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ON ELIMINATING AN UNWANTED AXIOM IN THE CHARACTERIZATION OF $\mathbf{R}^m$ USING TOPOLOGICAL GEOMETRIES

MICHAEL C. GEMIGNANI

The principal result in the author's paper [1] was a purely topological characterization of $\mathbf{R}^m$. The author expressed his complete set of topological invariants in terms of a "geometry", a set theoretic structure which was superimposed on a topological structure with axioms binding topology and geometry. There was, however, one axiom in that paper, axiom **3.9** in the definition of an m-arrangement, which was something of a blemish because of its lattice theoretic (rather than geometric-topological) nature. It did not really have anything to do with the topology of the underlying space. The purpose of the present paper is to replace axiom **3.9** with an axiom which will prove more satisfactory from this viewpoint. Axiom **3.9** will then be proved as a theorem.

The terminology and numbering of propositions of [1] will be followed throughout this present paper.

Theorem 1: Suppose X is a topological space with geometry G of length $m - 1 \geqslant 0$ *such that*

i) **3.1-3.8** *in the definition of an m-arrangement are satisfied by X and G;*
ii) *any point is a cut point of any 1-flat which contains it; and*
iii) *if f is a k-1-flat contained in a k-flat f' and $C(S)$ is a 2-simplex in f' such that f intersects the interior of one face of $C(S)$ in a single point, then f intersects another face of $C(S)$.*

Then **3.9** *is also satisfied, i.e.* $g \cap g' \neq \phi$ *implies* $\dim(g \vee g') + \dim(g \cap g') = \dim g + \dim g'$, *where g and g' are any two flats of G^*.*

It will easily be verified that **3.25, 3.26, 4.4, 4.4.1, 4.4.2,** and **4.4.3** are also satisfied by X and G. The proofs of these propositions go through either precisely as given in [1] or with at most changing "by **3.23**" to "by iii) of theorem 1".

Lemma 1: Under the hypotheses of theorem 1, if f is an m-1-flat and g is a k-flat such that $f \cap g \neq \phi$, but $g \not\subseteq f$, then $\dim(f \cap g) = k-1$.

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Proof: $\dim g \geqslant 1$. By **3.25** and **4.4.1** f disconnects X into convex, open components A and B. If $f \cap g$ does not disconnect g, then $g-f$ is contained in either A or B; assume A. Choose $x \epsilon g-f$ and $y \epsilon g \cap f$. Then $\mathbf{f}_1(x,y) \cap (\underline{g} \cup f) = \{y\}$, but y is a cut point of $\mathbf{f}_1(x,y)$ by ii), hence by **2.16** there are $u,v \epsilon \mathbf{f}_1(x,y)$ with $y \epsilon \operatorname{Int} \overline{uv}$. But $\overline{uv} \subseteq \mathbf{f}_1(x,y) \subseteq A \cup \{y\}$, hence A is not convex, contradicting **3.25**. Therefore $f \cap g$ disconnects g, hence $\dim(f \cap g) = k - 1$ by **4.4.3**.

Lemma 2: Let f be a k-flat, $-1 < k \leqslant m - 1$. Then there exist m-k distinct m-1-flats whose intersection is f.

Proof: If $k = m - 1$, the lemma is trivial. If $k < m - 1$, there is clearly at least one m-1-flat h^1 which contains f. Suppose $\{x_0, \ldots, x_k\}$ is a basis for f and $\{x_0, \ldots, x_k, x_{k+1}, \ldots, x_{m-1}\}$ is a basis for h^1. Choose $y_{m-1} \epsilon X - h^1$ and set $h^2 = f_{m-1}(\{x_0, \ldots, x_k, x_{k+1}, \ldots, x_{m-2}, y_{m-1}\})$. By **1.9.1** $f \subseteq h^1 \cap h^2$, and we also have that $\dim(h^1 \cap h^2) = m - 2$. If $k = m - 2$, then $f = h^1 \cap h^2$ by **1.4**. If $k < m-2$, then choose $x \epsilon h^1 - h^2$ and $y \epsilon h^2 - h^1$. Then $\overline{xy} \cap (h^1 \cup h^2) = \{x,y\}$, but $\operatorname{card} \overline{xy} \geqslant 3$, hence we can choose $y_{m-2} \epsilon \operatorname{Int} \overline{xy} \subset X - (h^1 \cup h^2)$. Set $h^3 = \mathbf{f}_{m-1}(\{x_0, \ldots, x_k, x_{k+1}, \ldots, x_{m-3}, y_{m-2}, y_{m-1}\})$. Then $f \subseteq h^1 \cap h^2 \cap h^3$ and $\dim(h^1 \cap h^2 \cap h^3) = m-3$. We can continue in like fashion until we have m-k distinct m-1-flats $h^1, \ldots, h^k$ with $f \subseteq h^1 \cap \ldots \cap h^{m-k}$ and $\dim(h^1 \cap \ldots \cap h^{m-k}) = k$. Therefore $f = h^1 \cap \ldots \cap h^{m-k}$.

Proof of theorem 1 completed: Suppose f and f' are arbitrary flats of G^* with $\dim f = k$ and $\dim f' = k'$, $k \geqslant k'$, and $f \cap f' \neq \phi$. Suppose $\dim(f \vee f') = q$, where $q = k + p$. We also let $k = k' + p'$. If $f' \subseteq f$, **3.9** is trivially verified, therefore assume $f' \not\subseteq f$. f is the intersection of p-1 distinct $(k+p-1)$-flats $h^1, \ldots, h^{p-1}$ contained in $f \vee f'$. Moreover, f' is the intersection of $p + p' - 1$ distinct $(k+p-1)$-flats $g^1, \ldots, g^{p+p'-1}$ also contained in $f \vee f'$. Now $g^i \neq h^j$ for any i,j for this would give $\dim(f \vee f') \leqslant k + p - 1$. Therefore $f \cap f' \neq \phi$ is the intersection of a $(k+p-1)$-flat h^1 with the intersection of $2p + p' - 2$ distinct $(k+p-1)$-flats, i.e. $h^2 \cap \ldots \cap h^{p-1} \cap \ldots \cap g^{p+p'-1}$. By the inductive use of lemma 1 we then have $\dim(f \vee f') + \dim(f \cap f') = k + p + (k - p - p') = k + (k - p') = \dim f + \dim f'$.

Condition ii) of theorem 1 is not very restrictive since if i) and iii) are satisfied, there is an open dense subset of X, $\operatorname{Int} X$, such that $\operatorname{Int} X$ and $G_{\operatorname{Int} X}$ satisfy i), ii), and iii) (cf. **4.10.1** and **4.11**). It is not as yet known if theorem 1 is true when condition ii) is replaced by the requirement that G be affine.

BIBLIOGRAPHY

[1] M. C. Gemignani; "Topological Geometries and a New Characterization of $\mathbf{R}^m$", *Notre Dame Journal of Formal Logic*, V. VII (1966), pp. 57-100.

State University of New York at Buffalo
Buffalo, New York