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A NOTE ON HALLDÉN-INCOMPLETENESS

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In [3], Halldén in effect suggests that a modal (or other) system S [1] is unreasonable if there are wffs A and B, each containing one propositional variable and such that the variable in A is different from the variable in B, with the property that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$. In the same paper, Halldén shows that any system intermediate between the Lewis systems S1 and S3 is unreasonable in this sense. McKinsey [9] relaxes the condition that the wffs A and B contain just one variable; following this approach, let us say that a system S is *Halldén-incomplete* (H-incomplete) iff there are wffs A and B with no variables in common such that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$, and *strongly* H-*incomplete* iff there are wffs A and B, with one variable in each and no variables in common, such that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$. Then evidently if S is strongly H-incomplete then S is H-incomplete; the converse, however, seems to be an open question. If a system is not H-incomplete, we say it is H-*complete*. McKinsey [9] also shows that S4, S5, and all extensions of S5 (closed under substitution and detachment) are H-complete, but that there is a system between S4 and S5 which is H-incomplete. More recently Kripke [4], p. 94, has shown additionally that the modal system T and the 'Brouwersche' system B are H-complete. Åqvist [1] claims that any system between S2 and T is H-complete; his proof, however, is faulty, as is pointed out in [7]; the Corollary to Theorem 2 below gives the result.

We begin by showing that for S to be H-incomplete it is necessary and sufficient that S be the 'intersection' of two disjoint extensions, in a sense we now explain. For a system S, let $T(\mathrm{S})$ be the class of theorems of S. We say that two systems S_1 and S_2 (with the same formation rules) are *disjoint* iff there are wffs A_1 and A_2 such that $\vdash_{S_1} A_1$, $\vdash_{S_2} A_2$, but not $\vdash_{S_2} A_1$ and not $\vdash_{S_1} A_2$, i.e. iff neither $T(\mathrm{S}_1) \subseteq T(\mathrm{S}_2)$ nor $T(\mathrm{S}_2) \subseteq T(\mathrm{S}_1)$. We prove:

Theorem 1. *A system* S *is* H-*incomplete iff there are disjoint systems* S_1 *and* S_2 *such that* $T(\mathrm{S}) = T(\mathrm{S}_1) \cap T(\mathrm{S}_2)$.

Proof. Suppose S is H-incomplete, and that wffs A, B, with no variables in common, are such that $\vdash_S A \vee B$ yet neither $\vdash_S A$ nor $\vdash_S B$. Let S_A (S_B) be the system whose axioms are all theorems of S together with all

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substitution-instances of $A(B)$, and sole rule of inference detachment. Then obviously $T(\mathrm{S}) \subseteq T(\mathrm{S}_A) \cap T(\mathrm{S}_B)$. Conversely, suppose $\vdash_{\mathrm{S}_A} C$ and $\vdash_{\mathrm{S}_B} C$, and let $A_1, \ldots, A_m$ $(B_1, \ldots, B_n)$ be the substitution-instances of $A(B)$ employed in the first (second) proof. Then we have

$$\vdash_{\mathrm{S}} A_1 \wedge \ldots \wedge A_m \rightarrow C,$$

$$\vdash_{\mathrm{S}} B_1 \wedge \ldots \wedge B_n \rightarrow C,$$

whence

$$\vdash_{\mathrm{S}} (A_1 \wedge \ldots \wedge A_m) \vee (B_1 \wedge \ldots \wedge B_n) \rightarrow C,$$

$$\vdash_{\mathrm{S}} (A_1 \vee B_1) \wedge (A_1 \vee B_2) \wedge \ldots \wedge (A_m \vee B_n) \rightarrow C,$$

by propositional calculus. However, since A and B have no variables in common and $\vdash_{\mathrm{S}} A \vee B$, we have $\vdash_{\mathrm{S}} A_i \vee B_j$ for all i, j $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$. Thus $\vdash_{\mathrm{S}} C$, and $T(\mathrm{S}_A) \cap T(\mathrm{S}_B) \subseteq T(\mathrm{S})$. It remains to show that S_A and S_B are disjoint. Suppose then that $\vdash_{\mathrm{S}_A} B$. Evidently

$$\vdash_{\mathrm{S}} A_1 \wedge \ldots \wedge A_m \rightarrow B$$

for substitution-instances $A_1, \ldots, A_m$ of A. But also $\vdash_{\mathrm{S}} A_i \vee B$, by substitution on $A \vee B$, for all i $(1 \leqslant i \leqslant m)$, whence

$$\vdash_{\mathrm{S}} (A_1 \vee B) \wedge \ldots \wedge (A_m \vee B),$$

$$\vdash_{\mathrm{S}} (A_1 \wedge \ldots \wedge A_m) \vee B.$$

Hence $\vdash_{\mathrm{S}} B$, contrary to our initial assumption. Thus not $\vdash_{\mathrm{S}_A} B$, and similarly not $\vdash_{\mathrm{S}_B} A$. This shows that S_A and S_B are disjoint.

Conversely, suppose that S_1 and S_2 are disjoint, and that $T(\mathrm{S}) = T(\mathrm{S}_1) \cap T(\mathrm{S}_2)$. Select A_1, A_2 such that $\vdash_{\mathrm{S}_1} A_1$, $\vdash_{\mathrm{S}_2} A_2$, and yet not $\vdash_{\mathrm{S}_2} A_1$, not $\vdash_{\mathrm{S}_1} A_2$. Let A_2' be an 'isomorphic' substitution-instance of A_2 (i.e. a substitution-instance such that in turn A_2 is a substitution-instance of it) having no variables in common with A_1. Clearly $\vdash_{\mathrm{S}_2} A_2'$ and not $\vdash_{\mathrm{S}_1} A_2'$. Now $\vdash_{\mathrm{S}_1} A_1 \vee A_2'$, $\vdash_{\mathrm{S}_2} A_1 \vee A_2'$, whence $\vdash_{\mathrm{S}} A_1 \vee A_2'$. However, not $\vdash_{\mathrm{S}} A_1$, not $\vdash_{\mathrm{S}} A_2'$, so that S is H-incomplete.

Corollary. S *has disjoint extensions*[2] *iff* S *has an* H-*incomplete extension.*

Proof. Suppose $T(\mathrm{S}) \subseteq T(\mathrm{S}_1)$, $T(\mathrm{S}) \subseteq T(\mathrm{S}_2)$, for disjoint S_1 and S_2; then $T(\mathrm{S}_1) \cap T(\mathrm{S}_2)$ will form the theorems of a system in the sense of footnote 1 which is H-incomplete by Theorem 1 and an extension of S. Conversely, if S has an H-incomplete extension S', by Theorem 1 $T(\mathrm{S}') = T(\mathrm{S}_1) \cap T(\mathrm{S}_2)$ for disjoint S_1, S_2 which clearly are extensions of S.

In view of Kripke's and McKinsey's results cited earlier, this Corollary throws some light on modal systems. Thus, since S5 has no H-incomplete extensions, S5 has no disjoint extensions, i.e. for any extensions S_1, S_2 of S5 either $T(\mathrm{S}_1) \subseteq T(\mathrm{S}_2)$ or $T(\mathrm{S}_2) \subseteq T(\mathrm{S}_1)$[3]. Conversely, since both S4 and T do notoriously have disjoint extensions (e.g. S5 and the system S4.1 of McKinsey [8] for S4, and S4 and B for T), then, even though both S4 and T are H-complete, they have H-incomplete proper extensions lying between S4 and S5, and between T and S4, respectively. Also, though the intuitionist propositional calculus is H-complete, as Halldén [3] points out, it too has an H-incomplete proper extension. The classical propositional calculus is H-complete and, since it has no extensions, has no H-incomplete ones.

We turn now to the task of showing that a number of particular modal systems are in fact H-incomplete. Let L be the modal logic defined by axiom-schemata appropriate to the classical propositional calculus (e.g. the three on Church [2], p. 149), together with the rule of (material) detachment and the rule:

$$\textbf{RE:}\quad \frac{A \longleftrightarrow B}{\Box A \longleftrightarrow \Box B}\ .$$

It is obvious that L is an extremely weak modal system, whose only modal assumption is the substitutivity in modal contexts of material equivalents; it is contained in S1, as well as the systems E2-E5, D2-D5 of [5] (though not in E1). Also let PC be the 'degenerate' modal system which results from adding to T (or even to E2) the schema:

$$A \rightarrow \Box A.$$

PC is by contrast very strong, and contains all the systems S1-S5, E1-E5, D1-D5, though not S6-S8. Let us put T for $p \rightarrow p$ and F for $-(q \rightarrow q)$. It is easy to show that

$$(1)\quad \vdash_{\mathrm{L}} \Diamond^n F \vee \Box^n T$$

for any n (use **RE**).

A modal system S *has necessity-gaps* iff for some n there is no wff A such that $\vdash_{\mathrm{S}} \Box^n A$; in this case, we say that n is a *necessity-gap* for S. It is known that the Lewis systems S1-S3, as well as E2-E5, D2-D5, have necessity-gaps: e.g. $n = 2$ for all these systems. Thus they all satisfy the antecedent of the next theorem, whence by it they are strongly H-incomplete.

Theorem 2. *If* S *is a modal system such that* $T(\mathrm{L}) \subseteq T(\mathrm{S}) \subseteq T(\mathrm{PC})$ *and* S *has necessity-gaps, then* S *is strongly* H*-incomplete* (*and so* H*-incomplete*).

Suppose $T(\mathrm{L}) \subseteq T(\mathrm{S}) \subseteq T(\mathrm{PC})$ and n is a necessity-gap for S. Since $T(\mathrm{L}) \subseteq T(\mathrm{S})$, $\vdash_{\mathrm{S}} \Diamond^n F \vee \Box^n T$ by (1). But not $\vdash_{\mathrm{S}} \Diamond^n F$, since $T(\mathrm{S}) \subseteq T(\mathrm{PC})$ and $\vdash_{\mathrm{PC}} -\Diamond^n F$; and not $\vdash_{\mathrm{S}} \Box^n T$, since n is a gap for S. Since $\Diamond^n F$ contains only q and $\Box^n T$ contains only p, S is strongly H-incomplete.

Corollary. If $T(\mathrm{E2}) \subseteq T(\mathrm{S}) \subset T(\mathrm{T})$, *then* S *is strongly* H*-incomplete.*

Proof. Since $T(\mathrm{L}) \subseteq T(\mathrm{E2})$, $T(\mathrm{T}) \subseteq T(\mathrm{PC})$, it suffices by Theorem 2 to show that, for S such that $T(\mathrm{E2}) \subseteq T(\mathrm{S}) \subset T(\mathrm{T})$, S has necessity-gaps. In fact we show that, if $T(\mathrm{E2}) \subseteq T(\mathrm{S})$ and S has no necessity-gaps, then $T(\mathrm{T}) \subseteq T(\mathrm{S})$. So suppose that for each n we have B_n such that $\vdash_{\mathrm{S}} \Box^n B_n$ and suppose $\vdash_{\mathrm{T}} A$. Then $\vdash_{\mathrm{E2}} \Box^m T \rightarrow A$ for some m, as is shown in [7], Lemma 2. Clearly $\vdash_{\mathrm{E2}} \Box^m B_m \rightarrow \Box^m T$, whence $\vdash_{\mathrm{E2}} \Box^m B_m \rightarrow A$. Thus $\vdash_{\mathrm{S}} A$, as was to be shown[4].

In correspondence, Åqvist has suggested to me that H-incompleteness may be connected with failure of the rule of necessitation: from A to infer $\Box A$. Certainly, if a system S has necessity-gaps, this rule fails; the converse, however, is not true, as is shown by the extension of S4 described in McKinsey-Tarski [10], Theorem 3.1, for which the rule fails but which has no gaps. Further, the H-incomplete extensions of T and S4 mentioned earlier both satisfy the rule of necessitation, since they are 'intersections' of systems that satisfy it. It is worth recalling that, though H-incompleteness was introduced by Halldén in reference to modal logics, its definition is quite general and makes no mention of modal operators.

It is clear from these results that a very wide class of modal logics are H-incomplete: for nearly all systems in the literature, apart from T and its extensions, contain L and are contained in PC and have necessity-gaps. It appears in general much harder to show that a system is H-complete. Since both the results of McKinsey and those of Kripke were obtained in the same way, it may be useful here to summarize the method. A matrix $\mathfrak{M}$ for a system S may be thought of as an (abstract) algebra consisting of a set of elements M, a designated subset of M, say D, and certain algebraic operations in terms of which the primitive symbols of S are to be interpreted. Since systems are here construed as containing the classical propositional calculus (*cf.* footnote 1), interesting matrices will contain (in an obvious sense) a Boolean algebra; let us call such matrices *normal*. Then for normal matrices it makes sense to speak of D being a *maximal filter* of M. In case this happens, let us say that $\mathfrak{M}$ is a maximal matrix. McKinsey's result (generalized a bit) is:

Theorem 3. *If* S *has a maximal characteristic matrix, then* S *is* H-*complete.*

Suppose that S has a maximal characteristic matrix, say $\mathfrak{M}$, and yet S is H-incomplete. Let A, B be such that $\vdash_S A \vee B$ yet not $\vdash_S A$ and not $\vdash_S B$, where A and B have no variables in common. Then for any assignment α from $\mathfrak{M}$, $\mathbf{V}_\alpha(A \vee B) \in D$[5]. On the other hand, since $\mathfrak{M}$ is characteristic, there are assignments α_1 and α_2 such that $\mathbf{V}_{\alpha_1}(A) \notin D$, $\mathbf{V}_{\alpha_2}(B) \notin D$. It is a property of maximal filters $\mathscr{F}$ that if $x \cup y \in \mathscr{F}$ then either $x \in \mathscr{F}$ or $y \in \mathscr{F}$. Since D is maximal, and since A and B have no variables in common, a composite assignment α' can be formed from α_1 and α_2 to the variables of both A and B such that $\mathbf{V}_{\alpha'}(A \vee B) \notin D$. This contradiction proves the theorem.

Both McKinsey and Kripke were able to show, though in different ways, that the systems they treated (between them, T, S4, B, S5 and its extensions) had maximal characteristic matrices. The demonstrations, however, are far from trivial, and the results rather isolated; so far as I know, apart from McKinsey's concerning the extensions of S5, there are no general results to the effect that all modal systems in a certain class are H-complete. Perhaps Theorem 1 can be harnessed to the job of finding some.

The converse of Theorem 3 seems at first sight plausible, since all systems known to be H-complete in fact have maximal characteristic matrices; however, I can find no proof of it, and now suspect it to be false.

NOTES

1. Throughout this paper, a logical *system* is understood to be a propositional logic whose class of theorems is closed with respect to substitution as well as detachment, and which contains (in some form or other) the classical propositional calculus. Thus the standard systems of modal logic, as well as the intuitionist propositional calculus, are systems in this sense.

2. By an extension S' of a system S we mean a system S' (cf. footnote 1) such that $T(\mathrm{S}) \subseteq T(\mathrm{S}')$.

3. Actually, however, this result (that extensions of S5 form a linear series) is implicit in Scroggs [11], so that we can also use the Corollary to obtain an independent proof of McKinsey's result that all extensions of S5 (including S5) are H-complete.

4. Since this proof makes no use of the fact that E2 has as theorem $\Box A \rightarrow A$, it can at once be extended to show that systems S such that $T(\mathrm{D2}) \subseteq T(\mathrm{S}) \subseteq T(T(D))$ or $T(\mathrm{C2}) \subseteq T(\mathrm{S}) \subseteq T(T(C))$ are strongly H-incomplete (for these systems, see [6]). Also, E1, though not covered by Theorem 2, is strongly H-incomplete, since $\vdash_{\mathrm{E1}} \Diamond F \vee \Box T$ but neither $\vdash_{\mathrm{E1}} \Diamond F$ nor $\vdash_{\mathrm{E1}} \Box T$.

5. I use the notation $\mathbf{V}_\alpha(\mathrm{B})$ for the value taken by the matrix-function associated with B under the assignment α.

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