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A NOTE ON BdX

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The following questions arise naturally in connection with the author's work in [1]:

1. What is a necessary and sufficient condition for a flat to be contained in BdX?

2. Is it true that any second countable space which can serve as the space for a closed m-arrangement, i.e. an m-arrangement in which every 1-flat has two non-cut points, is an m-manifold with boundary?

3. Is every space of a closed m-arrangement compact?

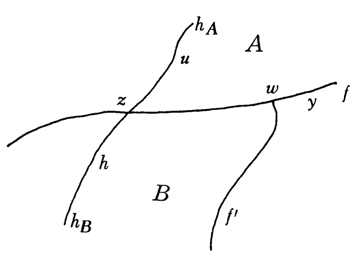
4. Is BdX for the space of a closed m-arrangement compact, and also connected if $m \geqslant 2$?

The purpose of this note is to answer these questions. The terminology and numbering of propositions in [1] will be followed throughout this paper. We also assume throughout that X is a topological space with geometry G such that X and G form an m-arrangement, $m \geqslant 1$.

Suppose $Y \subseteq X$. By BdY we denote the border of Y relative to G_Y, and set Int$Y$ = Y-BdY.

Lemma 1: Suppose $m=2$ and $w \in$ Int$f \cap$ BdX, where f is some 1-flat of G. Then $f \subseteq$ BdX.

Proof:



Since $w \in$ BdX, there is f', a 1-flat with $w \in$ Bdf'. Suppose $f \not\subseteq$ BdX. Then there are $z \in f$, and h, a 1-flat with $z \in$ Int$h$. Since $w \in$ Int$f$, there is $y \in f$ such that $w \in$ Int$\overline{zy}$. By **3.25** and **3.26** f disconnects X into convex components A and B, and f also disconnects h into components $h_A \subset A$ and $h_B \subset B$. We may label things so that h_B and f'-$\{w\}$ are both in B. Choose $u \in h_A$. Then $C(\{u,z,y\})$-$\overline{zy} \subseteq A$, hence $C(\{u,z,y\}) \cap f' = \{w\}$, a contradiction of **3.7**.

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Theorem: Suppose f is any k-flat and $w \in \mathrm{Int} f \cap \mathrm{Bd} X$. *Then* $f \subseteq \mathrm{Bd} X$.

Proof: If $k = 0$, the theorem is trivial. Assume $k \geq 1$. $w \in \mathrm{Int} f$ implies that for any 1-flat h in f with $w \in h$, $w \in \mathrm{Int} h$ (this follows at once from **3.24** and **3.20.1**). $w \in \mathrm{Bd} X$ implies that there is some 1-flat f' in X such that $w \in \mathrm{Bd} f'$. Therefore $w \in \mathrm{Bd}\,\mathbf{f}_2(h \cup f')$, hence by lemma 1, $h \subseteq \mathrm{Bd}\,\mathbf{f}_2(h \cup f')$, hence $h \subseteq \mathrm{Bd} X$. But h was an arbitrary 1-flat in f which contained w, and the union of all such 1-flats is f, hence $f \subseteq \mathrm{Bd} X$.

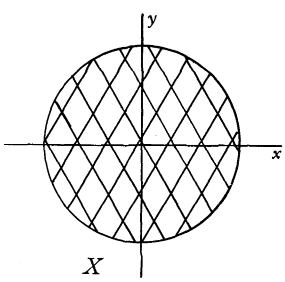
Cor. 1: Any given k-flat is contained entirely in BdX *iff* $\mathrm{Int} f \cap \mathrm{Bd} X \neq \phi$.

Cor. 2: If f is any 1-flat not contained in BdX, *then no more than two distinct points of f can be contained in* BdX, *i.e. the end points of f.*

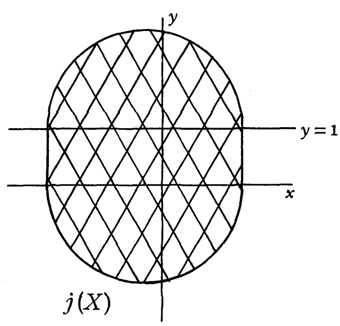
Cor. 3: If $x \in \mathrm{Bd} X$ *and f is any 1-flat which contains x and is not contained in* BdX, *then x is an end point of f.*

We have thus supplied at least one answer to question 1. The following example answers questions 2, 3, and 4 in the negative.

Example: $\mathbf{R}^2$ will represent both the topological space $\mathbf{R}^2$ and the usual Euclidean geometry on $\mathbf{R}^2$. Let $X = \{(x,y) \in \mathbf{R}^2 | x^2 + y^2 \leq 1\}$. X with geometry G_X induced from $\mathbf{R}^2$ and the subspace topology clearly forms a closed 2-arrangement.



X



j(X)

Define the map j as follows: $j((x,y)) = \begin{cases} (x,y) & \text{if } y \leq 0 \\ \left(x, \dfrac{y\sqrt{1-x^2} + 1}{\sqrt{1-x^2}}\right) & \text{if } y > 0. \end{cases}$

Then the set $j(X)$ with geometry $j(G_X)$ as defined in the epilogue of [1] and with the subspace topology from $\mathbf{R}^2$ forms a closed 2-arrangement. Noting that $j|\mathrm{Int} X$ is a homeomorphism onto $\mathrm{Int} j(X)$ and $j(\mathrm{Bd} X) = \mathrm{Bd}\, j(X)$, we readily see that $j(X)$ with geometry and topology as given furnish counterexamples for questions 2, 3, and 4.

BIBLIOGRAPHY

[1] M. C. Gemignani; "Topological Geometries and a New Characterization of $\mathbf{R}^m$", *Notre Dame Journal of Formal Logic*, Vol. VII (1966), pp. 57-100.

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