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# A SIMPLE VERSION OF THE GENERALIZED CONTINUUM HYPOTHESIS

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An understanding of the use of the following expressions of Zermelo-Fraenkel-Skolem set theory without the axiom of choice is presupposed: '$\mathsf{S}x$' ('The set of all subsets of $x$'), '2' ('The set whose only members are the empty set and the set whose only member is the empty set'), '$x^y$' ('The set of all functions from $y$ into $x$'), '$x \approx y$' ('There is a one-to-one correspondence between $x$ and $y$'), and '$x$ is finite'.

*Definition 1.* $x \preccurlyeq y$ *if and only if there is a* $z$ *in* $\mathsf{S}y$ *such that* $x \approx z$; $x \prec y$ *if and only if* $x \preccurlyeq y$ *and not* $y \preccurlyeq x$.

*Definition 2.* **AC** *if and only if, for any* $x$, *if the members of* $x$ *are non-empty and disjoint, then there is an* $s$ *such that, for any* $m$ *in* $x$, *there is exactly one* $d$ *in both* $m$ *and* $s$;
**GCH** *if and only if, for any* $x$, *if* $x$ *is not finite, then there is no* $y$ *such that both* $x \prec y$ *and* $y \prec 2^x$.

The following theorems are well-known:

*Theorem 1.* **AC** *if and only if, for any* $x$ *and* $y$, *either* $x \preccurlyeq y$ *or* $y \preccurlyeq x$.

*Theorem 2.* *If either* $x$ *or* $y$ *is finite, then either* $x \preccurlyeq y$ *or* $y \preccurlyeq x$.

*Theorem 3.* *If* **GCH**, *then* **AC**.

From theorems 1 and 2 we have

*Theorem 4.* **AC** *if and only if, for any* $x$ *and* $y$, *if both* $x$ *and* $y$ *are not finite, then either* $x \preccurlyeq y$ *or* $y \preccurlyeq x$.

This version of the axiom of choice suggests a nearly identical version of the generalized continuum hypothesis.

*Definition 3.* **G** *if and only if, for any* $x$ *and* $y$, *if both* $x$ *and* $y$ *are not finite, then either* $x \preccurlyeq y$ *or* $\mathsf{S}y \preccurlyeq x$.

*Theorem 5.* **G** *if and only if* **GCH**.

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Assume first **G**, $x$ is not finite, and $x \prec y$. Hence, $x \preccurlyeq y$ and not $y \preccurlyeq x$. But then $y$ is not finite and so $\mathsf{S}x \preccurlyeq y$ by **G**; that is, since $2^x \approx \mathsf{S}x$, $2^x \preccurlyeq y$ and so not $y \prec 2^x$.

Assume on the other hand **GCH**, both $x$ and $y$ are not finite, and not $x \preccurlyeq y$. By theorems 3 and 4, $y \prec x$; hence, by **GCH**, not $x \prec 2^y$ and so, by theorems 3 and 1, $2^y \preccurlyeq x$. But $\mathsf{S}y \approx 2^y$ and so $\mathsf{S}y \preccurlyeq x$.

This version of the generalized continuum hypothesis is claimed to be simple both because it is economical in its symbolism (having only '$\preccurlyeq$' and '$\mathsf{S}y$' where the generalized continuum hypothesis has the less fundamental '$\prec$' and '$2^x$') and because it is very easy to prove.

*Theorem 6. If* **G**, *then* **AC**.

Assume **G**. Since $y \prec \mathsf{S}y$ for any $y$, we then have $x \preccurlyeq y$ or $y \preccurlyeq x$ for any $x$ and $y$ which are not finite; that is, by theorem 4, **AC**.

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