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A PROPOSITIONAL CALCULUS INTERMEDIATE BETWEEN THE MINIMAL CALCULUS AND THE CLASSICAL

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1. *Introduction.* As is well known, the minimal calculus can be simply formulated by adding to the positive propositional calculus ([1], pp. 140-41) a symbol f to represent a statement supposed to be false, and the standard definition of $-S$ as $S \supset f$. This makes its incompleteness immediate: the theorems of the minimal calculus are not only valid in the ordinary sense, with f interpreted as false, but they are also valid if f is interpreted as true. If the minimal calculus is formulated alternatively with negation as primitive and the axiom $p \supset q \;.\supset:\; p \supset \bar{q} \;.\supset \bar{p}$, this is tantamount to interpreting -- as expressing not negation but the tautological truth-function of one argument. Let us call a formula (whether formulated with f or with '—') *pseudo-valid* if it is valid under this queer interpretation.

Evidently any negation-free formula of propositional logic is valid if and only if it is pseudo-valid. The class of formulae which is both valid and pseudo-valid is thus larger than the class of theorems of the minimal calculus and includes some formulae not intuitionistically valid. It is the purpose of this note to obtain an axiomatic characterization of this class (theorem 2). As one might expect, it is related to the class of negation-free tautologies in the same way as the class of theorems of the minimal calculus is related to the class of theorems of positive logic. We obtain on the way an axiomatization of the negation-free tautologies (theorem 1).

2. We consider formulations of the propositional calculus with the rules of substitution and modus ponens. By **PC** (positive propositional calculus) we mean the system with the axioms[1]

1. $p \supset .\; q \supset p$
2. $p \supset .\; q \supset r :\supset: p \supset q \;.\supset.\; p \supset r$
3. $p \supset .\; p \vee q$
4. $q \supset .\; p \vee q$
5. $p \supset r \;.\supset\therefore\; q \supset r \;.\supset:\; p \vee q \;.\supset r$
6. $p\, q \supset p$
7. $p\, q \supset q$
8. $p \supset .\; q \supset p\, q$

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The minimal calculus **MC** is obtained by either adding f and no axioms or $-$ and the axiom

9. $p \supset q \;.\!\supset\!: p \supset \bar{q} \;.\!\supset \bar{p}$

The system resulting from **PC** by adding the axiom

10. $p \vee . \, p \supset q$

will be called **PCC**. The system resulting from adding axiom 10 to **MC** will be called **MCC**.

Lemma 1. In **PCC** *or* **MCC**, *axiom* 10 *may be replaced by Peirce's law* $p \supset q \;.\!\supset p \;:\!\supset p$.

Proof. First, Peirce's law is provable in **PCC**. For by axiom 1,

$\vdash p \supset\therefore p \supset q \;.\!\supset p \;:\!\supset p$ in **PC**.[2] [i]

Assume $p \supset q$ and $p \supset q \;.\!\supset p$. By modus ponens p follows, and by the deduction theorem

$p \supset q \vdash p \supset q \;.\!\supset p :\!\supset p$ in **PC**.

By the deduction theorem we have

$\vdash p \supset q \;.\!\supset\!\because p \supset q \;.\!\supset p \;:\!\supset p$

and therefore by axioms 5 and 10 and [i]

$\vdash p \supset q \;.\!\supset p \;:\!\supset p$ in **PCC**.

Conversely, assume $p \vee . \, p \supset q \;:\!\supset q$. Now assume p. Then $p \vee . \, p \supset q$ follows by axiom 3, and q by modus ponens. Therefore by the deduction theorem

$p \vee . \, p \supset q \;:\!\supset q \vdash p \supset q$ in **PC** and by axiom 4

$p \vee . \, p \supset q \;:\!\supset q \vdash p \vee . \, p \supset q$ in **PC** and therefore by the deduction theorem

$\vdash p \vee . \, p \supset q \;:\!\supset q \;\because\!\supset\!: p \vee . \, p \supset q$ in **PC**

and hence by modus ponens

$p \vee . \, p \supset q \;:\!\supset q \;\because\!\supset\!: p \vee . \, p \supset q \;::\!\supset\!: p \vee . \, p \supset q$

$\vdash p \vee . \, p \supset q$ in **PC**.

The premiss is an instance of Peirce's law.

Theorem 1. All tautologies without negation are provable in **PCC**.

Proof. According to a well-known result of Tarski and Bernays ([4], p. 52), all tautologies containing only the conditional are provable by means of axioms 1 and 2 and Peirce's law, and therefore in **PCC**. Theorem 1 will follow easily from this fact and the following two lemmas.

Lemma 2. Let S' be the formula obtained from S by eliminating alternation by the explicit definition $T \vee U$ for $T \supset U \;.\!\supset U$ ([1], p. 78).[3] *Then* $S \equiv S'$ *in* **PCC**, *or, if S contains negation, in* **MCC**.

Proof. It can be shown in the standard way that **MC** and **PC** satisfy the usual replacement principle: If $\vdash S_1 \equiv S_2$, and T_2 is like T_1 except for containing occurrences of S_2 at one or more places where T_1 contains S_1, then $\vdash T_1 \equiv T_2$. It then suffices to show that $\vdash T \vee U .\equiv: T \supset U. \supset U$ in **PCC**.

By axiom 1

$$\vdash U \supset : T \supset U . \supset U$$

and by modus ponens and the deduction theorem

$$\vdash T \supset : T \supset U . \supset U$$

and hence by axiom 5 we have even in **PC**

$$\vdash T \vee U . \supset : T \supset U \; . \supset U. \qquad [ii]$$

Assuming $T \supset U . \supset U$ and $T \supset U$, U follows by modus ponens and therefore by axiom 4, $T \vee U$ follows. By the deduction theorem

$$T \supset U . \supset U \vdash T \supset U . \supset . T \vee U.$$

By axiom 3, $\vdash T \supset . T \vee U$. Hence by axioms 5 and 10, $T \supset U . \supset U \vdash T \vee U$, and by the deduction theorem, $\vdash T \supset U . \supset U : \supset . T \vee U$. Hence by [ii] and axiom 8, $\vdash T \vee U . \equiv : T \supset U \; . \supset U$, q.e.d.

Lemma 3. Let S be any formula without negation. Then there is a formula S′ which is a conjunction of formulae containing neither negation nor conjunction, such that $\vdash S \equiv S'$ *in* **PCC**.

Proof. By induction on the structure of S.

1. S is a statement letter. $S' = S$. Since $\vdash p \supset p$ in **PC**, $\vdash p \equiv p$, and $\vdash S \equiv S'$.

2. S is $T \; U$. Then there are T', U' of the required form so $\vdash T \equiv T'$ and $\vdash U \equiv U'$. Evidently $T' \; U'$ is of the required form, and by the replacement principle $\vdash T \; U \equiv T' \; U'$.

3. S is $T \vee U$. Then there are T', U' of the required form so $\vdash T \equiv T'$ and $\vdash U \equiv U'$. Let S' be the result of distributing alternation through conjunction in $T' \vee U'$. Evidently $\vdash S \equiv S'$ if the distributive laws hold in **PCC**. It suffices to show $\vdash p \vee q \; r . \equiv . p \vee q . p \vee r$ and $\vdash q \; r \vee p . \equiv . q \vee p . r \vee p$. In fact it is easy to show that these hold for **PC**.

4. S is $T \supset U$. Again, there are T', U' of the required form so $\vdash T \equiv T'$ and $\vdash U \equiv U'$.

Suppose T' is $A_1 . \text{---} . A_n$ and U' is $B_2 . \text{---} . B_m$, where the A_i and B_j do not contain conjunction. Since evidently $\vdash p \supset q \; r . \equiv . p \supset q . p \supset r$ already in **PC**, by repeated substitutions and applications of the replacement principle

$$\vdash S \equiv . T' \supset U' . \equiv . T' \supset B_1 . \text{---} . T' \supset B_m . \qquad [iii]$$

But since evidently $\vdash p \; q \supset r . \equiv : p \supset . q \supset r$ in **PC**, by substitutions and replacements we have for each j

$$\vdash T' \supset B_j . \equiv (A_1 \supset [A_2 \supset (\ldots \supset . A_n \supset B_j \ldots)])$$

where the right side does not contain conjunction. If we write this right side as S_j, then by [iii] and the replacement principle

$$\vdash S \equiv . S_1 . \ldots . S_m$$

where the right side is the required S'.

We note that the argument shows that if S does not contain alternation, then $\vdash S \equiv S'$ in **PC**.

Proof of theorem 1. Let S be a negation-free tautology. By lemma 3, $\vdash S \equiv . A_1 . \cdots . A_n$ in **PCC**, where $A_1 \cdots A_n$ do not contain conjunction. In view of lemma 2 and the replacement principle, they can be assumed not to contain alternation either. Hence each A_i contains only the conditional. Since evidently it is a tautology, it is provable in **PCC**. But then by repeated applications of axiom 8, $\vdash A_1 . \ldots . A_n$, and therefore $\vdash S$, q.e.d.[4]

Theorem 2. Let S be any formula of propositional logic. Then S is both valid and pseudo-valid if and only if S is provable in **MCC**.

Proof. The 'if' is obvious. Conversely, consider first the case where negation is expressed by f. S's being both valid and pseudo-valid is equivalent to the validity of S where f is viewed as an ordinary statement letter. But in that case, by theorem 1, S is provable in **MCC**. Now suppose S is a formula in the $-$ formulation which is both valid and pseudo-valid. S is provable in the classical calculus, i.e. the result of adding to **MC** the axiom $\bar{\bar{p}} \supset p$. Hence there are formulae $B_1 \ldots B_m$ such that

$$--B_1 \supset B_1 \ldots --B_m \supset B_m \vdash S \qquad [iv]$$

in **MCC**. Now let $A_1 \ldots A_n$ by the outermost negated formulae in S. From the pseudo-validity of S it follows that if each $-A_i$ is replaced by a distinct statement letter α_i, then the resulting formula S' is implied by $\alpha_1 . \cdots . \alpha_n$ and does not contain negation. By theorem 1

$$\alpha_1 \ldots \alpha_n \vdash S'$$

in **PCC**, and therefore

$$-A_1 \ldots -A_n \vdash S$$

in **MCC**.[5]

From this and [iv] we can infer

$$--A_1 \supset A_1 \ldots --A_n \supset A_n,\ --B_1 \supset B_1 \ldots --B_m \supset B_m \vdash S$$

$$-A_1 \ldots -A_n, -B_1 \ldots -B_m \vdash S$$

and therefore by the deduction theorem and axiom 5

$$-A_1 \vee . --A_1 \supset A_1 \ldots -A_n \vee . --A_n \supset A_n,$$
$$-B_1 \vee . --B_1 \supset B_1 \ldots -B_m \vee . --B_m \supset B_m \vdash S$$

To obtain $\vdash S$, it suffices to show $\vdash \bar{p} \vee . \bar{\bar{p}} \supset p$. By axiom 9, $\vdash p \supset p . \supset : p \supset \bar{p} . \supset \bar{p}$ and hence $\vdash p \supset \bar{p} . \supset \bar{p}$. By axiom 3, $\vdash \bar{p} \supset : \bar{p} \vee . \bar{\bar{p}} \supset p$, and hence $\vdash p \supset \bar{p} . \supset : \bar{p} \vee . \bar{\bar{p}} \supset p$. [v]

By axiom 1, $p \vdash \bar{\bar{p}} \supset p$ and hence by axiom 4

$$p \vdash \bar{p} \vee . \bar{\bar{p}} \supset p$$

and therefore by the deduction theorem and [v]

$$\vdash p \vee . p \supset \bar{p} :\supset: \bar{p} \vee . \bar{\bar{p}} \supset p$$

Since by axiom 10 $p \vee . p \supset \bar{p}$, we have $\bar{p} \vee . \bar{\bar{p}} \supset p$, q.e.d.

3. If either $p \vee \bar{p}$ or $\bar{\bar{p}} \supset p$ is added as axiom to the intuitionistic propositional calculus, the classical is obtained. The same is true if $\bar{\bar{p}} \supset p$ is added to the minimal calculus. But since $p \vee \bar{p}$ is pseudo-valid, its addition to **MC** does not yield the full classical calculus, but a subsystem of **MCC**. That it does not include all of **MCC** follows from

Theorem 3. Let S be a formula without negation and suppose S is provable in the system **MC**$^+$ *obtained from* **MC** *by adding* $p \vee \bar{p}$ *as an axiom. Then S is provable in* **PC**.

(Thus, for example, $p \vee . p \supset q$ is not probable in **MC**$^+$.)
Proof. If S is provable in **MC**$^+$, there are formulae $S_1 \ldots S_n$ such that $S_1 \vee - S_1 \ldots S_n \vee - S_n \vdash S$ in **MC**. Let α be a statement letter not occurring in any S_i or in S. Then $S_1 \vee . S_1 \supset \alpha$ --- $S_n \vee . S_n \supset \alpha \vdash S$ in **PC**. But for each i, $\alpha \vdash S_i \vee . S_i \supset \alpha$ in **PC** by axioms 1 and 4. Therefore $\alpha \vdash S$ in **PC**. In this deduction, a provable formula, say $\alpha \supset \alpha$, could have been substituted for α, leaving S unaffected. Therefore $\vdash S$.
Remark. Relative to some intuitionist concept of model (e.g. that of [3]) we could define intuitionistic pseudo-validity. Theorem 3 could have been proved by observing that since $p \vee \bar{p}$ is intuitionistically pseudo-valid, so is every theorem of **MC**$^+$; therefore every negation-free theorem of **MC**$^+$ is intuitionistically valid.

Every theorem of **MC** is intuitionistically valid and pseudo-valid, but the converse does not hold. To see this intuitively, consider the f formulation. If S is intuitionistically valid, it is valid modulo the interpretation of f as absurd; if it is pseudo-valid, it is valid moduls the interpretation of f as true; but if S is provable in **MC** it can be seen to be valid without the assumption that f is either true or absurd. Thus we would not expect $f \vee . f \supset p$, which is intuitionistically valid and pseudo-valid, to be provable in **MC**. Indeed it is not, for if it were $q \vee . q \supset p$ would be provable in **PC**, which it is not since it is not intuitionistically valid. In the – formulation, $\bar{p} \vee : \bar{p} \supset . p \supset q$ is a ready example of a formula which is intuitionistically valid and pseudo-valid, but not provable in **MC**, for if it were $p \supset r . \vee : p \supset r . \supset . p \supset q$ would be provable in **PC**, but it is not intuitionistically valid. A less contrived example is $-(p \supset q) \supset \bar{\bar{p}}$. If it is provable in **MC**, $p \supset q . \supset r : \supset : p \supset r . \supset r$ is provable in **PC**. But then so is

$$p \supset q . \supset : p \vee . p \supset q .\because. \supset :: p \supset : p \vee . p \supset q .\because. \supset : p \vee . p \supset q.$$

But since $\vdash p \supset q . \supset : p \vee . p \supset q$ and $\vdash p \supset : p \vee . p \supset q$ by axioms 4 and 3, this implies $\vdash p \vee . p \supset q$, which is impossible.

Evidently the formulae which are intuitionistically both valid and pseudo-valid are exactly those provable in **MC** with the additional axiom $f \vee . f \supset p$ in the f formulation or $\bar{p} \vee : \bar{p} \supset . p \supset q$ in the -- formulation.

We note finally that the full classical calculus is obtained from **MCC** by adding the intuitionistically valid axioms $\bar{p} \supset . p \supset q$ or $f \supset p$ according as the – or the f formulation is chosen.

NOTES

1. We omit axioms for the biconditional, since it can be introduced by the explicit definition $S \equiv T$ for $S \supset T \cdot T \supset S$.

2. The symbol '$\vdash$' is used on the model of [1], pp. 82-83, 86-87.

3. Note that $p \supset q . \vee p$ becomes Peirce's law under this definition, while $p \vee . p \supset q$ becomes $p \supset . p \supset q : \supset . p \supset q$, which is provable in **PC**.

4. Theorem 1 could have been proved in the following fashion: If S is a tautology without negation, it is provable by Gentzen rules (e.g. in the classical system **G1** of [2], pp. 442-43) without the use of the rules for negation. One can show by induction on the derivation that if S_1 --- $S_n \rightarrow T_1$ --- T_m is provable in this system, then S_1 --- $S_n \vdash T_1 \vee$ --- $\vee T_m$ in **PCC**. $p \vee . p \supset q$ is needed to cover the rule $\rightarrow \supset$, i.e. to show that if

$$U, S_1 \text{---} S_n \vdash V \vee W$$

in **PCC**, then $S_1 \ldots S_n \vdash V \vee . U \supset W$. In dealing with formulae without conjunction, one can replace **PCC** by the system with axioms 1 and 2 and Peirce's law by defining alternation as in lemma 2. One obtains in this way a proof of Tarski-Bernays theorem.

5. This shows that if S is pseudo-valid, S is provable in **MCC** with the additional axiom '$\bar{p}$'.

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