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A PROPERTY OF SENTENCES THAT DEFINE QUASI-ORDER

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In this paper we show that any sentence in a first order language without identity and in prenex conjunctive normal form which states that a binary predicate is reflexive and transitive has a disjunction with more than two terms. This answers negatively the question mentioned in 3.4 of [1].

Let $\mathcal{L}$ be the set of formulas of a first order predicate logic without identity, without function symbols, and without individual constants, but with exactly one predicate letter **P**, a binary one. Let α, β, α_i, β_i, α_{ij}, β_{ij}, i, $j = 1, 2 \ldots$ be variables ranging over the set of atomic formulas and negations of atomic formulas which occur in $\mathcal{L}$. For any α we let α' be the negation of α if α is positive and the atomic formula occurring in α if α is negative. A *binary disjunction* is any disjunction of the form $\alpha \vee \beta$. For any α, β we let $\alpha \dot{\vee} \beta$ be a variable ranging over the set $\{\alpha \vee \beta, \beta \vee \alpha\}$. A *chain* from α to β is any finite set of binary disjunctions of the form as $\alpha_1 \dot{\vee} \alpha_2, \alpha_2' \dot{\vee} \alpha_3, \ldots, \alpha_{n-1}' \dot{\vee} \alpha_n$ where $\alpha_1 = \alpha$ and $\alpha_n = \beta$. The following lemma appears in obviously equivalent but slightly different forms as Corollary 2.2 and as theorem 2.1 in [2]. We refer the reader to [2] for its proof which is an application of mathematical induction.

Lemma 1. *For any set Σ of conjunctions of binary disjunctions, Σ is inconsistent if and only if there are two chains formed with these binary disjunctions, one from α to α and one from α' to α' for some α.*

To determine consistency of quantified formulas we use a system of quantificational deduction described on page 111 of [3]. This system has two rules of derivation, called **UI** and **EI**. Let $X(x)$ be any formula in which x occurs free and let $X(y)$ be like $X(x)$ except that $X(y)$ has y free everywhere that $X(x)$ has x free. Then **UI** and **EI** are the rules whereby we respectively pass from a formula of the form $(\forall x)X(x)$ or $(\exists x)X(x)$ to the corresponding formula of the form $X(y)$. These rules enable one to extend any finite sequence of prenex formulas by successively adjoining formulas that follow by **UI** or **EI** from some predecessor. But the restriction is imposed that the variable y of the formula $X(y)$ introduced in an **EI** step must not be free in any previous formula of the sequence. Any sequence

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obtained by such an extension will be called a *derivation* from the original finite sequence. A set of formulas is *derivable* from a given finite sequence of prenex formulas if there is a derivation from the finite sequence in which each formula of the set occurs. A given finite set of prenex formulas is inconsistent if and only if a truth functionally inconsistent set of quantifierless formulas is derivable from any finite sequence in which each member of the given set occurs (cf. page 111 [3]).

Theorem 1[1]. *There does not exist a formula X in $\mathcal{L}$ which is in prenex conjunctive normal form in which all disjunctions are binary and which is logically equivalent to*

$$Y = (\forall x)\mathsf{P}xx \wedge (\forall x)(\forall y)(\forall z)[\mathsf{P}xy \wedge \mathsf{P}yz \supset \mathsf{P}xz].$$

Proof. For reductio ad absurdum, suppose that X is a formula with the properties mentioned in the theorem. (1) Let X^* be obtained from X by conjoining the two formulas (i) $(\forall x)\mathsf{P}xx \vee (\forall x)(\forall y)\mathsf{P}xy$ and (ii) $(\forall x)\mathsf{P}xx \vee (\forall x)(\forall y)\neg\mathsf{P}xy$ and then exporting quantifiers to the prefix changing individual variables to avoid collision. Since $(\forall x)\mathsf{P}xx$ is a logical consequence of X, it follows that X^* is logically equivalent to X and thus X^* also satisfies the properties mentioned in the theorem.

Let $Z = (\exists x)(\exists y)(\exists z)[[\neg\mathsf{P}xx \vee \mathsf{P}xy] \wedge [\neg\mathsf{P}xx \vee \mathsf{P}yz] \wedge [\neg\mathsf{P}xx \vee \neg\mathsf{P}xz]]$.

(2) Observe[2] that Z is logically equivalent to the negation of Y and that if any one of the conjuncts of the matrix of Z is deleted the resulting formula is consistent with Y and thus also with X^*. It follows that there is a derivation $\mathcal{D}$ from Z and X^* of an inconsistent set of quantifierless formulas.

Let $Z' = [\neg\mathsf{P}aa \vee \mathsf{P}ab] \wedge [\neg\mathsf{P}aa \vee \mathsf{P}bc] \wedge [\neg\mathsf{P}aa \vee \neg\mathsf{P}ac]$ where a, b and c are distinct new individual variables not otherwise occurring in $\mathcal{D}$. We modify $\mathcal{D}$ as follows. First delete all occurrences of formulas obtained from Z in $\mathcal{D}$, except for the initial occurrence of Z. Then in the resulting derivation replace all occurrences of any free individual variable that also occurs free in any formula that was deleted, with the corresponding variable a, b or c. We see that the "corresponding variable" is well defined by noting that a variable occurring free in a formula obtained from Z is introduced by rule EI and by comparing Z' with the matrix of Z. Finally we introduce into the resulting sequence of formulas the three line derivation of Z' just following the remaining occurrence of Z. It follows that the result is a derivation $\mathcal{E}$ of an inconsistent set of quantifierless formulas in which Z' is the only quantifierless formula obtained from Z. To see that the set $\mathcal{J}_{\mathcal{E}}$ of quantifierless formulas of $\mathcal{E}$ is inconsistent, observe that it is obtained from the set of quantifierless formulas of $\mathcal{D}$ by replacing all occurrences of some individual variables with a, b or c in such a way that any coincidences of individual variables are retained.

By Lemma 1 it follows that there are two chains, c and c' in $\mathcal{J}_{\mathcal{E}}$ from α to α and from α' to α' for some signed atomic formula α. For any signed atomic formulas α and β we say that α is *joined* to β by the chain d in case d is a chain from α' to β'. We will call $\mathsf{P}ab$, $\mathsf{P}bc$, and $\neg\mathsf{P}ac$, which occur in Z', *terminal disjuncts*. Let $\mathcal{J}_{\mathcal{C}}{}^*$ be the set obtained from $\mathcal{J}_{\mathcal{C}}$ by deleting Z'. (3) There does not exist a chain in $\mathcal{J}_{\mathcal{C}}{}^*$ which joins two terminal disjuncts or one terminal disjunct to itself. For, suppose that e were such a chain. Then e could be extended, using no more than two binary disjunctions from Z', to form a chain from $\neg \mathsf{P}aa$ to $\neg \mathsf{P}aa$. Then, by (1) (i), we could augment our derivation to obtain an additional binary disjunction from X^* which, by itself, forms a chain from $\mathsf{P}aa$ to $\mathsf{P}aa$. But these chains would not use one of the binary disjunctions of Z'. So we would contradict (2); that is, we would be able to derive an inconsistent set of quantifierless formulas from X^* together with a formula obtained by deleting one conjunct of the matrix of Z.

We see that in each of the chains c and c' all occurrences of binary disjunctions of Z' are directed the same way. That is, the terminal disjunct of all occurrences of binary disjunctions from Z' are toward the same end of the chain, otherwise there would be a portion of the chain in $\mathcal{J}_{\mathcal{C}}$ which achieved a change of direction and this would contradict (3) above.

In c and c', if there are any occurrences of binary disjunctions from Z', then the *last* Z'*-disjunct* is the last occurrence toward the end to which terminal disjuncts are directed, of a disjunction from Z'. Let $\bar{c}$ and $\overline{c'}$ be the subchains of c and c', respectively, which consist of the last Z'*-disjunction,* if any, together with the remaining portion of the chain to the end toward which the terminal disjuncts are directed or which consist of all of c or c', respectively, if it contains no occurrences of binary disjunctions from Z'. It follows that neither $\bar{c}$ nor $\overline{c'}$ contains more than one occurrence of a binary disjunction from Z' and also, by (1), (i), (ii), it follows that they may be extended, if necessary, to form chains from α to α and from α' to α' respectively, by introducing only binary disjunctions which can be obtained from X^*. Thus we are able to derive an inconsistent set of quantifierless formulas from X^* together with a formula obtained by deleting one conjunct of the matrix of Z. This contradicts (2) above, so we conclude that there is no formula X as supposed.

Corollary 1. *There does not exist a formula W in L which is in prenex conjunctive normal form in which all disjunctions are binary and which is logically equivalent to* $T = (\forall x)(\forall y)(\forall z)[\mathsf{P}xy \wedge \mathsf{P}yz \supset \mathsf{P}xz]$.

Proof. Given a formula W with the properties mentioned above we could conjoin $(\forall x)[\mathsf{P}xx \vee \mathsf{P}xx]$ obtaining a formula logically equivalent to Y of Theorem 1, then by exporting quantifiers we would obtain a formula with the properties mentioned for X in Theorem 1.

NOTES

1. If this theorem were not true then, by the results of [1], the class of prenex conjunctive formulas in which all disjunctions are binary in a pure first order logic with an extra binary predicate symbol, would form a reduction class for satisfiability.

2. Z is obtained from Y by first rewriting Y in a prenex disjunctive normal form, affixing a negation symbol, and then importing the negation symbol.

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