

THE MENGER ALGEBRAS OF 2-PLACE FUNCTIONS
 IN THE 2-VALUED LOGIC

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A Menger algebra of 2-place functions over a set Δ is (cf. [5]) a set \mathfrak{C} of functions mapping $\Delta \times \Delta$ into Δ , which is closed with respect to substitution or composition, i.e., has the property that, for any three functions F, G, H belonging to \mathfrak{C} , the composite function $F(G,H)$ belongs to \mathfrak{C} . Here, $F(G,H)$ is the function assuming the value $F(G(x,y), H(x,y))$ for each (x,y) in $\Delta \times \Delta$. The purpose of this paper is to list all the Menger algebras of 2-place functions over $\{0,1\}$. The correspondence between these functions and the binary operators of the 2-valued logic is obvious.

We denote the 16 functions (cf. [1], [2]) by

	A	B	C	D	E	I	J	I	A'	B'	C'	D'	E'	I'	J'	0
(0,0)	1	0	1	1	1	0	0	1	0	1	0	0	0	1	1	0
(0,1)	1	1	1	0	0	0	1	1	0	0	0	1	1	1	0	0
(1,0)	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0
(1,1)	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

The smallest Menger algebra containing, as a subset, the set of functions $\{F_1, \dots, F_k\}$ is said to be *generated*¹ by $\{F_1, \dots, F_k\}$ (cf. [1] and [3]) and is denoted by $[F_1, \dots, F_k]$. If $[F] = \{F\}$, or, which amounts to the same, if $F(F,F) = F$, then F is called *idempotent* (cf. [1]). We denote the set of all 16 functions as well as the algebra consisting of these functions by \mathfrak{F} . The set of the idempotent functions, that is $\{I, 0, I, J, A', B\}$, is denoted by \mathfrak{F}_0 . We further introduce an operator ν in \mathfrak{F} by defining νF as the function whose value for each (x,y) in $\{0,1\} \times \{0,1\}$ is 0 or 1 according as $F(x,y)$ is 1 or 0. An algebra is said to be ν -closed if, whenever F is in it, νF is also.

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¹It should be remembered that in this purely algebraic sense of generation, the function A (corresponding to Sheffer's stroke) generates only 4 of the 16 functions in \mathfrak{F} (cf. [1]).

The 122 subalgebras of \mathfrak{F} listed in our Principal Theorem are found by the following procedure.

- 1) All subsets of \mathfrak{F}_0 are algebras (Theorem 2)².
- 2) All ν -closed algebras, with 2 exceptions, are generated by pairs of functions (Theorem 5).
- 3) All subalgebras of \mathfrak{F} not in 1) or 2) are subsets of 2 octuples of functions both of which are algebras containing 5 idempotent functions (Theorem 6).

We further define three operators by setting, for each F in \mathfrak{F} ,

$$F(J, I) = \sigma F, \quad \nu F(I', J') = \delta F, \quad \nu F(J', I') = \rho F.$$

If ι is the identity operator, then clearly,

$$\sigma^2 = \delta^2 = \rho^2 = \iota \quad \text{and} \quad \sigma\delta = \delta\sigma = \rho$$

For each subset \mathfrak{G} of \mathfrak{F} , we denote by $\sigma\mathfrak{G}$, $\delta\mathfrak{G}$, and $\rho\mathfrak{G}$ the sets of the functions σF , δF , and ρF for all F in \mathfrak{G} respectively. These sets will be said to be *symmetric*, *dual*, and *dual-symmetric* to \mathfrak{G} , respectively, and will also be called the three *transforms* of \mathfrak{G} . Clearly, an algebra that is identical with two of its transforms is also identical with the third. If G is a transform of F , then F is a transform of G . We also set $\mathfrak{G} = \iota\mathfrak{G}$.

The subalgebras of \mathfrak{F} can be classified in various ways, e.g., according to the numbers of functions contained in the algebras (these numbers will be seen to be 1, 2, 3, 4, 5, 6, 8, 16), or according to the minimum numbers of functions generating the algebras (these numbers will be seen to be 1, 2, 3, 4, 5, 6). In our Principal Theorem, we arrange the algebras according to their invariance properties with regard to the operators σ , δ , ρ . We list first the algebras that are identical with their three transforms; then the algebras that are identical with exactly one of their transforms; and finally the algebras that belong to a quadruple of different transforms. Sets that are mutual transforms will be listed on one and the same line. For the first algebra in each line, irreducible generating sets will be given. Here, a set $\{F_1, \dots, F_k\}$ is said to be *irreducible* if $[F_1, \dots, F_k]$ is not generated by a proper subset of $\{F_1, \dots, F_k\}$. From the irreducible generating set of the first algebra in each line, one obtains the irreducible generating sets of the transforms of that algebra by virtue of the following

Theorem 1. *If $\mathfrak{G} = [F_1, \dots, F_k]$, then $\tau\mathfrak{G} = [\tau F_1, \dots, \tau F_k]$ or, briefly, $\tau[F_1, \dots, F_k] = [\tau F_1, \dots, \tau F_k]$, where $\tau = \sigma, \delta, \rho$.*

It will be shown that, besides the 15 pairs of idempotent functions, 47 algebras are generated by irreducible pairs, and 12, by irreducible triples. All these irreducible pairs and triples will be listed in the Principal Theorem. But we point out that the set $\{A, A', B, B', E, E', I, \theta\}$, for instance, has not only irreducible generating pairs, such as $\{A, B\}$ but is also

²For $\Delta = \{0, 1, 2\}$, Mr. H. I. Whitlock has found sets of idempotent functions over Δ that are not closed under substitution and hence are not Menger algebras.

generated by the irreducible triples $\{E, E', A'\}$ and $\{E, E', B'\}$. A systematic study of all irreducible generating sets by H. Skala is contained in [4].

In order to facilitate the reading of the following, we list the 16 functions with their transforms:

$$\begin{aligned} \iota: & A \ B \ C \ D \ E \ I \ J \ I \ A' \ B' \ C' \ D' \ E' \ I' \ J' \ 0 \\ \sigma: & A \ B \ D \ C \ E \ J \ I \ I \ A' \ B' \ D' \ C' \ E' \ J' \ I' \ 0 \\ \delta: & B' \ A' \ D' \ C' \ E' \ I \ J \ 0 \ B \ A \ C \ C \ E \ I' \ J' \ 1 \\ \rho: & B' \ A' \ C' \ D' \ E' \ J \ I \ 0 \ B \ A \ C \ D \ E \ J' \ I' \ 1 \end{aligned}$$

Moreover we shall use operator variables as follows:

$$\tau = \iota, \sigma, \delta, \text{ or } \rho; \quad \tau_1 = \iota \text{ or } \sigma; \quad \tau_2 = \iota \text{ or } \delta; \quad \tau_3 = \iota \text{ or } \rho.$$

Principal Theorem. *There are exactly 122 Menger algebras of 2-place functions in the 2-valued logic.*

A. 59 algebras that are not subsets of \mathfrak{F}_0 . They fall into three classes:

I. 7 algebras that are symmetric, self-dual, and dual-symmetric, namely, \mathfrak{F} and

- (1) $\{I, I', J, J'\} = [I', J'] = \tau_1[I', J]$
- (2) $\{I, I', J, J', I, 0\} = \tau[I', J', I] = \tau[I', J, I]$
- (3) $\{E, E', I, 0\} = [E, E'] = \tau_2[E, 0]$
- (4) $\{I, I', J, J', E, E', I, 0\} = \tau[E, I']$
- (5) $\{A, A', B, B', E, E', I, 0\} = [A, B'] = \tau_2[A, B] = \tau_2[A, E] = \tau_2[A, E']$
- (6) $\{C, C', D, D', E, E', I, 0\} = \tau[C, D'] = \tau[C, E']$

II. 24 algebras, each identical with exactly one of its transforms, namely,

II_σ. 9 pairs of symmetric but not self-dual algebras:

$$\mathfrak{C} = \sigma\mathfrak{C} \qquad \delta\mathfrak{C} = \rho\mathfrak{C}$$

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|--|---------------------------|
| (1) $\{A, A', I, 0\} = [A]$ | $\{B, B', I, 0\}$ |
| (2) $\{E, I\} = [E]$ | $\{E', 0\}$ |
| (3) $\{E, I, A', B\} = [E, A'] = [E, B]$ | $\{E', 0, B, A'\}$ |
| (4) $\{E, I, I, J\} = \tau_1[E, I]$ | $\{E', 0, J, I\}$ |
| (5) $\{C, D, I\} = [C, D]$ | $\{D', C', 0\}$ |
| (6) $\{E, C, D, I\} = \tau_1[C, E]$ | $\{E', D', C', 0\}$ |
| (7) $\{C, D, B, I\} = [C, D, B]$ | $\{D', C', A', 0\}$ |
| (8) $\{C, D, B, I, J, I\} = \tau_1[C, I, J]$ | $\{D', C', A', I, J, 0\}$ |

$$\begin{aligned}
 (9) \quad \{C, D, B, I, J, E, A', I\} &= [C, D, A'] & \{D', C', A', I, J, E', B, 0\} \\
 &= \tau_1[C, A', B] = \tau_1[C, A', J] = \tau_1[E, A', I] \\
 &= \tau_1[E, B, I] = \tau_1[E, C, I] = \tau_1[E, D, I] \\
 &= \tau_1[E, A', C] = \tau_1[E, B, C].
 \end{aligned}$$

II_δ. 2 pairs of self-dual, nonsymmetric algebras:

$$\mathfrak{G} = \delta \mathfrak{G}. \qquad \qquad \qquad \sigma \mathfrak{G} = \rho \mathfrak{G}$$

$$(10) \quad \{I, I'\} = [I'] \qquad \qquad \qquad \{J', J\}$$

$$(11) \quad \{I, I', I, 0\} = [I', I] \qquad \qquad \qquad \{J', J, I, 0\}$$

II_ρ. 1 pair of dual-symmetric, nonsymmetric algebras:

$$\mathfrak{G} = \rho \mathfrak{G}. \qquad \qquad \qquad \sigma \mathfrak{G} = \rho \mathfrak{G}$$

$$(12) \quad \{C, C', I, 0\} = [C, C'] = \tau_3[C, 0] \qquad \qquad \qquad \{D, D', I, 0\}$$

III. 28 algebras having mutually distinct transforms:

\mathfrak{G}	$\sigma \mathfrak{G}$	$\delta \mathfrak{G}$	$\rho \mathfrak{G}$
(1) $\{C, I\} = [C]$	$\{D, I\}$	$\{D', 0\}$	$\{C', 0\}$
(2) $\{C, I, B\} = [C, B]$	$\{D, I, B\}$	$\{D', 0, A'\}$	$\{C', 0, A\}$
(3) $\{C, I, B, J\} = [C, J]$	$\{D, I, B, I\}$	$\{D', 0, A', J\}$	$\{C', 0, A', I\}$
(4) $\{C, I, B, I, D\} = [C, D, I]$ $= [B, C, I]$	$\{D, I, B, J, C\}$	$\{D', 0, A', I, C'\}$	$\{C', 0, A', J, D'\}$
(5) $\{C, I, I\} = [C, I]$	$\{D, I, J\}$	$\{D', 0, I\}$	$\{C', 0, J\}$
(6) $\{C, I, A', I\} = [C, A']$	$\{D, I, A', J\}$	$\{D', 0, B, J\}$	$\{C', 0, B, I\}$
(7) $\{C, I, A', I, C', 0, A, I'\}$ $= [I', A'] = [A, \mu I]$	$\{D, I, A', J, D'$ $0, A, J'\}$	$\{D', 0, B, I, D,$ $I, B', I'\}$	$\{C', 0, B, J, C',$ $I, B', J'\}$
$= [A, \mu C] = [I', \mu C]$, where $\mu = \iota$ or ν .			

B. The 63 nonempty subsets of \mathfrak{F}_0 , each being its own irreducible generating set.

I₀. 7 algebras that are symmetric, self-dual, and dual-symmetric:
 $\mathfrak{F}_0, \{I, 0\}, \{I, J\}, \{A', B\}, \{I, 0, I, J\}, \{I, 0, A', B\}, \{I, J, A', B\}.$

II₀. 32 algebras each identical with exactly one of its transforms, namely,

II_{0,σ}. 24 symmetric but not self-dual algebras, namely, the 12 proper subalgebras of $\{0, I, A', B\}$ with the exception of the two mentioned in I₀, and their dual algebras.

II_{0,δ}. 8 self-dual but nonsymmetric algebras, namely, $\{I\}, \{I, I, 0\}, \{I, A', B\}, \{I, 0, A', B, I\}$ and their 4 symmetric algebras.

III₀. 24 algebras, namely, the following 6 having altogether 18 transforms: $\{I, I\}$, $\{I, A'\}$, $\{I, I, A'\}$, $\{I, I, B\}$, $\{I, 0, A', I\}$, $\{I, A', B, I\}$.

The Principal Theorem includes two assertions: 1) that each of the 122 listed sets is an algebra (that is, closed under substitution); 2) that no other subalgebras of \mathfrak{F} exist.

For the 63 subsets of \mathfrak{F}_0 , Assertion 1) is an immediate consequence of Theorem 2. *If F, G, H belong to \mathfrak{F}_0 , then $F(G, H)$ belongs to $\{F, G, H\}$.*

This is evident for $F = I, J, 1, 0$. If $F = A'$, then $F(G, H) = F(H, G)$ and moreover, $A'(G, G) = A'(G, I) = G$ and $A'(G, 0) = 0$ for each G in \mathfrak{F}_0 . Furthermore $A'(A', B) = A'(A', I) = A'(A', J) = A'(I, J) = A'$, $A'(B, I) = I$, and $A'(B, J) = J$. The case $F = B$ can be treated similarly.

By the value-sum of the function F we mean the number

$$V_F = F(0, 0) + F(0, 1) + F(1, 0) + F(1, 1).$$

We say the function F is *even* if $V_F \equiv 0 \pmod{2}$, and we prove

Theorem 3. *The set $l(4) = \{E, E', I, I', J, J', 1, 0\}$ consisting of all even functions is an algebra.*

All that requires a proof is that $E(H, K)$ and $E'(H, K)$ belong to $l(4)$ if H and K do. Since $l(4)$ is ν -closed and $E'(H, K) = \nu(E(H, K))$, only E has to be considered. $E(H, K)(x, y) = 1$ or 0 according as $H(x, y) =$ or $\neq K(x, y)$. Hence all that has to be shown is that $H(x, y) \neq K(x, y)$ for an even number of arguments (x, y) . Now the number $V_H - V_K$ is even and equals $\sum(H(x, y) - K(x, y))$, where the summation is over the four arguments (x, y) . Omitting here those of the four terms, if any, that are 0 , we retain just the terms for which $H(x, y) \neq K(x, y)$ and each contributes either 1 or -1 to the even sum. Hence an even number of terms is not zero.

That each of the other nonidempotent sets are algebras can be shown by examining them and by using the following lemma, whose proof is straightforward.

Lemma 1. *Let (a, b) and (c, d) be two elements of Δ , and \mathcal{L} the octuple of functions Q in \mathfrak{F} for which $Q(a, b) = Q(c, d)$. If Q_1 and Q_2 are two functions in \mathcal{L} , then for any function F in \mathfrak{F} , the function $F(Q_1, Q_2)$ belongs to \mathcal{L} . In particular $[Q_1, Q_2]$ is a subset of \mathcal{L} . (cf. [3], p. 40).*

By virtue of Lemma 1, for example, the octuples I(5), I(6) and the four octuples III(7) are algebras since they consist of the functions F satisfying $F(0, 1) = F(1, 0)$, $F(0, 0) = F(1, 1)$ and the other four conditions $F(i, i) = F(m, n)$ ($m \neq n$), respectively. The two octuples II(9) are closed since they consist of the functions F for which $F(1, 1) = 1$ and $F(0, 0) = 0$, respectively.

We now turn to the proof of Assertion 2): all subalgebras of \mathfrak{F} have been listed.

Lemma 2. *For a subalgebra \mathfrak{G} to be ν -closed, it is sufficient that \mathfrak{G} contain, as a subset, at least one of the following ten sets:*

$$\{A\}, \{B\}, \{I'\}, \{J'\}, \{E, 0\}, \{E', I\}, \{C, 0\}, \{C', I\}, \{D, 0\}, \{D', I\}.$$

Clearly, $A(F,F) = B'(F,F) = I'(F,F) = J'(F,F) = E(F,0) = E'(F,1) = C(F,0) = C'(F,1) = D(F,0) = D'(F,1) = F'$.

Theorem 4. \mathfrak{F} has no proper subalgebra containing more than 8 functions.

We abbreviate $\{G, G'\}$ to $|G|$ for any G in \mathfrak{F} , and assume that the algebra \mathfrak{G} contains at least 9 functions. Then \mathfrak{G} contains at least one pair $|G|$.

We first show that \mathfrak{G} is ν -closed. By Lemma 2, only the case where $|G| = \{I, 0\}$ need be considered. Since \mathfrak{G} contains at least 9 functions, one of the functions $I, J', A, B', D, D', C, C', E, E'$ must be in \mathfrak{G} . But any one of them, in conjunction with $I, 0$, is sufficient for \mathfrak{G} to be ν -closed.

The remainder of the proof will be continued after observing that since $F \neq \nu F$, from the definition of ν -closed sets, one immediately obtains

Lemma 3. *The number of functions in a ν -closed set is even.*

According to Lemma 3, the ν -closed set \mathfrak{G} considered above contains at least 10 functions and thus 5 pairs $|F|$. One of them must be $\{I, 0\}$ since there are only two pairs that do not generate $\{I, 0\}$, namely, $|I|$ and $|J|$. Moreover, \mathfrak{G} must contain at least one of the pairs $|A|, |B|, |I|, |J|$ since there are only 8 pairs in \mathfrak{F} .

If $|A|$ is in \mathfrak{G} , then at least one of $|C|, |D|, |I|, |J|$ is in \mathfrak{G} . First suppose $|A|$ and $|C|$ are in \mathfrak{G} . Then at least one of $|B|, |D|, |E|, |J|$ is in \mathfrak{G} . Now $[A, C]$ contains $|I|$ and $[B, I'] = [D, I']$. Therefore, if $|A|, |C|$, and $|B|$ are in \mathfrak{G} , so are $|I|$ and $|D|$. But $[A, B]$ contains $|E|$, and $[E, I]$ contains $|J|$, whence $\mathfrak{G} = \mathfrak{F}$. If $|A|, |C|$, and $|D|$ are in \mathfrak{G} so is $|I|$. But $[D, I']$ contains $|B|$, and $\mathfrak{G} = \mathfrak{F}$. Since $[A, E]$ and $[C, J]$ contain $|B|$ we have $\mathfrak{G} = \mathfrak{F}$ if $|A|$ and $|C|$ as well as $|E|$ and/or $|J|$ belong to \mathfrak{G} . Hence $\mathfrak{G} = \mathfrak{F}$ whenever $|A|$ and $|C|$, and, similarly, whenever $|A|$ and $|D|$ are in \mathfrak{G} . Next suppose that \mathfrak{G} , besides $|A|$, contains $|I|$ and/or $|J|$. Now $[A, I]$ contains $|C|$, and $[A, J]$ contains $|D|$. Consequently $\mathfrak{G} = \mathfrak{F}$ whenever \mathfrak{G} contains $|A|$ or, for similar reasons, whenever \mathfrak{G} contains $|B|$. If $|I|$ is in \mathfrak{G} while $|A|$ and $|B|$ are not, then at least one of $|C|, |D|$ must be in \mathfrak{G} . But $[I', C]$ contains $|A|$, and $[I', D]$ contains $|B|$. Thus if $|I|$ (or $|J| = \sigma|I|$) is in \mathfrak{G} , then again $\mathfrak{G} = \mathfrak{F}$, which completes the proof of Theorem 4.

Theorem 5. *The only ν -closed subalgebras of \mathfrak{F} that cannot be generated by less than three functions are \mathfrak{F} and $I(2) = \{I, I', J, J', I, 0\}$.*

In view of Lemma 1 and Theorem 4, only ν -closed algebras consisting of 4, 6, or 8 functions need be considered.

1) Suppose \mathfrak{G} contains 4 functions. If \mathfrak{G} contains $|A|$ or $|B|$, then $\mathfrak{G} = [A]$ or $= [B]$. If \mathfrak{G} contains $|I|$ or $|J|$ as well as $|G|$ then $\mathfrak{G} = [I', G]$ or $= [J', G]$. For the remaining nonconstant functions, $[F, F'] = \{F, F', I, 0\}$.

2) Suppose \mathfrak{G} contains 8 functions. Since there are only 2 pairs $|F|$ that do not generate $\{I, 0\}$, namely $|I|, |J|$, \mathfrak{G} contains $\{I, 0\}$ and also one of the 5 pairs $|A|, |B|, |C|, |D|, |E|$. By combining any of these 5 pairs with a pair other than $|I|$, we obtain one of the ν -closed octuples $I(4), (5), (6)$ or $III(2)$, which can be generated by 2 functions.

3) If \mathfrak{G} has 6 elements, \mathfrak{G} cannot contain any of the 5 pairs just mentioned. Thus \mathfrak{G} must be $I(2) = \{I, I', J, J', I, 0\} = [I', J', I]$.

Since it can be verified by inspection that all algebras generated by 2 functions have been listed, it follows from Theorem 5 that all ν -closed subalgebras of \mathfrak{F} have been listed.

Theorem 6. *Any non-idempotent subalgebra \mathfrak{G} of \mathfrak{F} that is not ν -closed is a subalgebra of $II(9) = \{A', B, C, D, E, I, J, I\}$ or of $\delta II(9) = \{A', B, C', D', E', I, J, 0\}$.*

Since \mathfrak{G} is not ν -closed, by Lemma 3, A, B', I', J' are not in \mathfrak{G} . Being non-idempotent, \mathfrak{G} must contain at least one of the functions C, C', D, D', E, E' and, therefore, either I or 0 . If \mathfrak{G} contains I (or 0), \mathfrak{G} cannot contain C', D', E' (or C, D, E) and hence $\mathfrak{G} \subseteq II(9)$ (or $\mathfrak{G} \subseteq \delta II(9)$).

The remainder of the proof is based on the following

Lemma 4. *There are no proper subalgebras of $II(9)$ containing $II(6) = [C, E] = \{C, D, E, I\}$ or $III(6) = [A', C] = \{A', C, I, I\}$ or $\sigma III(6) = [A', D] = \{A', D, J, I\}$.*

Clearly, $[C, E, F] = II(6)$ if $F = C, D, E, I$ and $= II(9)$ if $F = A', B, I, J$. Similarly, $[A', C, F] = III(6)$ if $F = A', C, I, I$, and $= II(9)$ if $F = B, D, E, J$. Hence there are no proper subalgebras of $II(9)$ containing $II(6)$, $III(6)$ or $\sigma III(6)$.

Since C, D, E are the only nonidempotent functions in $II(9)$, and D belongs to $[C, E]$ we need only consider 1) algebras which contain E and neither C nor D and 2) algebras containing at least one of C, D but not E .

1) Any subalgebra of $II(9)$ containing neither C nor D must be a subset of $\{A', B, E, I, J, I\}$. If the subalgebra is nonidempotent it must contain E . But $[E, A'] = [E, B] = [E, A', B, I] = II(3)$. Adjoining I or J yields $II(9)$. Therefore \mathfrak{G} cannot contain A' or B . But $[E, I] = [E, J] = \{E, I, J, I\} = II_{\sigma}(4)$. Hence all subalgebras of $\{A', B, E, I, J, I\}$ have been listed.

2) From Lemma 4 it follows that any subalgebra of $II(9)$ containing C (or $D = C$) but not E must be a subalgebra of $II_{\sigma}(8) = \{B, C, D, I, J, I\}$. Any nonidempotent subalgebra \mathfrak{G} must contain C or D . Suppose \mathfrak{G} contains C and B . Examining the algebras $[C, B, F]$ where F is in $II_{\sigma}(8)$, we see that we obtain $II_{\sigma}(7)$, $III(2)$, (3), (4) according as $F = D, I, J, I$. Likewise all subalgebras of $II_{\sigma}(8)$ have been listed.

This completes the proof of Assertion 2) of the Principal Theorem.

In conclusion we prove the following

Theorem 7. *Two functions in \mathfrak{F} generate at most 8 functions.*

Let F and G be two functions. If at least one of them, say F , has an odd value-sum, then F assumes one and the same value for 3 arguments (x, y) . For at least 2 of these arguments G must also assume one and the same value. Hence there are two arguments (a, b) and (c, d) such that $F(a, b) = F(c, d)$ and $G(a, b) = G(c, d)$. According to Lemma 1, F, G is a subset of the octuple of functions each of which assumes the same value for (a, b) and (c, d) . If both F and G are even, they belong to the octuple $I(4)$, which has been proved to be closed.

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