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SINGLE AXIOM SCHEMATA FOR D AND S

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Given below* are the axiom schemata **D** and **S**, which are each sufficient for a complete propositional calculus based on a Sheffer functor,* with the appropriate rule of detachment, **I** or **I***. The schemata were inspired by the axiom of Łukasiewicz [2] for a propositional calculus with variables for propositional functions. But here we axiomatize precisely propositional calculus, not the slightly larger system of Łukasiewicz, and therefore have no need for a rule of substitution for functional variables. Further, it should be remarked that although the correspondence between the two approaches is close, such an axiomatization as that of Meredith [3], viz., $C\delta\delta O\delta p$, does not admit of an easy interpretation as an axiom schema.

Łukasiewicz' axiom is essentially a "law of bivalency": $C\delta COOC\delta O\delta p$. We will modify this to adapt it to our purposes, to the forms:

D $D(\alpha:\mathfrak{a}/\beta)DD(\alpha:\mathfrak{a}/D\beta\beta)D(\alpha:\mathfrak{a}/\gamma)(\alpha:\mathfrak{a}/\gamma)D(\alpha:\mathfrak{a}/D\beta\beta)D(\alpha:\mathfrak{a}/\gamma)(\alpha:\mathfrak{a}/\gamma)$

S $SSS(\alpha:\mathfrak{a}/\beta)(\alpha:\mathfrak{a}/\beta)SSS(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/\gamma)SS(\alpha:\mathfrak{a}/S\beta\beta)$
$(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/\gamma)SS(\alpha:\mathfrak{a}/\beta)(\alpha:\mathfrak{a}/\beta)SSS(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/S\beta\beta)$
$(\alpha:\mathfrak{a}/\gamma)SS(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/S\beta\beta)(\alpha:\mathfrak{a}/\gamma)$

And our rules of detachment are to be the relatively weak rules

$$\mathbf{I}\quad \begin{array}{c} D\alpha D\beta\beta \\ \alpha \\ \hline \beta \end{array} \qquad\qquad \mathbf{I}^*\quad \begin{array}{c} SSS\alpha\alpha\beta SS\alpha\alpha\beta \\ \alpha \\ \hline \beta \end{array}$$

In our axiom schemata, and throughout this paper, we use the following conventions: Lower case Greek letters are variables for well formed formulas; lower case German letters are variables for the propositional variables (which are lower case Latin letters); an expression of the kind $\alpha:\mathfrak{a}/\beta$ means the formula resulting from the formula α by substitution of the formula β for every occurrence of the variable $\mathfrak{a}$ in α; an asterisk indicates replacement of D by S throughout the formula in question.

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*The terminology here used is largely explained in [4]. The suggestion that these axiomatizations be treated as schemata is due to Prof. Sobociński.

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§*1*. *Consequences of* **D**. To make the steps in the deductions to follow more perspicuous, we will use only particular instances of the schema D, and a rule of substitution for propositional variables. The latter quite obviously holds in this system, as well as the rule

$$\mathrm{II}\qquad \frac{\begin{array}{c}\alpha:\mathfrak{a}/\beta\\ \alpha:\mathfrak{a}/D\beta\beta\end{array}}{\alpha:\mathfrak{a}/\mathfrak{b}}$$

Only two instances of D are needed, *viz.*:

1. $DDpDDDppDqqDppDqq$ [$\alpha = p$, $\mathfrak{a} = p$]
2. $DpDDpDppDpDpp$ [$\alpha = p$, $\mathfrak{a} = q$]

from which we continue as follows:

3. $DpDDqDqqDqDqq$ [1: q/Dpp; 2; II]
*4. $DpDDqDppDqDpp$ [1: q/p; 2; II]
*5. $DpDpp$ [3: $p/3$, q/p; 3; I]
6. $DDppDDpqDpq$ [5: p/Dpp; 3: p/Dpp, q/p; II]
7. $DrDDDppDDpqDpqDDppDDpqDpq$ [3: $p/6$, q/r; 6; I]
*8. $DDpDDqrDqrDDqDDprDprDqDDprDpr$
[5: $p/DpDDprDpr$; 7: $r/DpDDDpprDDppr$, q/r; II]
*9. $DDqpDDpqDpq$ [5: p/Dpp; 3: $p/DpDDDpprDDppr$, q/r; II]
10. $DrDDqDDpDppDpDppDqDDpDppDpDpp$ [4: $p/4$, q/r; 4; I]
11. $DDpDDppDppDDDppqDDpqDpqDDDppqDDpqDpq$
[10: $r/DpDDppDpp$, $q/DDppDpp$; 4: $p/DpDDppDpp$, $q/DDppq$; II]
*12. $DDpDqqDDDqrDDprDprDDqrDDprDpr$ [3: $p/DpDqq$, q/Dpr; 11; II]
13. $DDpqDDDpDqqDDprDprDDpDqqDDprDpr$
[4: $q/DpDqq$, p/Dpq; 3: p/Dpq, $q/DpDqq$; II]
*14. $DDpDqqDDDpDrrDDpDqrDpDqrDDpDrrDDpDqrDpDqr$
[2: $p/DpDqq$; 13: q/Dqq, $r/DpDqq$; II]
15. $DDppp$ [9: q/p, p/Dpp; 5; I]
16. $DqDDDpppDDppp$ [4: $p/15$; 15; I]
17. $DrDDqDDDpppDDpppDqDDDpppDDppp$ [4: $p/16$, q/r; 16; I]
18. $DDpDqqDDDpqDDppDppDDpqDDppDpp$
[3: $p/DpDqq$, q/Dqq; 4: $p/DDqqDqq$, $q/DDqqq$; II]
*19. $DDpDqqDDDrqDDrpDrpDDrqDDrpDrp$ [18; 17: $r/DpDqq$, $q/DDppq$; II]
*20. $DDrDpqDDrDppDrDpp$ [19: p/Dpp, q/Dpq; 6; I]
21. $DDppDqpDqp$ [5: p/Dpp; 16: q/Dpp; II]
*22. $DDrDqpDDrDppDrDpp$ [19: p/Dpp, q/Dqp; 21; I]

To describe a normal form for propositions, we make use of an idea of Gentzen [1]. If Γ is a (finite) sequence of well-formed formulas, $\Gamma\rightarrow$ denotes the negation of the conjunction of all of them. Then this serves as a kind of generalization of the Sheffer functor D. Formally, we give an inductive definition:

$$\begin{array}{ll}\alpha,\beta\rightarrow & \text{for } D\alpha\beta\\ \Gamma,\alpha,\beta\rightarrow & \text{for } \Gamma,\ DD\alpha\beta D\alpha\beta\rightarrow \text{ (if } \Gamma \text{ not empty)}\end{array}$$

The properties which we can demonstrate of these "sequents" will be investigated below, in particular that the order of the members is immaterial, and that this simplification of the members of the sequent is possible, *salva veritate*:

III $$\frac{\Gamma,\alpha,DD\beta\gamma D\delta\epsilon\rightarrow}{\Gamma,\alpha,\beta,\gamma\ \rightarrow\ \Gamma,\alpha,\delta,\epsilon\rightarrow}$$

Use of **III** in reducing the complexity of a sequent will end at the point that all the members are of the form α or $D\alpha\alpha$. In that case, we say that the sequent is in normal form.

Theorem: If a sequent in normal form is a tautology, it follows from **D** *by use of the rule* **I**.

Given the demonstrability of the rule **III**, this theorem (to be proved below) yields the completeness of the propositional calculus. Note that this also yields a simple decision procedure—reduce a sequent by **III** to normal form, and then check whether in the normal form both α and $D\alpha\alpha$, for some variable α, occur. In such a case the formula is a tautology (since α and $D\alpha\alpha$ are incompatable), otherwise, it is not a tautology (for any number of distinct variables are mutually compatable). As an example of the application of this method, begin with the axiom of Nicod for D:

$$\begin{array}{c} DDpDqrDDtDttDDsqDDpsDps \\ DpDqr,\ DDtDttDDsqDDpsDps \rightarrow \\ \hline \begin{array}{cc} \begin{array}[t]{c} DpDqr,\ t,\ Dtt \rightarrow \\ \text{(tautology)} \end{array} & \begin{array}[t]{c} DpDqr,\ Dsq,\ p,\ s \rightarrow \\ \hline \begin{array}{cc} \begin{array}[t]{c} Dpp,\ Dsq,\ p,\ s \rightarrow \\ \text{(tautology)} \end{array} & \begin{array}[t]{c} q,\ r,\ Dsq,\ p,\ s \rightarrow \\ \hline \begin{array}{cc} \begin{array}[t]{c} q,\ r,\ Dss,\ p,\ s \rightarrow \\ \text{(tautology)} \end{array} & \begin{array}[t]{c} q,\ r,\ Dqq,\ p,\ s \rightarrow \\ \text{(tautology)} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array}$$

To prove completeness, we require only the ten starred laws above, and the rules of substitution and detachment, so the proof may be readily adapted to other axiomatizations of D.

First we will define the notion of equivalence of formulas, which will mean that they not only have the same truth value, but also that they follow equally well from our axioms.

i) $\alpha \sim \beta$ *if and only if* $D\alpha D\beta\beta$ *and* $D\beta D\alpha\alpha$ *are both laws*

It is clear that this notion of equivalence is, in fact, an equivalence relation:

$\alpha \sim \alpha$	[*by* 5]
$\alpha \sim \beta$ *implies* $\beta \sim \alpha$	[*by definition*]
$\alpha \sim \beta$ *and* $\beta \sim \gamma$ *imply* $\alpha \sim \gamma$	[12]
$\alpha \sim \beta$ *implies* $D\alpha\gamma \sim D\beta\gamma$	[13]
$\alpha \sim \beta$ *implies* $D\gamma\alpha \sim D\gamma\beta$	[19]

and also that if $\alpha \sim \beta$, then α is a law (or tautology) if and only if β is a law (or tautology).

For an inductive step for definition of equivalence of a set of formulas with another, we may generalize from this case:

ij) $\alpha \sim \beta;\gamma$ *if and only if* $D\alpha D\beta\beta$, $D\alpha D\gamma\gamma$, *and* $D\beta DD\gamma D\alpha\alpha D\gamma D\alpha\alpha$ *are all laws*.

For this notion of equivalence, we need these properties:

$\alpha \sim \beta;\gamma$ *implies*

a) $\Gamma, D\alpha\alpha \rightarrow \sim \Gamma, D\beta\beta \rightarrow; \Gamma, D\gamma\gamma \rightarrow$

b) α *is a law* (*or tautology*) *if and only if* β *and* γ *are laws* (*or tautologies*)

(Esthetic reasons will have us drop the arrow from now on in such expressions as in (*a*) above.)

We may now prove:

Lemma 1: $\Gamma,\alpha,\Delta \sim \Gamma,\Delta,\alpha$

Proof: By induction on the length of Δ

i)
$$\begin{aligned} \Gamma,\alpha,\beta &= \Gamma, DD\alpha\beta D\alpha\beta \\ &\sim \Gamma, DD\beta\alpha D\beta\alpha \quad [9;\ \textit{properties of} \sim] \\ &= \Gamma,\beta,\alpha \end{aligned}$$

ij)
$$\begin{aligned} \Gamma,\alpha,\beta,\Delta &= \Gamma, DD\alpha DD\beta\Delta D\beta\Delta D\alpha DD\beta\Delta D\beta\Delta \\ &\sim \Gamma, DD\beta DD\alpha\Delta D\alpha\Delta D\beta DD\alpha\Delta D\alpha\Delta \\ &\qquad\qquad [8;\ \textit{properties of} \sim] \\ &= \Gamma, \beta,\alpha,\Delta \\ &\sim \Gamma,\beta,\Delta,\alpha \quad [\textit{induction hypothesis}] \end{aligned}$$

Note that this follows also if Γ or Δ were empty.

Lemma 2: *Every formula is equivalent to a set of formulas in normal form*.

Proof: The essential part of this is in verifying the applicability of the rule **III**. Say that $\Delta, D\beta\gamma, \Theta$ is the formula in question. By Lemma 1, it is equivalent to $\Delta,\Theta,D\beta\gamma$.

For completeness, consider first the case that $D\beta\gamma$ is the only formula in the sequence—then by definition, $D\beta\gamma = \beta,\gamma$. So we may assume that we have $\Gamma,\alpha, D\beta\gamma$.

$$\begin{aligned} \Gamma,\alpha,D\beta\gamma &= \Gamma, DD\alpha D\beta\gamma D\alpha D\beta\gamma \\ &\sim \Gamma, DD\alpha D\beta\beta D\alpha D\beta\beta;\ \Gamma, DD\alpha D\gamma\gamma D\alpha D\gamma\gamma \\ &\qquad\qquad [14, 20, 22] \\ &= \Gamma,\alpha,D\beta\beta;\ \Gamma,\alpha,D\gamma\gamma \end{aligned}$$

if β or γ is not a variable, then it is of the form $D\delta\epsilon$, and by definition,

$$\Gamma,\alpha,DD\delta\epsilon D\delta\epsilon = \Gamma,\alpha,\delta,\epsilon$$

Whence the rule **III** is shown correct.

To show the completeness of **S** with **I***, we use the theorem of [4], p. 215, that $S\alpha^*\beta^*$ is a tautology if and only if α and β are tautologies. By following the steps above with the appropriate changes for *S*, it can be shown that every tautology of the form $S\alpha^*\beta^*$ follows from the axiom schema. One tautology of the form is

$$SSSSppSppSSSSqqSqqSpqSSSqqSqqSpqSSSppSppSSSSqqSqqSpqSSSqqSqqSpq$$

If α and β are any tautologies, then $S\alpha^*\alpha^*$ and $S\beta^*\beta^*$ are laws, so if we substitute in this formula: p/α^*, q/β^*, by I* we have that $S\alpha^*\beta^*$ is a law. Hence the S system is complete.

If we were to form a direct proof of completeness with S, it would be appropriate to take our sequents of the form $\rightarrow \Gamma$, i.e., that the alternation of the members of Γ is true. (Since S is negation of conjunction, this recalls the disjunctive normal form.) It is interesting to note that because of the duality of D and S, reduction rules for D formulas to the left of an arrow are identical to reduction rules for S formulas to the right of an arrow, and the criterion for tautology of an S sequent $\rightarrow \Gamma$, where every member of Γ is either α or $S\alpha\alpha$ is simply that for some variable α, either α or $S\alpha\alpha$ appears in Γ, the identical rule as for D. The general reduction rules for D and S are:

$$\frac{\Gamma, D\alpha\beta \rightarrow \Delta}{\Gamma \rightarrow \alpha, \Delta \quad \Gamma \rightarrow \beta, \Delta} \qquad \frac{\Gamma \rightarrow \Delta, D\alpha\beta}{\Gamma, \alpha, \beta \rightarrow \Delta}$$

$$\frac{\Delta \rightarrow \Gamma, S\alpha\beta}{\Delta, \alpha \rightarrow \Gamma \quad \Delta, \beta \rightarrow \Gamma} \qquad \frac{\Delta, S\alpha\beta \rightarrow \Gamma}{\Delta \rightarrow \alpha, \beta, \Gamma}$$

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