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A SET OF AXIOMS FOR THE PROPOSITIONAL CALCULUS WITH IMPLICATION AND NON-EQUIVALENCE

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It is well-known that implication and non-equivalence constitute a complete system of independent primitive connectives for the propositional calculus. In this article it is the intention of the author to give an independent set of axioms by means of the two connectives mentioned above, the rules of inference being substitution and *modus ponens*.

In §1 we state the axioms and prove some preliminary theorems. In §2 we solve the decision problem. Finally, we establish the independence of the axioms and rules in §3. In the matter of notation we shall follow Alonzo Church[1].

§1. *Axioms and Preliminary Theorems*. The axioms of our logistic system, say P, are the seven following:

Axiom 1. $p \supset \cdot q \supset p$

Axiom 2. $s \supset [p \supset q] \supset \circ\, s \supset p \supset \cdot s \supset q$

Axiom 3. $p \supset q \supset p \supset p$

Axiom 4. $p \supset [p \not\equiv q] \supset \cdot q \supset \cdot p \not\equiv q$

Axiom 5. $p \not\equiv q \supset \cdot p \supset q \supset q$

Axiom 6. $p \not\equiv q \supset \cdot p \supset \cdot q \supset s$

Axiom 7. $p \not\equiv q \supset \cdot q \not\equiv p$

In fact, as is evident from the above set, any formulation of the implicational propositional calculus and Axioms 4-7 will suffice. We note that from the present formulation the deduction theorem—to be henceforth referred to as D.T.—follows immediately.

We now go on to prove some theorems.

1. Church, A. *Introduction to Mathematical Logic,* I. Princeton, N. J., 1956.

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Theorem 1. $p \not\equiv q \supset \cdot q \supset \cdot p \supset s$

Proof:

By Axiom 6, $p \not\equiv q, q, p \vdash s$
Hence by D.T., $\vdash p \not\equiv q \supset \cdot q \supset \cdot p \supset s$

Theorem 2. $r \supset \cdot p \not\equiv r \supset \cdot p \supset q$

Proof:

By Theorem 1, $r, p \not\equiv r \vdash p \supset q$
Hence by D.T., $\vdash r \supset \cdot p \not\equiv r \supset \cdot p \supset q$

Theorem 3. $r \supset \cdot p \supset \cdot q \not\equiv r \supset \cdot p \supset q \not\equiv r$

Proof:

We have, $r, p, q \not\equiv r, p \supset q \vdash q$
Again by Theorem 2, $r, p, q \not\equiv r, p \supset q \vdash q \supset \cdot p \supset q \not\equiv r$
Hence, $r, p, q \not\equiv r, p \supset q \vdash p \supset q \not\equiv r$
Hence by D.T., $r, p, q \not\equiv r \vdash p \supset q \supset \cdot p \supset q \not\equiv r$
Hence by Axiom 4, $r, p, q \not\equiv r \vdash r \supset \cdot p \supset q \not\equiv r$
Again we have, $r, p, q \not\equiv r \vdash r$
Hence, $r, p, q \not\equiv r \vdash p \supset q \not\equiv r$
Hence by D.T., $\vdash r \supset \cdot p \supset \cdot q \not\equiv r \supset \cdot p \supset q \not\equiv r$

Theorem 4. $r \supset \cdot p \supset \cdot q \supset \cdot p \not\equiv q \not\equiv r$

Proof:

By Theorem 2, $p, q, p \not\equiv q \vdash p \not\equiv q \not\equiv r$
Hence by D.T., $p, q, \vdash p \not\equiv q \supset \cdot p \not\equiv q \not\equiv r$
Hence by Axiom 4, $p, q \vdash r \supset \cdot p \not\equiv q \not\equiv r$
Hence, $r, p, q \vdash p \not\equiv q \not\equiv r$
Hence by D.T., $\vdash r \supset \cdot p \supset \cdot q \supset \cdot p \not\equiv q \not\equiv r$

Theorem 5. $q \supset [p \not\equiv q] \supset \cdot p \supset \cdot p \not\equiv q$

Proof:

We have, $q \supset [p \not\equiv q], q \vdash p \not\equiv q$
Hence by Axiom 7, $q \supset [p \not\equiv q], q \vdash q \not\equiv p$
Hence by D.T., $q \supset [p \not\equiv q] \vdash q \supset \cdot q \not\equiv p$
Hence by Axiom 4, $q \supset [p \not\equiv q] \vdash p \supset \cdot q \not\equiv p$
Hence, $q \supset [p \not\equiv q], p \vdash q \not\equiv p$
Hence by Axiom 7, $q \supset [p \not\equiv q], p \vdash p \not\equiv q$
Hence by D.T., $\vdash q \supset [p \not\equiv q] \supset \cdot p \supset \cdot p \not\equiv q$

Theorem 6. $r \supset \cdot p \supset \cdot q \not\equiv r \supset \cdot p \not\equiv q$

Proof:

By Theorem 1, $r, q \not\equiv r \vdash q \supset \cdot p \not\equiv q$
Hence by Theorem 5, $r, q \not\equiv r \vdash p \supset \cdot p \not\equiv q$
Hence, $r, p, q \not\equiv r \vdash p \not\equiv q$
Hence by D.T., $\vdash r \supset \cdot p \supset \cdot q \not\equiv r \supset \cdot p \not\equiv q$

Theorem 7. $r \supset \cdot q \supset \cdot p \not\equiv r \supset \cdot p \not\equiv q$

Proof:

By Theorem 1, $r, p \not\equiv r \vdash p \supset \cdot p \not\equiv q$
Hence by Axiom 4, $r, p \not\equiv r \vdash q \supset \cdot p \not\equiv q$
Hence, $r, q, p \not\equiv r \vdash p \not\equiv q$
Hence by D.T., $\vdash r \supset \cdot q \supset \cdot p \not\equiv r \supset \cdot p \not\equiv q$

Theorem 8. $r \supset \cdot p \not\equiv r \supset \cdot q \not\equiv r \supset \cdot p \not\equiv q \not\equiv r$

Proof:

By Theorem 2, $r, p \not\equiv r,\ q \not\equiv r,\ p \not\equiv q \vdash p \supset q$
Again by Axiom 5, $r, p \not\equiv r,\ q \not\equiv r,\ p \not\equiv q \vdash p \supset q \supset q$
Hence, $r, p \not\equiv r,\ q \not\equiv r,\ p \not\equiv q \vdash q$
Again by Theorem 2, $r, p \not\equiv r,\ q \not\equiv r,\ p \not\equiv q \vdash q \supset \cdot p \not\equiv q \not\equiv r$
Hence, $r, p \not\equiv r,\ q \not\equiv r,\ p \not\equiv q \vdash p \not\equiv q \not\equiv r$
Hence by D.T., $r, p \not\equiv r,\ q \not\equiv r \vdash p \not\equiv q \supset \cdot p \not\equiv q \not\equiv r$
Hence by Axiom 4, $r, p \not\equiv r,\ q \not\equiv r \vdash r \supset \cdot p \not\equiv q \not\equiv r$
Again we have, $r, p \not\equiv r,\ q \not\equiv r \vdash r$
Hence, $r, p \not\equiv r,\ q \not\equiv r \vdash p \not\equiv q \not\equiv r$
Hence by D.T., $\vdash r \supset \cdot p \not\equiv r \supset \cdot q \not\equiv r \supset \cdot p \not\equiv q \not\equiv r$

Theorem 9. $p \not\equiv q \supset s \supset \cdot p \supset s \supset \cdot q \supset s$

Proof:

We have, $p \supset s,\ s \supset [p \not\equiv q],\ p \vdash p \not\equiv q$
Hence by D.T., $p \supset s,\ s \supset [p \not\equiv q] \vdash p \supset \cdot p \not\equiv q$
Hence by Axiom 4, $p \supset s,\ s \supset [p \not\equiv q] \vdash q \supset \cdot p \not\equiv q$
Hence, $p \supset s,\ q,\ s \supset [p \not\equiv q] \vdash p \not\equiv q$
Hence, $p \not\equiv q \supset s,\ p \supset s,\ q,\ s \supset [p \not\equiv q] \vdash s$
Hence by D.T., $p \not\equiv q \supset s,\ p \supset s,\ q \vdash s \supset [p \not\equiv q] \supset s$
Hence by Axiom 3, $p \not\equiv q \supset s,\ p \supset s,\ q \vdash s$
Hence by D.T., $\vdash p \not\equiv q \supset s \supset \cdot p \supset s \supset \cdot q \supset s$

§2. *The Decision Problem*

Metatheorem 1. Every Theorem of P is a tautology.

This Metatheorem can be easily established. We omit the proof.

Metatheorem 2. Let $\mathbf{B}$ be a wff of P, let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be distinct variables among which are all the variables occurring in $\mathbf{B}$, and let $a_1, a_2, \ldots, a_n$ be truth-values. Let $\mathbf{C}$ be any theorem of P i.e., $\vdash \mathbf{C}$. Further, let $\mathbf{A}_i$ be $\mathbf{a}_i$ or $\mathbf{a}_i \not\equiv \mathbf{C}$ according as $\mathbf{a}_i$ is t or f; and let $\mathbf{B}'$ be $\mathbf{B}$ or $\mathbf{B} \not\equiv \mathbf{C}$ according as the value of $\mathbf{B}$ for the values $a_1, a_2, \ldots, a_n$ of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is t or f. Then $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n \vdash \mathbf{B}'$.

Proof: In order to prove that

(1) $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n \vdash \mathbf{B}'$

we proceed by mathematical induction with respect to the number of occurrences of $\supset$ and $\neq$ in B. If there are no occurrences of $\supset$ and $\neq$ in B, then B is one of the variables α_i. Hence B' is the same wff as A_i, and (1) follows trivially. Suppose that there are occurrences of $\supset$ or $\neq$ or both in B. Then B is either $B_1 \supset B_2$ or $B_1 \neq B_2$. By the hypothesis of induction

(2) $\quad A_1, A_2, \ldots, A_n \vdash B_1'$

(3) $\quad A_1, A_2, \ldots, A_n \vdash B_2'$

where B_1' is B_1 or $B_1 \neq C$ according as the value of B_1 for the values $a_1, a_2, \ldots, a_n$ of $\alpha_1 \alpha_2, \ldots, \alpha_n$ is t or f, and B_2' is B_2 or $B_2 \neq C$ according as the value of B_2 for the values $a_1, a_2, \ldots, a_n$ of $\alpha_1, \alpha_2, \ldots, \alpha_n$ is t or f.

Case 1. If B is $B_1 \supset B_2$

(In the treatment of this and the next case it shall be tacit that $\vdash C$.)

In case B_2' is B_2, we have that B' is $B_1 \supset B_2$, and (1) follows from (3) by Axiom 1. In case B_1' is $B_1 \neq C$, we have again that B' is $B_1 \supset B_2$ and (1) follows from (2) by Theorem 2. There remains only the case that B_1' is B_1 and B_2' is $B_2 \neq C$, and in this case B' is $B_1 \supset B_2 \neq C$, and (1) follows from (2) and (3) by Theorem 3.

Case 2. If B is $B_1 \neq B_2$

In case B_1' is B_1 and B_2' is B_2, we have that B' is $B_1 \neq B_2 \neq C$, and (1) follows from (2) and (3) by Theorem 4. In case B_1' is $B_1 \neq C$ and B_2' is $B_2 \neq C$, we have again that B' is $B_1 \neq B_2 \neq C$, and (1) follows from (2) and (3) by Theorem 8. In Case B_1' is B_1 and B_2' is $B_2 \neq C$, we have that B' is $B_1 \neq B_2$, and (1) follows from (2) and (3) by Theorem 6. There remains only the case that B_1' is $B_1 \neq C$ and B_2' is B_2, and in this case again B' is $B_1 \neq B_2$, and (1) follows from (2) and (3) by Theorem 7. Therefore Metatheorem 2 is proved by mathematical induction.

Metatheorem 3. If B is a tautology, $\vdash B$.

Proof: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the variables of B, and for any system of values $a_1, a_2, \ldots, a_n$ of $\alpha_1, \alpha_2, \ldots, \alpha_n$ let $A_1, A_2, \ldots, A_n$ be as in Metatheorem 2. The B' of Metatheorem 2 is B, because B is a tautology. Therefore, by Metatheorem 2,

$$A_1, A_2, \ldots, A_n \vdash B$$

This holds for either choice of a_n, i.e., whether a_n is f or t, and so we have both

$$A_1, A_2, \ldots, A_{n-1}, \quad \alpha_n \neq C \vdash B$$

and

$$A_1, A_2, \ldots, A_{n-1}, \alpha_n \vdash B$$

By the deduction theorem,

$$A_1, A_2, \ldots, A_{n-1} \vdash \alpha_n \neq C \supset B$$
$$A_1, A_2, \ldots, A_{n-1} \vdash \alpha_n \supset B$$

Hence, by Theorem 9,

$$\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1} \vdash \mathbf{C} \supset \mathbf{B}$$

Hence, since $\vdash \mathbf{C}$,

$$\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1} \vdash \mathbf{B}$$

This shows the elimination of the hypothesis $\mathbf{A}_n$. The same process may be repeated to eliminate the hypothesis $\mathbf{A}_{n-1}$, and so on, until all the hypotheses are eliminated. Finally we obtain $\vdash \mathbf{B}$.

In Metatheorem 1 and Metatheorem 3, together with the algorithm for determining whether a wff is a tautology, we have a solution of the decision problem of P. The consistency and completeness of P, now follows as corollaries of this solution of the decision problem.

§3. *Independence*. The independence of each of the axioms and rules of inference, with the exception of the rule of substitution, is established by the standard device of generalized systems of truth-values (see tables below).

For the proof of independence of *modus ponens*, it is necessary to supply also an example of a theorem of P which is not a tautology according to the truth-table used. One such example is $p \supset p$. The independence of the rule of substitution can be established by a well-known argument. Finally, since the calculations required to establish the independence of Axiom 2 are extremely long, the author wishes to point out for the convenience of the reader that when s, p, q take the values 4, 5, 3 respectively, the axiom yields an undesignated value according to the truth-table used.

MODUS PONENS

⊃	0	1	2
*0	0	0	0
1	0	2	0
2	0	0	0

≢	0	1	2
*0	2	2	2
1	2	2	2
2	2	2	2

AXIOM 1

⊃	0	1	2	3	4
*0	0	1	2	3	4
*1	0	1	3	3	4
*2	0	1	0	3	4
3	0	1	0	0	1
4	0	1	0	0	1

≢	0	1	2	3	4
*0	4	4	4	0	0
*1	4	4	4	0	0
*2	4	4	4	2	2
3	0	0	2	4	4
4	0	0	2	4	4

AXIOM 2

⊃	0	1	2	3	4	5
*0	0	1	2	3	5	5
*1	0	1	2	3	5	5
*2	2	1	0	3	5	5
3	0	1	0	2	4	4
4	0	0	0	3	0	0
5	1	1	1	1	1	1

≢	0	1	2	3	4	5
*0	5	5	5	5	3	0
*1	5	5	5	5	3	0
*2	5	5	5	5	3	0
3	5	5	5	5	0	3
4	3	3	3	0	5	5
5	0	0	0	3	5	5

AXIOM 3

⊃	0	1	2
*0	0	1	2
1	0	0	2
2	0	0	0

≢	0	1	2
*0	2	2	0
1	2	2	1
2	0	1	2

AXIOM 4

⊃	0	1
*0	0	1
1	0	0

≢	0	1
*0	1	1
1	1	1

AXIOM 5

⊃	0	1
*0	0	1
1	0	0

≢	0	1
*0	1	0
1	0	0

AXIOM 6

⊃	0	1
*0	0	1
1	0	0

≢	0	1
*0	0	0
1	0	1

AXIOM 7

⊃	0	1
*0	0	1
1	0	0

≢	0	1
*0	1	1
1	0	1

Remark. Ax. 1, Ax. 2, Ax. 5, Ax. 6, Ax. 7, Th. 9 also constitute a complete set. For, (1) Ax. 4 follows immediately from Th. 9 by substitution and *modus ponens* ($p \supset p$ is deducible from Ax. 1 and Ax. 2), and (2) in order to prove the completeness of P, we need Ax. 3 only in one place: to prove Th. 9.

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