Notre Dame Journal of Formal Logic
Volume VII, Number 4, October 1966

THE IRREDUCIBLE GENERATING SETS OF 2-PLACE FUNCTIONS IN THE 2-VALUED LOGIC

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A set $\mathfrak{S}$ of functions is said to be an irreducible generating set—or briefly, *irreducible*—if no proper subset of $\mathfrak{S}$ generates the set generated by $\mathfrak{S}$ (cf. [2] and [1]). For instance, among the 16 2-place functions in the 2-valued logic the set $\{C,0\}$ which generates $\{C,C',1,0\}$ is irreducible since this latter set is not generated by any proper subset of $\{C,0\}$. The set $\mathfrak{S} = \{C,0,1\}$ is not irreducible since $\mathfrak{S}$ likewise generates $\{C,C',1,0\}$, which is also generated by a proper subset of $\mathfrak{S}$.

In [1], Calabrese, whose notation we adopt, lists all subalgebras of the Menger algebra $\mathfrak{F}$ of the 16 functions mentioned above. For each of those 122 algebras, with the exception of $\mathfrak{F}$ itself, Calabrese also lists all irreducible generating sets with the minimum number of functions. He furthermore gives an example of a subalgebra generated by irreducible pairs, which, however, can also be generated by irreducible triples. The purpose of the present note is to present a complete list of the irreducible subsets of $\mathfrak{F}$. Those listed in [1] will be referred to without being restated. But all irreducible sets generating $\mathfrak{F}$ or an algebra that can also be generated by sets with fewer functions will be explicitly listed.

We further simplify the exposition by the following two abbreviations:

1) If 2 or 4 algebras are transforms of one another we only list the irreducible sets generating one of them, say $\mathfrak{S}$, and prefix an operator variable as defined in [1], namely,

τ, if $\mathfrak{S}$ is one of 4 unlike transforms, $\mathfrak{S}$, $\delta\mathfrak{S}$, $\sigma\mathfrak{S}$, $\rho\mathfrak{S}$;
τ_1, if $\mathfrak{S} = \delta\mathfrak{S} \neq \sigma\mathfrak{S} = \rho\mathfrak{S}$;
τ_2, if $\mathfrak{S} = \sigma\mathfrak{S} \neq \delta\mathfrak{S} = \rho\mathfrak{S}$.

For instance, if $\{F_1, F_2\}$ is irreducible and generates $\mathfrak{S} \neq \sigma\mathfrak{S}$, then $\sigma\{F_1,F_2\} = \{\sigma F_1, \sigma F_2\}$ is irreducible and generates $\sigma\mathfrak{S}$. We would only list $\tau_1\{F_1, F_2\}$. For example, $\tau_1\{C,0\}$ would indicate that not only $\{C,0\}$ but also $\{\sigma C, \sigma 0\} = \{D,0\}$ is an irreducible set.

*The author wishes to thank Professor K. Menger for his suggestions in making the exposition of this paper clearer.

Received February 28, 1966

2) We write $\langle F_1 F_2 \ldots F_k \rangle_n$, where n is a positive integer less than k, in order to express that any n consecutive functions in the sequence $F_1, F_2, \ldots, F_k$ constitute an irreducible set. For example, $\langle 0,C,C',1 \rangle_2$ indicates that $\{0,C\}$, $\{C,C'\}$, and $\{C',1\}$ are irreducible pairs. (These are the three irreducible sets generating the algebra $\mathrm{II}_\rho(12)$ listed in [1].) For the sake of uniformity, we also write $\langle F_1 F_2 \ldots F_k \rangle_k$ or, briefly, $\langle F_1 F_2 \ldots F_k \rangle$ in order to indicate that the entire set $\{F_1, F_2, \ldots, F_k\}$ is irreducible.

As in [1], the set of the 6 idempotent functions $1, 0, I, J, A', B$ will be denoted by $\mathfrak{F}_0$.

Theorem. There are 491 *irreducible subsets of* $\mathfrak{F}$.

I. 16 *singles, each consisting of one of the functions.*

II. 106 *pairs, namely, all except the following* 14 *pairs:*

$$\tau\{C,1\},\ \tau_1\{I,I'\},\ \tau_2\{A,1\},\ \tau_2\{B',1\},\ \tau_2\{A,A'\},\ \tau_2\{E,1\}.$$

III. 288 *triples, namely,*

- **a.** *The* 20 *triples contained in* $\mathfrak{F}_0$
- **b.** *The* 54 *triples listed in* [1] *generating algebras not generated by any pair of functions.*
- **c.** 42 *irreducible triples generating subalgebras of* $\mathfrak{F}$ *that can also be generated by pairs of functions, namely,*

$$\tau\langle IE0CA'C'1ID'D\rangle_3,\ \tau_1\langle EE'I\rangle,\ \tau_2\langle CD0\rangle,\ \tau_2\langle 0EBE'1\rangle_3.$$

- **d.** 172 *triples generating* $\mathfrak{F}$, *namely,*

$$\tau\langle AB'C\rangle, \tau\langle DIABCD'AIJ'BAC'J'A'CE'AIEC'JI'CJ'DE'BI'ECAJ'E'B'CEJ'D'\rangle_3$$
$$\tau_1\langle IAB'I'\rangle_3,\ \tau_1\langle ICD'I'\rangle_3,\ \tau_1\langle JCD'J'\rangle_3,\ \tau_1\langle A'BI'\rangle$$
$$\tau_2\langle AIJ\rangle,\ \tau_2\langle ACDB'\rangle_3,\ \tau_2\langle AI'J'A'\rangle_3.$$

IV. 75 *quadruples, namely,*

- **a.** *The* 15 *quadruples contained in* $\mathfrak{F}_0$.
- **b.** 60 *quadruples generating* $\mathfrak{F}$,

$$\tau\langle BD0A'EIE'\rangle_4,\ \tau\langle C'A'CJ0EBD\rangle_4,\ \tau\langle BDJ0CIE\rangle_4,$$
$$\tau_1\langle A'BCC'\rangle,\ \tau_1\langle CC'IJ\rangle,$$
$$\tau_2\langle A'C0DB\rangle_4.$$

V. *The* 5 *quintuples contained in* $\mathfrak{F}_0$.

VI. *The sextuple* $\mathfrak{F}_0$.

That each of the listed sets is irreducible can be readily verified; that there are no other irreducible sets follows from the method used in determining irreducible sets. If to the set $[F]$ a function G not contained in $[F]$ is adjoined, one obtains an irreducible pair unless $[G]$ contains F. In this way one easily determines all irreducible pairs. In order to obtain an irreducible triple, consider an irreducible pair $\{F,G\}$ and adjoin a function H not contained in $[F,G]$ such that $[F,G,H] \neq [F,H]$, $[G,H]$. If, for example, to $[A',E] = \{A',B,E,1\}$ one adjoins E' one obtains an irreducible triple since

$[A,E,E'] = \{A,A',B,B',E,E',1,0\}$ is not equal to either $[A',E'] = \{A',E',B,0\}$ or $[E,E'] = \{E,E',1,0\}$. If, however, to $\{A',E\}$ one adjoins B', the triple is not irreducible since $[A',E,B'] = [A',B'] = \{A,A',B,B',E,E',1,0\}$. By thus extending all irreducible pairs we have obtained all irreducible triples. Similarly, by adjoining to any such triple $\{F,G,H\}$ a function K not contained in $[F,G,H]$ we obtain an irreducible quadruple provided $[F,G,H,K] \neq [F,G,K]$, $[F,H,K]$, $[G,H,K]$. By adjoining A' to $[B,D,J] = \{B,D,J,C,1\}$, for instance, one does not obtain an irreducible quadruple since the resulting $[B,D,J,A'] = [B,D,A']$. Adjoining D' one obtains $\mathfrak{F}$, not generated by any subset of $\{B,D,J,D'\}$.

In conclusion, we remark:

1) *There are no irreducible quadruples generating proper subsets of* $\mathfrak{F}$ *other than subsets of* $\mathfrak{F}_0$; *in other words, all irreducible quadruples including at least one nonidempotent function generate* $\mathfrak{F}$.

2) *Consequently, there are no irreducible quintuples including nonidempotent functions.*

3) *All triples generating* $\mathfrak{F}$ *are necessarily irreducible since, by Theorem* 7 *in* [1], *no pair of functions generates* $\mathfrak{F}$.

4) *All irreducible sets of the form* IIIc *or* IVb *include either* 1 *or* 0 *or a pair* $\{F,F'\}$; *they furthermore generate* v-*closed algebras.*

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