, then $\langle Su,Sz\rangle \in F$; if $u=y$, since $\langle y,v\rangle \in F$ and $F$ is a function, one has $z=v$, and then $\langle Su,Sv\rangle = \langle Sy,Sz\rangle \in H$. In any case, $\langle Su,Sv\rangle \in H$. (v) $\langle Sy,Sz\rangle \in H$. (vi) Property (vi) in Definition 2.29 is satisfied by $F$ and, if $\langle u',v'\rangle \in H-F$, $\langle u,v\rangle \in F$, then $u \in u'=Sy$, and $v \in v'=Sz$.

(b) (i) Suppose that the assumptions of the implication (b) hold; then $H' \in V_3$ and $x,y,Pv \in N$; note also that from Constr $+$ $(H',x,Sy,v)$, and from the properties of $V_3$ it follows that there is an $A \in V_3$ such that $A \approx H'-\{\langle Sy,v\rangle\}$; (ii) since $H'$ is a function from $SSy$ to $v$, and $A \approx H'-\{\langle Sy,v\rangle\}$, it is easily seen that $A$ is a function from $Sy$ to $Pv$; (iii) $\langle\phi,x\rangle \in H$, $\langle\phi,x\rangle \neq \langle Sy,v\rangle$, so $\langle\phi,x\rangle \in A$; (iv) if $u \in y$ and $\langle u,w\rangle \in A$, then $\langle u,w\rangle \in H'$ and $u \neq Sy$ (so $w \neq v$); it follows $\langle Su,Sv\rangle \in H'$; but from $u \in y$ it follows $Su \in Sy$ (by Theorem 2.27); so $\langle Su,Sv\rangle \neq \langle Sy,v\rangle$, therefore $\langle Su,Sv\rangle \in A$; (v) $\langle y,Pv\rangle \in A$; indeed, there is a $w \in v$ such that $\langle y,w\rangle \in A$; thus $\langle Sy,Sw\rangle \in H'$, $Sw=v$, since $H'$ is a function, and finally $w=Pv$, and $\langle y,Pv\rangle \in H'$; moreover, $\langle y,Pv\rangle \neq \langle Sy,v\rangle$, therefore $\langle y,Pv\rangle \in A$. Condition (vi) in Definition 2.29 is obvious.

**Proposition 2.35** (Add5) $x+y=z \to x+Sy=Sz$.

*Proof:* Let $F \in V_3$ be such that Constr + $(F,x,y,z)$, and let $H = F \cup \{\langle Sy, Sv\rangle\}$. By Lemma 2.34(a), Constr + $(H,x,Sy,Sz)$. If Constr + $(H',x,Sy,v)$, let $A \in V_3$ be such that $A \approx H' - \{\langle Sy, v\rangle\}$. By Lemma 2.34(b), Constr + $(A,x,y,Pv)$. Now, from $x + y = z$, Constr + $(A,x,y,Pv)$ and Constr + $(F,x,y,z)$ we deduce $F \approx A$ (hence $z = Pv$); last, from $H = F \cup \{\langle Sy, Sz\rangle\}$, $H' \approx A \cup \{\langle Sy, v\rangle\} = A \cup \{\langle Sy, Sz\rangle\}$ and $F \approx A$, we conclude $H \approx H'$. So $x + Sy = Sz$.

**Proposition 2.36** (Add6) $x + Sy = Sz \rightarrow x + y = z$.

*Proof:* Let $F \in V_3$ be such that Constr + $(F,x,Sy,Sz)$. Let $H \in V_3$ be such that $H \approx F - \{\langle Sy, Sz\rangle\}$. By Lemma 2.34(b), Constr + $(H,x,y,z)$; moreover, if Constr + $(H',x,y,w)$, letting $G = H' \cup \{\langle Sy, Sw\rangle\}$ we get, by Lemma 2.34(a), that Constr + $(G,x,Sy,Sw)$. From $x + Sy = Sz$ it follows $F \approx G$, therefore $Sw = Sz$; from this we conclude $H \approx F - \{\langle Sy, Sz\rangle\} \approx G - \{\langle Sy, Sz\rangle\} \approx H'$. So $x + y = z$.

**Definition 2.37** Let $U$ be a subuniverse of $N$ defined by a formula; we say that $U$ is $N$-like if $\phi \in U$, $x \in U \rightarrow Sx \in U$, and $Sx \in U \rightarrow x \in U$.

**Lemma 2.38** *Let $U$ be an $N$-like subuniverse of $N$; let $U^+ = \{x \in U : \forall y \in U((y + x) \in U)\}$; then $U^+$ is $N$-like.*

*Proof:* $\phi \in U^+$ is obvious since, for all $y \in U$, $y + \phi = y$. If $x \in U^+$ then $Sx \in U^+$. Indeed, let $y \in U$; $y + x \in U$, since $x \in U^+$. But then $y + Sx = S(y + x) \in U$; so $Sx \in U^+$. Finally, if $Sx \in U^+$, then $x \in U^+$. Indeed, let $y \in U$; then, $y + Sx \in U$, since $Sx \in U^+$; but then $P(y + Sx) = PS(y + x) = y + x \in U$ as $U$ is closed under $P$. It follows $x \in U^+$.

**Definition 2.39** $N_1 = \{x \in (N^+)^+ : \forall y,z \in N^+ \, [z + (y + x) = (z + y) + x]\}$.

**Remark 2.40** If $x,y,z \in (N^+)^+$, then $z + y \in N^+$ and $y + x \in N^+$, therefore $z + (y + x) \in N$ and $(z + y) + x \in N$.

We prove that $N_1$ is $N$-like:

**Proposition 2.41** $(N_1 1)$ $\phi \in N_1$.

*Proof:* $\phi \in (N^+)^+$; moreover for $y,z \in (N^+)^+$, $(z + y) + \phi = z + (y + \phi) = z + y$.

**Proposition 2.42** $(N_1 2)$ *If $x \in N_1$, then $Sx \in N_1$.*

*Proof:* If $y,z \in (N^+)^+$, then $(z + y) + Sx = S[(z + y) + x] = S[z + (y + x)]$ (the last follows from $x \in N_1$) $= z + S(y + x) = z + (y + Sx)$; so, $Sx \in N_1$.

**Proposition 2.43** $(N_1 3)$ *If $Sz \in N_1$ then $z \in N_1$.*

*Proof:* If $y,z \in (N^+)^+$, then $S((z + y) + x) = (z + y) + Sx = z + (y + Sx)$ (since $Sx \in N_1$) $= z + S(y + x) = S(z + (y + x))$, therefore $z + (y + x) = PS(z + (y + x)) = PS((z + y) + x) = (z + y) + x$; so $x \in N_1$.

**Definition 2.44** $N_2 = \{x \in N_1 : \phi + x = x\}$.

We prove that $N_2$ is $N$-like:

**Proposition 2.45** $(N_2 1)$ $\phi \in N_2$.

*Proof:* Obvious.

**Proposition 2.46** $(N_2 2)$ *If* $x \in N_2$ *then* $Sx \in N_2$.

*Proof:* First of all, $Sx \in N_1$, as $N_1$ is closed under $S$. Moreover, $\phi + Sx = S(\phi + x) = Sx$, since $x \in N_2$.

**Proposition 2.47** $(N_2 3)$ *If* $Sx \in N_2$, *then* $x \in N_2$.

*Proof:* First of all, $x \in N_1$, as $N_1$ is closed under $P$. Moreover, $\phi + Sx = Sx$, since $Sx \in N_2$. So $Sx = \phi + Sx = S(\phi + x) \in N_1$. Thus, $\phi + x = PS(\phi + x) = PSx = x$.

**Definition 2.48** $N_3 = \{x \in N_2 : S\phi + x = Sx\}$.

We prove that $N_3$ is $N$-like:

**Proposition 2.49** $(N_3 1)$ $\phi \in N_3$.

*Proof:* Obvious.

**Proposition 2.50** $(N_3 2)$ *If* $x \in N_3$, *then* $Sx \in N_3$.

*Proof:* First of all, $Sx \in N_2$, as $N_2$ is closed under $S$. Moreover, $S\phi + Sx = S(S\phi + x) = SSx$, since $x \in N_3$.

**Proposition 2.51** $(N_3 3)$ *If* $Sx \in N_3$, *then* $x \in N_3$.

*Proof:* First of all, $x \in N_2$, as $N_2$ is closed under $P$. Moreover, $S\phi + Sx = SSx$, since $Sx \in N_3$. So $SSx = S\phi + Sx = S(S\phi + x) \in N_2$; so, $S\phi + x = PS(S\phi + x) = PSSx = Sx$. Thus, $x \in N_3$.

**Definition 2.52** $N_4 = \{x \in N_3^+ : \forall y \in N_3^+ (y + x = x + y)\}$. (Note: since $x, y \in N_3^+$, we obtain $y + x \in N_3$, and $x + y \in N_3$.)

We prove that $N_4$ is $N$-like:

**Proposition 2.53** $(N_4 1)$ $\phi \in N_4$.

*Proof:* Clearly, $\phi \in N_3^+$. If $y \in N_3^+$, then $y = y + \phi = \phi + y$, since $y \in N_2$.

**Proposition 2.54** $(N_4 2)$ *If* $x \in N_4$, *then* $Sx \in N_4$.

*Proof:* First of all, $Sx \in N_3^+$, as $N_3^+$ is closed under $S$. If $y \in N_3^+$, then $y + Sx = S(y + x) = S(x + y)$ (since $x \in N_4$) $= x + Sy = x + (S\phi + y)$ (since $y \in N_3$) $= (x + S\phi) + y$ (since $y \in N_1$) $= Sx + y$.

**Proposition 2.55** $(N_4 3)$ *If* $Sx \in N_4$, *then* $x \in N_4$.

*Proof:* First of all, $x \in N_3^+$, as $N_3^+$ is closed under $P$. If $y \in N_3^+$, then $S(y + x) = y + Sx = Sx + y$ (since $Sx \in N_4$) $= (x + S\phi) + y = x + (S\phi + y)$ (as $y \in N_1$) $= x + Sy$ (since $y \in N_3$) $= S(x + y)$. Thus, $x + y = PS(x + y) = PS(y + x) = y + x$.

**Lemma 2.56** *If $U$ is an $N$-like subuniverse of $N_4$, then $U^+$ is an $N$-like subuniverse of $N_4$ closed under $+$; moreover, $+$ is commutative and associative on $U^+$.*

*Proof:* That $U^+$ is $N$-like follows from Lemma 2.38; if $x, y, z$, $x + y$, $y + x$, $(z + y) + x$ belong to $U^+$, then $x + y = y + x$ and $(z + y) + x = z + (y + x)$, because $U^+$ is a subuniverse of $N_4$ (hence of $N_1$). We prove that $U^+$ is closed under $+$; if $y, x \in U^+$ and $z \in U$ then $z + y \in U$ (as $y \in U^+$), therefore, $(z + y) + x \in U$ (as $x \in U^+$); but then $z + (y + x) = (z + y) + x \in U$; by the definition of $U^+$, we conclude that $y + x \in U^+$.

In the following, we shall construct $N$-like subuniverses $U$ of $N_4$ that have some additional property, but that can possibly loose closure under $+$; Lemma 2.56 tells us that we can always save closure under $+$, simply taking $U^+$ instead of $U$.

**Definition 2.57** $N_5 = \{x \in N_4^+ : \forall y, z \in N_4^+\ y \in z \to x + y \in x + z\}$.

We prove that $N_5$ is $N$-like:

**Proposition 2.58** $(N_5 1)$ $\phi \in N_5$.

*Proof:* Clearly, $\phi \in N_4^+$. If $y, z \in N_4^+$ and $y \in z$, then $\phi + y = y \in z = \phi + z$, *since* $y, z \in N_2$.

**Proposition 2.59** $(N_5 2)$ $x \in N_5 \to Sx \in N_5$.

*Proof:* Clearly, $Sx \in N_4^+$. If $y, z \in N_4^+$ and $y \in z$, then $x + y \in x + z$, as $x \in N_5$; so, $Sx + y = y + Sx = S(y + x) = S(x + y) \in S(x + z) = S(z + x) = z + Sx = Sx + z$.

**Proposition 2.60** $(N_5 3)$ $Sx \in N_5 \to x \in N_5$.

*Proof:* Clearly, $x \in N_4^+$. If $y, z \in N_4^+$ and $y \in z$, then, $S(x + y) = S(y + x) = y + Sx = Sx + y \in Sx + z$ (as $Sx \in N_5$) $= S(x + z)$, therefore $x + y \in x + z$.

**Remark 2.61** (a) If $y \neq \phi$ and $y \in N_5$, then $\phi \in y$, therefore, for all $x \in N_5$, $x = x + \phi \in x + y$.

(b) If $x, y, y', z \in N_5$ and $x + y = x + y' = z$, then $y = y'$; indeed, if $y \in y'$, then $x + y \in x + y'$, and if $y' \in y$, then $x + y' \in x + y$.

**Notation** We define a binary partial map $-$ on $N_5$ letting $x - y = z$ if either: ($x \in y$ or $x = y$) and $z = \phi$ or: $y \in x$ and $y + z = x$. By the Remark above, $-$ is a partial function. Note that, if $x, y \in N_5^+$, then $(x + y) - y = x$, and that, if $y \in x$ and $x - y \in N_5^+$, then $(x - y) + y = x$.

**Lemma 2.62** *Let $U$ be an $N$-like subuniverse of $N_5$; let:* $U^- = \{x \in U^+ : \forall y \in U^+[(x \in y \to y - x \in U^+) \wedge (y \in x \to x - y \in U^+)\}$. *Then $U^-$ is $N$-like and is closed under $+$ and $-$ (which is total on $U^-$).*

*Proof:* Throughout this proof, we tacitly use the fact that $U^+$ contains $\phi$ and is closed under $S, P, +$, therefore in parts a), b), c), d) below we do not need to prove that the desired elements are in $U^+$. We prove successively:

a) $\phi \in U^-$.

*Proof:* Let $y \in U^+$; $y \in \phi$ is impossible; if $\phi \in y$, or $\phi = y$, then $\phi + y = y$.

b) $x \in U^- \to Sx \in U^-$.

*Proof:* Let $y \in U^+$; if $Sx \in y$, then $x \in y$, therefore there is a $w \in U^+$ such that $x + w = y$; $w \neq \phi$, since $x + \phi = x \neq y$; so, $SPw = w$; it follows $Sx + Pw =$

$Pw + Sx = S(Pw + x) = S(x + Pw) = x + SPw = x + w = y$, and $Pw \in U^+$ is the required element; if $y \in Sx$, then either $y \in x$ or $y = x$; in both cases, from $x \in U^-$ it follows that there is a $z \in U^+$ such that $y + z = x$; therefore, $y + Sz = Sx$, and $Sz \in U^+$ is the required element.

c) $Sx \in U^- \rightarrow x \in U^-$.

*Proof:* Let $y \in U^+$; if $x \in y$, then $Sx \in y$ or $Sx = y$; in both cases, from $Sx \in U^-$ it follows that there is a $z \in U^+$ such that $Sx + z = y$ therefore $x + Sz = S(x + z) = S(z + x) = z + Sx = Sx + z = y$, and $Sz \in U^+$ is the required element. If $y \in x$, then $y \in Sx$ therefore, there is a $w \in U^+$ such that $y + w = Sx$; $w \neq \phi$, since $y + \phi = y \neq Sx$; so $SPw = w$; it follows $S(y + Pw) = y + SPw = y + w = Sx$, therefore $y + Pw = x$, and $Pw \in U^+$ is the required element.

d) $x, y \in U^- \rightarrow x + y \in U^-$.

*Proof:* Let $v \in U^+$; if $x + y \in v$, then, by part a) of the Remark preceding this Lemma, $x \in v$ and $y \in v$, therefore, from $x \in U^-$ it follows that there is a $w \in U^+$ such that $x + w = v$; it is not the case that $w \in y$ or $w = y$, otherwise it would follow respectively $v = x + w \in x + y$ or $v = x + w = x + y$; so $y \in w$. Since $w \in U^+$ and $y \in U^-$, there is a $z \in U^+$ such that $y + z = w$; it follows $(x + y) + z = x + (y + z) = x + w = v$, and $z \in U^+$ is the required element. Now, suppose $v \in x + y$; if $v \in x$ or $v = x$, there is a $w \in U^+$ such that $v + w = x$, therefore $v + (w + y) = (v + w) + y = x + y$ and $w + y \in U^+$ is the required element; if $x \in v$, from $x \in U^-$ it follows that there is a $w \in U^+$ such that $x + w = v$; it is not the case that $y \in w$ or $w = y$, otherwise we would get respectively $x + y \in x + w = v$ or $v = x + w = x + y$; so $w \in y$. Since $w \in U^+$ and $y \in U^-$, there is a $z \in U^+$ such that $w + z = y$. So, $v + z = (x + w) + z = x + (w + z) = x + y$, and $z$ is the required element.

e) *If* $x, y \in U^-$, *and* $y \in x$, *then* $x - y \in U^-$.

*Proof:* First of all, $x - y \in U^+$; this follows from the definition of $U^-$. Now, suppose $v \in U^+$, $v \in x - y$; then by the Remark preceding this Lemma, $v \in x - y \in (x - y) + y = x$; since $x \in U^-$, there is a $w \in U^+$ such that $v + w = x$; if $w \in y$ or $w = y$, it would follow either $x = v + w \in (x - y) + w \in (x - y) + y = x$ or $x = v + w \in (x - y) + w = (x - y) + y = x$. So, $y \in w$, and from $y \in U^-$ it follows that there is a $z \in U^+$ such that $y + z = w$; thus, $x = v + w = v + (y + z) = (v + z) + y$; so $x - y = v + z$ and $z \in U^+$ is the required element. Now, suppose $x - y \in v$; then, $x = (x - y) + y = y + (x - y) \in y + v$; since $y + v \in U^+$ and $x \in U^-$, there is a $z \in U^+$ such that $x + z = y + v$, therefore $((x - y) + z) + y = ((x - y) + y) + z = x + z = y + v = v + y$, and finally $(x - y) + z = v$, by the Remark preceding this Lemma, and $z \in U^+$ is the required element.

Lemma 2.62 allows us to obtain, from any $N$-like subuniverse of $N_5$, an $N$-like subuniverse closed under $+$ and $-$, and in which $+$ is associative, commutative, and compatible with the order. Such subuniverses have all desirable properties of $+$ and $-$; thus, we are ready first to define the product as a partial map on $N_5^-$, and then to obtain a subuniverse of $N_5^-$ closed also under product.

**Definition 2.63** Constr$^*(P,x,y,z)$ denotes the conjunction of the following formulas: (i) $P \in V_3, x,y,z \in N_5^-$; (ii) $P$ is a function from $Sy$ to $Sz$; (iii) $\langle \phi, \phi \rangle \in P$; (iv) $(u \in y \wedge \langle u,v \rangle \in P \rightarrow v+x \in N \wedge \langle Su, v+x \rangle \in P])$; (v) $\langle y,z \rangle \in P$; (vi) $(x \neq \phi \wedge \langle u,v \rangle \in P \wedge \langle u',v' \rangle \in P \wedge u \in u') \rightarrow v \in v'$.

**Definition 2.64** Prod$(x,y,z)$ denotes the following formula:

$$\exists P[\mathrm{Constr}^*(P,x,y,z) \wedge \forall P' \forall z'(\mathrm{Constr}^*(P',x,y,z') \rightarrow P \approx P')].$$

We prove the following properties:

**Proposition 2.65** (Prod1) $\mathrm{Prod}(x,y,z) \wedge \mathrm{Constr}^*(E,x,y,t) \rightarrow z = t$.

*Proof:* From our assumptions it follows that there is an $F$ such that Constr$^*(F,x,y,z)$ and for all $F',z'$, if Constr$^*(F',x,y,z')$ then $F \approx F'$; from this and Constr$^*(E,x,y,t)$ we get $F \approx E$; from Constr$^*(F,x,y,z)$ and Constr$^*(E,x,y,t)$ we deduce $\langle y,z \rangle \in F \wedge \langle y,t \rangle \in E$; since $F$ and $E$ are functions and $F \approx E$, we conclude $z = t$.

**Proposition 2.66** (Prod2) $\mathrm{Prod}(x,y,z) \wedge \mathrm{Prod}(x,y,t) \rightarrow z = t$.

*Proof:* Obvious, by Proposition 2.65.

**Notation.** By Proposition 2.66, Prod defines a partial function, which will be denoted by $*$; we shall extend to $*$ the terminology of partial functions, exactly as we did for $+$ and for $-$.

**Proposition 2.67** (Prod3) $\forall x \in N_5^-(x^*\phi = \phi)$.

*Proof:* Let $F = \{\langle \phi, \phi \rangle\}$; we get: $F \in V_3$, since $\phi \in V_3$, and $V_3$ is closed under the operations $\langle\ ,\ \rangle$ and $\{\ \}$; $x, \phi \in N_5^-$; $F$ is a function from $S\phi$ to $S\phi$; $\langle \phi, \phi \rangle \in F$; the other conditions in Definition 2.63 follow from the fact that $F$ has only one element. So, Constr$^*(F,x,\phi,\phi)$. Moreover, if Constr$^*(F',x,\phi,t)$ then, since $\phi$ is the unique element of $S\phi$, the unique element of $F'$ is $\{\langle \phi, \phi \rangle\}$; so $F \approx F'$.

**Lemma 2.68**
(a) $\mathrm{Constr}^*(F,x,y,z) \wedge H = F \cup \{\langle Sy, z+x \rangle\} \rightarrow \mathrm{Constr}^*(H,x,Sy,z+x)$.
(b) $\mathrm{Constr}^*(H',x,Sy,z) \wedge A \in V_3 \wedge A \approx H' - \{\langle Sy,z \rangle\} \rightarrow \mathrm{Constr}^*(A,x,y,z - x)$.

*Proof:* (a) Suppose Constr$^*(F,x,y,z) \wedge H = F \cup \{\langle Sy, z+x \rangle\}$; then: (i) $H \in V_3$, and $x, Sy, z+x \in N_5^-$; (ii) since $Sy \notin Sy$ and $F$ is a function from $Sy$ to $Sz$, and $z+x \in S(z+x)$, $H = F \cup \{\langle Sy, z+x \rangle\}$ is a function from $SSy$ to $S(z+x)$; (iii) $\langle \phi, \phi \rangle \in H$, since $\langle \phi, \phi \rangle \in F$ and $\langle \phi, \phi \rangle \neq \langle Sy, z+x \rangle$; (iv) if $u \in Sy$ and $\langle u,v \rangle \in H$, then, since $u \neq Sy$, one has $\langle u,v \rangle \in F$. If $u \in y$, then $\langle Su, v+x \rangle \in F$; if $u = y$, since $\langle y,z \rangle \in F$ and $F$ is a function, we get $v = z$, therefore, $\langle Su, v+x \rangle = \langle Sy, z+x \rangle \in H$. In any case, $\langle Su, v+x \rangle \in H$. (v) $\langle Sy, z+x \rangle \in H$. (vi) If $x \neq \phi$, $\langle u,v \rangle \in H$, $\langle u',v' \rangle \in H$ and $u \in u'$, then, from $u \in u' \in SSy$, it follows $u \in y$ and either $u' \in Sy$ or $u' = Sy$; in the first case, $\langle u,v \rangle \in F$, $\langle u',v' \rangle \in F$, therefore $v \in v'$; in the second case, $\langle u,v \rangle \in F$, therefore $v \in Sz \in S(z+x)$ and $v \in z+x$. On the other hand, from $u' = Sy$ it follows $v' = z+x$; thus, $v \in v'$. We conclude Constr$^*(H,x,Sy,Sz)$.

(b) Suppose Constr$^*(H',x,Sy,z)$, $A \in V_3$ and $A \approx H' - \{\langle Sy,z \rangle\}$; then: (i) note that, by the properties of $V_3$, if $H' \in V_3$, there is an $A \in V_3$ such that

$A \approx H' - \{\langle Sy, z\rangle\}$; moreover, $x, y, z - x \in N_5^-$; (ii) let $w$ be such that $\langle y, w\rangle \in H'$; then, $\langle Sy, w + x\rangle \in H'$, whence $w + x = z$, $w = z - x$ (as $H'$ is a function); if $\langle y', w'\rangle \in A$, then either $y' = y$, in which case $w' = z - x$, or $y' \in y$, in which case $w' \in w = z - x$; in any case, $w' \in S(z - x)$; thus, $A$ is a function from $Sy$ to $S(z - x)$. (iii) $\langle \phi, \phi\rangle \in H'$ and $\langle \phi, \phi\rangle \neq \langle Sy, z\rangle$, therefore $\langle \phi, \phi\rangle \in A$; (iv) if $u \in y$ and $\langle u, w\rangle \in A$, then $\langle u, w\rangle \in H'$ and $Su \in Sy$; so, $\langle Su, w + x\rangle \in H'$ and $\langle Su, w + x\rangle \neq \langle Sy, z\rangle$, therefore $\langle Su, w + x\rangle \in A$; (v) we already saw that $\langle y, z - x\rangle \in A$; (vi) follows from Constr*$(H', x, Sy, z)$.

**Proposition 2.69** (Prod4) $x^*y = z \rightarrow x^*Sy = z + x$.

*Proof:* Let $F$ be such that Constr*$(F, x, y, z)$, and let $H = F \cup \{\langle Sy, z + x\rangle\}$. By Lemma 2.68(a), Constr*$(H, x, Sy, z + x)$; now, if Constr*$(H', x, Sy, v)$, let $A \in V_3$ be such that $A \approx H' - \{\langle Sy, v\rangle\}$. By Lemma 2.68(b), Constr*$(A, x, y, v - x)$; from this, from Constr*$(F, x, y, z)$ and from $x^*y = z$ we deduce $F \approx A$ (so, $z = v - x$); since $H \approx F \cup \{\langle Sy, z + x\rangle\}$, $H' \approx A \cup \{\langle Sy, v\rangle\} = A \cup \{\langle Sy, (v - x) + x\rangle\} = A \cup \{\langle Sy, z + x\rangle\}$, we conclude $H \approx H'$. Thus, we have proved $x^*Sy = z + x$.

**Proposition 2.70** (Prod5) $x^*Sy = z \rightarrow x^*y = z - x$.

*Proof:* Let $F$ be such that Constr*$(F, x, y, z)$, and let $H \in V_3$ be such that $H \approx F - \{\langle Sy, z\rangle\}$. By Lemma 2.68(b), Constr*$(H, x, y, z - x)$; moreover if Constr*$(H', x, y, w)$, letting $G = H' \cup \{\langle Sy, w + x\rangle\}$ we get, by Lemma 2.68(a), Constr*$(G, x, Sy, w + x)$. From $x^*Sy = z$ it follows $F \approx G$, therefore $w + x = z$; from this we conclude $H \approx F - \{\langle Sy, z\rangle\} \approx G - \{\langle Sy, w + x\rangle\} \approx H'$.

**Lemma 2.71** *Let $U$ be an $N$-like subuniverse of $N_5^-$, and let $U^* = \{v \in U^- : \forall w \in U^- (w^*v \in U^-)\}$; then, $U^*$ is $N$-like, and, if $v, w \in U^*$, then $v^*w \in U^-$.*

*Proof:* (a) $\phi \in U^*$, since $v^*\phi = \phi$ for all $v \in U^-$.

(b) If $v \in U^*$, then, for all $w \in U^-$, $w^*v \in U^-$, therefore by Proposition 2.69 $w^*Sv = w^*v + w \in U^-$, as $U^-$ is closed under $+$.

(c) If $Sv \in U^*$, then for all $w \in U^-$, $w^*Sv \in U^-$, therefore by Proposition 2.69 $w^*v = w^*Sv - w \in U^-$, as $U^-$ is closed under $-$.

(d) That, if $v, w \in U^*$, then $v^*w \in U^-$ follows from the Definition of $U^*$.

**Definition 2.72** $N_6 = (N_5^-)^*$.

$$N_7 = \{x \in N_6^- : \forall y, z \in N_6^- (y^*(z + x) = y^*z + y^*x)\}.$$

**Remark 2.73** By Lemmas 2.62 and 2.71, if $x, y, z \in N_6^-$, $z + x \in N_6^-$, and $y^*z$, $y^*x \in N_5^-$, therefore $y^*(z + x)$ and $y^*z + y^*x \in N_5^-$.

We prove that $N_7$ is $N$-like:

**Proposition 2.74** $(N_7 1)$ $\phi \in N_7$.

*Proof:* If $y, z \in N_6^-$, then $y^*(z + \phi) = y^*z = y^*z + \phi = y^*z + y^*\phi$.

**Proposition 2.75** $(N_7 2)$ $x \in N_7 \rightarrow Sx \in N_7$.

*Proof:* If $y, z \in N_6^-$, then $y^*(z + Sx) = y^*S(z + x) = y^*(z + x) + y = (y^*z + y^*x) + y$ (since $x \in N_7$) $= y^*z + (y^*x + y) = y^*z + (y^*Sx)$.

**Proposition 2.76** $(N_7 3)$ $Sx \in N_7 \to x \in N_7$.

*Proof:* Let $y, z \in N_6^-$; then $y^*(z + Sx) = y^*z + (y^*Sx)$ (since $Sx \in N_7$). It follows $y^*(z + x) = y^*(z + Sx) - y = (y^*z + (y^*Sx)) - y = (y^*z + (y^*x + y)) - y = ((y^*z + y^*x) + y) - y = y^*z + y^*x$.

**Definition 2.77** $N_8 = (N_7^-)^*$.

$$N_9 = \{x \in N_8^- : \forall y, z \in N_8^-(y^*(z^*x) = (y^*z)^*x)\}$$

**Remark 2.78** (By Lemma 2.68, if $x, y, z \in N_8^-$ then $z^*x \in N_7^-$, and $y^*z \in N_7^-$, therefore $y^*(z^*x)$ and $(y^*z)^*x \in N_5^-$.)

We prove that $N_9$ is $N$-like:

**Proposition 2.79** $(N_9 1)$ $\phi \in N_9$.

*Proof:* If $y, z \in N_8^-$, then $y^*(z^*\phi) = (y^*z)^*\phi = \phi$.

**Proposition 2.80** $(N_9 2)$ $x \in N_9 \to Sx \in N_9$.

*Proof:* Let $y, z \in N_8^-$; then, $y^*(z^*Sx) = y^*(z^*x + z) = y^*(z^*x) + y^*z$ (as $y, z^*x \in (N_6^-)^*$ and $z \in N_7$) $= (y^*z)^*x + y^*z$ (as $x \in N_9$) $= (y^*z)^*Sx$.

**Proposition 2.81** $(N_9 3)$ $Sx \in N_9 \to x \in N_9$.

*Proof:* If $y, z \in N_8^-$, then $y^*(z^*Sx) = (y^*z)^*Sx$ (since $Sx \in N_9$). So, $(y^*z)^*x = (y^*z)^*Sx - y^*z = y^*(z^*Sx) - y^*z = y^*(z^*x + z) - y^*z = y^*(z^*x) + y^*z - y^*z = y^*(z^*x)$.

**Definition 2.82** $N_{10} = (N_9^-)^*$.

**Theorem 2.83** *$N_{10}$ is N-like and closed under + and *; moreover $N_{10}$ is a model of* Q *(more precisely, if $N_{10}(x)$ defines $N_{10}$, then the relativisation to $N_{10}(x)$ of all axioms of* Q *is provable in* N*). Thus,* Q *is interpretable in* N.

*Proof:* That $N_{10}$ is $N$-like follows from Lemma 2.68. We prove that $N_{10}$ is closed under $+$; let $x, y \in N_{10}$, and let $z \in N_9^-$. Then, $z^*(x + y) = z^*x + z^*y$, and $z^*x, z^*y \in N_9^-$; since $N_9^-$ is closed under $+$, $z^*(x + y) \in N_9^-$; so, $x + y \in N_{10}$. Now, we prove that $N_{10}$ is closed under $*$; let $x, y \in N_{10}$, and let $z \in N_9^-$; then, $(z^*y) \in N_9^-$ (since $x \in N_{10}$). So $(z^*x)^*y \in N_9^-$ (since $y \in N_{10}$), and finally $z^*(x^*y) = (z^*x)^*y \in N_9^-$. Thus, $x^*y \in N_{10}$. Last, the operations $+$, $*$, $S$, $P$ satisfy the axioms of Q on their domain (each axiom of Q is satisfied in a universe of which $N_{10}$ is a subuniverse), therefore $N_{10}$ is a model of Q.

**Remark 2.84** If either we drop one of the axioms of N or we replace the second axiom by either $\forall x \forall y \forall z (x \in y \cup \{z\} \to x \in y \vee x = z)$ or $\forall x \forall y \forall z [(x \in y \vee x = z) \to x \in y \cup \{z\}]$, then we get theories that are undecidable (as N is hereditarily undecidable), but not essentially undecidable; indeed, let $T_1$,$T_2$, denote N minus the axiom on the empty set, and N minus the axiom of addition of one element, respectively, and let $T_3$,$T_4$ denote $T_2$ plus $\forall x \forall y \forall z (x \in y \cup \{z\} \to x \in y \vee x = z)$ and $T_2$ plus $\forall x \forall y \forall z [(x \in y \vee x = z) \to x \in y \cup \{z\}]$, respectively; then, $T_1$ plus $\forall x \forall y (x = y)$ is consistent and complete (its only model has only one element $x$ such that $x \in x$); hence $T_1$ plus $\forall x \forall y (x = y)$ is consistent and decidable. $T_2$ plus $\forall x \forall y (x = y)$ and $T_3$ plus $\forall x \forall y (x = y)$ are consistent and complete (their only model has only one element $x$ such that $\neg x \in x$). Thus, these extensions of $T_2$ and $T_3$ are decidable. Finally, $T_4$ plus $\forall x \forall y[(x \neq \phi \wedge y \neq \phi) \rightarrow x = y]$ is consistent and complete: its only model consists of $\phi$ plus another element $x$ such that $\phi \in x$, $x \in x$ and, for any $u, v$ in the model, $u \cup \{v\} = x$. So, Q is not interpretable in any of $T_1, \ldots, T_4$. This justifies the adjective "minimal" occurring in the title of this paper.

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