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AN EXTENSION OF VENN DIAGRAMS

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Part One: World-state Diagrams

In *Methods of Logic* Quine mentions two limitations of Venn-Diagrammatic techniques. The first is the well-known difficulty of constructing a diagram for a large number of terms. But Venn himself suggested a way to get around this difficulty, viz. to renounce all hope of generating a k-term diagram by superimposing k simple closed curves and instead to subdivide a rectangle into the requisite number of sub-compartments or bins, i.e. 2^k of them.[1] Marquand's rectangular graphs seem simply to incorporate this suggestion. Despite the fact that Marquand-graphs (hereafter **M**-graphs) are readily available for any finite number of terms, they seem to be no more capable than Venn-diagrams of representing arguments which involve, in Quine's phraseology, "an admixture of truth functions"[2] and which present "another place where the unaided method of diagrams bogs down".[3] Quine cites the following as an example of an argument form involving an admixture of truth functions:

(*All FG are H*) $\supset$ (*Some F are not G*)
(*All F are G*) $\vee$ (*All F are H*)
Thus, (*All FH are G*) $\supset$ (*Some F which are not H are G*).[4]

Referring to this example Quine rhetorically asks "just how may we splice the two techniques in order to handle a combined inference of the above kind?"[5]. (The two techniques mentioned are Venn-diagrams and truth-value analysis.) This paper is an answer to Quine's rhetorical question. It shows how to splice Venn-diagrammatic and truth-tabular techniques so as to get a *diagrammatic* decision procedure applicable to all arguments of the above kind, i.e. to uniform quantification theory.[6] If, furthermore, one appends to it some simple but non-truth-functional transformations, the decision procedure becomes applicable to the whole of monadic quantification theory.[7] In addition these diagrams, which will be called *world-state diagrams* (hereafter **WSD**s), provide an intuitive basis on which to define the notions of validity and semantical completeness of both uniform and monadic quantification theory.

Before introducing **WSD**s, I wish to point out a little-noticed fact, viz.

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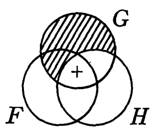
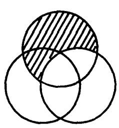
that Venn-diagrams are commonly used to depict the content of some *truth functions* of Venn-diagrammable formulas. In validating syllogisms, for example, one usually constructs a Venn-diagram for the conjunction of the two syllogistic premisses and then inspects it to determine whether the content of the Venn-diagram for the conclusion is contained in it. The Venn-diagram for the conjunction of Venn-diagrammable formulas is obtained merely by stacking or superimposing the Venn-diagrams for all the conjuncts. Consider the argument form Datisi:

All G are H.
Some G are F.
Some F are H.

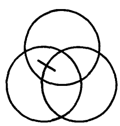
A Venn-diagram for the conjunction of the premisses of this syllogism is



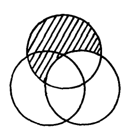
which results upon superimposing the Venn-diagrams



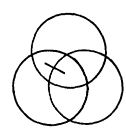
and



. (Notice, in particular, how superimposition changes the bar into '+'.) When one appreciates just what is involved in getting a Venn-diagram for the conjunction of Venn-diagrammable formulas, a method of diagramming disjunctions of Venn-diagrammable formulas suggests itself almost immediately. Why not, as a start at least, string out the Venn-diagrams of the disjunctions to signify that at least one of them depicts the world? Thus the disjunction of the two categorical premisses above might be diagrammed somewhat like this



v



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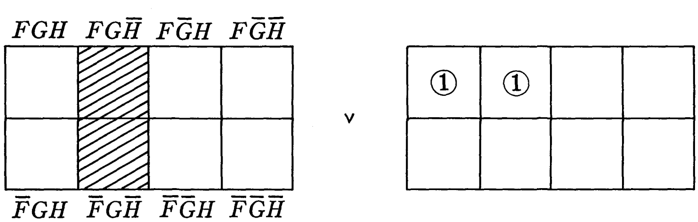
Supplanting Venn-diagrams by **M**-graphs, one gets the following string of **M**-graphs

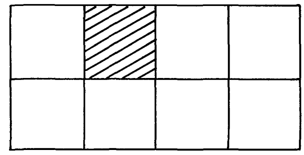


where a circled-numeral in an **M**-graph signifies the nonemptiness of at least one of the bins in which it appears. Such strings of one or more **M**-graphs will be called *world-state pictures* (hereafter **WSP**s). Look again at Quine's sample argument exhibited above. Using truth-tabular methods to eliminate '⊃' in favor of '∼' and 'v', one reduces it to the following form:

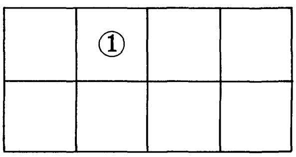
~(*All FG are H*) v (*Some F are not G*)
(*All F are G*) v (*All F are H*)
Thus, ~ (*All FH are G*) v (*Some F which are not H are G*).

Since the expression

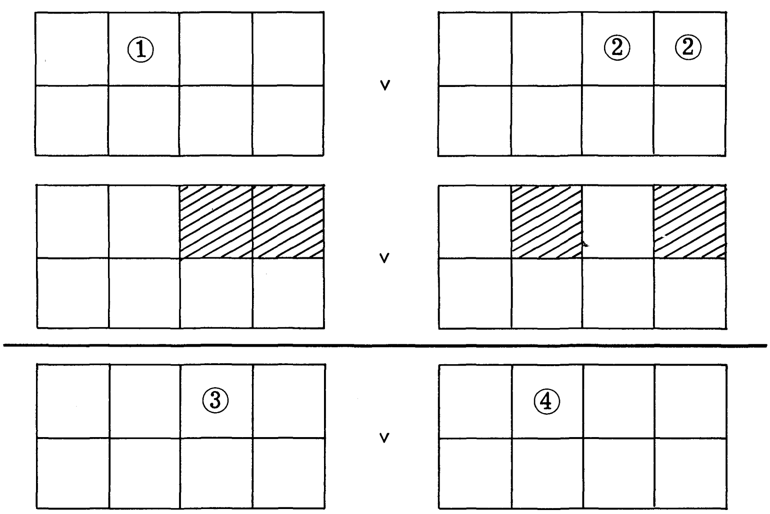
'All FG are H' has the **M**-graph



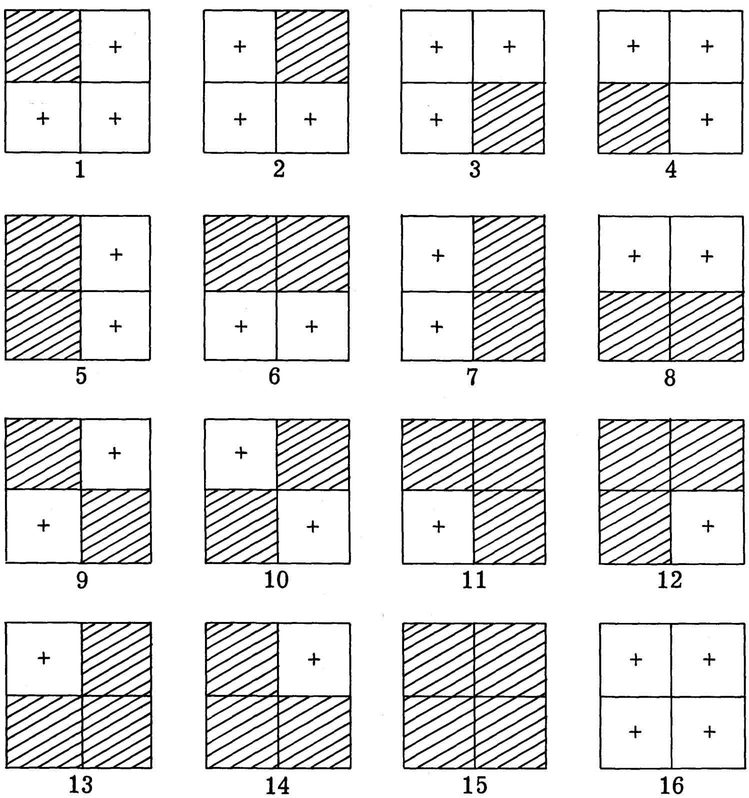
it evidently follows that its negation '~(*All FG are H*)' has the **M**-graph

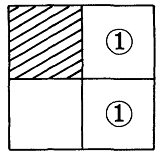


which embodies the necessary and sufficient condition for the falsity of the unnegated formula. Proceeding in this manner, one can quickly construct the following **WSD** for the argument, a **WSD** being simply a finite sequence of **WSP**s:

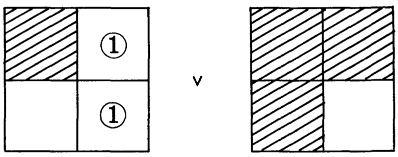


It is now, as shall soon be shown, an easy task to inspect this **WSD** to determine whether or not the argument is valid. An argument form is usually said to be valid if every model of the premisses is also a model of the conclusion. Referring to a **WSD** for an argument, one could say simply that the argument is *valid* if every world-state depicted by the premiss **WSP**s is also depicted by the conclusion **WSP**. Before explaining this definition, I must say a few words about world-states. Suppose that k distinct monadic predicates occur in an argument. If one says of each of these predicates whether it is true or false of a certain individual, one has completely specified that individual in terms of those predicates. It is easily shown that there are 2^k ways to so specify an individual. A complex predicate which completely specifies an individual in this fashion will be called a *specific-predicate.* Thus k distinct predicates determine exactly 2^k specific-predicates. Note, too, that the 2^k bins of an **M**-graph correspond to the 2^k specific-predicates. A specific-predicate will be said to be *empty* if there are no individuals of which it is true; otherwise it is said to be *nonempty.* One can show that, given k distinct predicates, there are 2^{2^k} ways to ascribe emptiness or nonemptiness to the 2^k specific-predicates which those k predicates determine. Each of these ways of ascribing emptiness or nonemptiness to the specific-predicates amounts to a complete description of a state of the world. I.e. each ascription purports to describe the world completely in regard to the emptiness or non-emptiness of the specific-predicates. Since there are 2^{2^k} such ascriptions, there are 2^{2^k} corresponding possible states of the world. Thus, corresponding to 4 distinct predicates there are 65,536 possible world-states, i.e. 65,536 ways in which the world might be with respect to the emptiness or nonemptiness of the 16 specific-predicates determined by those 4 predicates. To illustrate, two predicates *'F'* and *'G'* determine 16 possible world-states, viz. those depicted by the following completely determinate **M**-graphs:

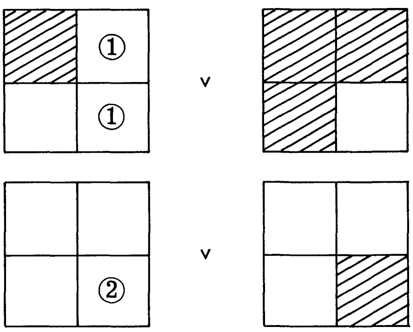


I have, of course, yet to say under what conditions an **M**-graph depicts a world-state. An **M**-graph will be said *to depict* a world-state if the specific-predicates which the **M**-graph shows to be empty are empty in that world-state and those which it shows to be nonempty are nonempty in that world-state. For example, the **M**-graph 

depicts 6 world-states, viz. 1, 5, 6, 9, 12, and 14. A **WSP** will be said *to depict* a world-state if at least one of its **M**-graphs depict that world-state. Thus the **WSP**



depicts 7 world-states, viz. 1, 5, 6, 9, 12, 14, and 15. A sequence of **WSPs** will be said *to depict* those world-states which each of its members depicts. Thus the two **WSPs**



depict exactly 7 world-states, viz. the same 7 that the first **WSP** depicts, for the second **WSP** depicts all 16 world-states.

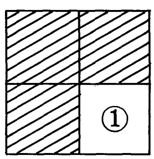
It should now be obvious how, once in possession of a **WSD** for an argument, one could set about quite mechanically, albeit laboriously, to discover whether the argument were valid. He could simply consider all the possible world-states, find out which ones the premiss **WSPs** depict, then check to see if the conclusion **WSP** also depicts them. Happily the same end can be achieved in a much less onerous way. One can show that the premiss **WSPs** depict a world-state if and only if some **M**-graph which results upon superimposing exactly one **M**-graph from each premiss **WSP** depicts it. An **M**-graph which results from the superimposition of exactly one **M**-graph from each premiss WSP will be called a *superimposition* **M**-graph (hereafter **S-M**-graph). One can prove that a world-state which a **S-M**-graph depicts is also depicted by the conclusion **WSP** if and only if at least one of the conclusion **M**-graphs depicts it. Two rules suffice to decide this matter. In expositions of diagrammatic procedures such rules are usually left unstated. For example, when explaining how to use Venn-diagrams to validate syllogisms, Quine remarks that, once having inscribed the content of the premisses in the Venn-diagram, "we inspect the diagram to see whether the content of the conclusion has automatically appeared in the diagram as a result".[8]

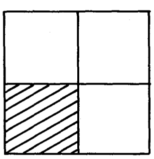
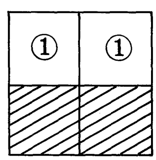
Before the rules of content can be given, a few definitions are needed. A bin of an **M**-graph will be said to be *marked* if it is shaded or if it contains an occurrence of a circled-numeral. An **M**-graph will be said to be *vacuous* if none of its bins are marked and it is not crossed through as inconsistent (see below). The shading of an **M**-graph G_1 will be said to be a *subsystem of* the shading of an **M**-graph G_2 if, for every bin of G_1 that is shaded, the corresponding bin of G_2 is also shaded. A system of plus-marks in bins of an **M**-graph G will be said to *satisfy G's existential commitments* if, for each distinct circled-numeral of G some occurrence of it appears in some bin containing a plus-sign. A system of plus-marks and/or shading in bins of an **M**-graph will be said to be *consistent* if no bin exhibits both shading and a plus-mark. (In a system of plus-marks in bins, a bin receives at most one plus-mark.) And, finally, an **M**-graph G will be said to *contain the content of* a **WSP** W if:

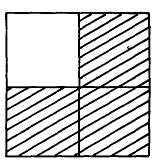
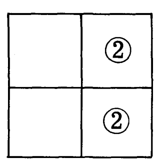
(1) G is an inconsistent **M**-graph; or

(2) W contains a vacuous **M**-graph; or

(3) for each consistent way of putting plus-marks and/or shading in the content-bins of G in such a way that G's existential commitments are satisfied, there is some consistent **M**-graph of W whose shading is a subsystem of G's resulting shading and whose existential commitments are also satisfied by that way of putting plus-marks in its bins (where a bin of G is understood to be a *content-bin of* G if it is marked in G or if the corresponding bin of one of the consistent **M**-graphs of W is marked).

For example, the **M**-graph 

contains the content of the **WSP** . As an additional example, the **M**-graph 

contains the content of the **WSP**  v 

, as the reader may readily verify.

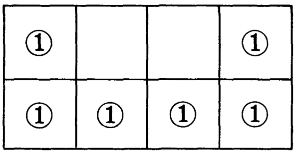
The three clauses of the foregoing definition may be considered the rules of content of the method of **WSDs**. Actually, since the third rule makes the second redundant, the reader may prefer to regard as primitive just the first and third rules and to establish the second as a derived rule of content.[9]

One can readily prove that a **WSP** *W* depicts all the world-states depicted by an **M**-graph *G* if and only if *G* contains the content of *W*. It follows, accordingly, that the conclusion **WSP** of a **WSD** depicts all the world-states depicted by the premiss **WSP**s if and only if every **S-M**-graph contains the content of the conclusion **WSP**. (Implicit in this last assertion is the assumption that a **WSD** for an argument always contains premiss **WSP**s, even when the argument is one from no premisses. This matter will be disposed of shortly.) If all its **S-M**-graphs contain the content of the conclusion **WSP**, a **WSD** will be said to be *diagrammatically valid.* One can decide diagrammatic validity effectively by applying the rules of content a finite number of times, once for each **S-M**-graph. The connection between diagrammatic validity and validity should now be apparent and is simply this: an argument is valid (in all domains whether empty or not) if and only if its **WSD** is diagrammatically valid. (Actually there are many **WSD**s for the same argument. They have in common, however, the all-important feature that whenever one of them is valid they are all valid and, consequently, that whenever one of them is not valid none of them are valid. Hence, so far as validity is concerned, one may talk as if there were only one **WSD** for a given argument.)

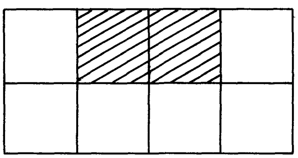
In connection with inspecting a **WSD** for validity, there are 4 subtle points which must yet be made. First, it is assumed throughout this paper that all the **M**-graphs which appear in a **WSD** are alike both in the numbering and in the labelling of their bins, a requirement easily enough satisfied. Second, in constructing a **WSD**, whenever a circled-numeral is needed to indicate nonemptiness of some system of bins of an **M**-graph, one must use a circled-numeral not yet appearing anywhere in the diagram. Third, a word or two must be said about the process of superimposing **M**-graphs to get a **S-M**-graph. In the process of superimposing **M**-graphs, shading in a bin erases or deletes an occurrence of a circled-numeral in that bin unless that circled-numeral appears only in bins that are shaded. In the latter case the diagram is inconsistent and is so marked by drawing a large cross through it. As remarked above, an inconsistent **M**-graph contains the content of any **WSP**. Fourth and finally, solely in the interest of tidiness, superimposition may be taken to erase or delete all occurrences of a circled-numeral which appears in every bin in which some one other circled-numeral appears. One must, of course, regard this last statement to apply to the circled-numerals one at a time.

Part Two: Proof that **WSD**s Constitute a Decision Procedure For Uniform Quantification Theory

Before disposing of the problem of diagramming arguments from no premisses, I will show that every argument of uniform (closed) quantification theory is world-state-diagrammable. Closely following Quine, by a *uniform open* (quantificational) *schema* I shall understand a quantifier-free wff that contains only monadic predicates, grouping indicators, truth-functional connectives, and a single individual variable which, of course, may have many occurrences in that wff. For example, '$Fz \supset (Gz \equiv Hz)$' and '$Gy \vee Hy$' are uniform open schemata. Given a uniform open schema which appears in some argument of uniform quantification theory, truth-tabular techniques suffice to get an equivalent uniform open schema in full disjunctive normal form and which contains every predicate which occurs in the argument. Thus, given the uniform open schema '$Fz \supset (Gz \equiv Hz)$' appearing in an argument which contains only the predicates 'F', 'G', and 'H', a truth-table yields the following equivalent full disjunctive normal schema '$FzGzHz \vee Fz\overline{G}z\overline{H}z \vee \overline{F}zGzHz \vee \overline{F}zGz\overline{H}z \vee \overline{F}z\overline{G}zHz \vee \overline{F}z\overline{G}z\overline{H}z$'. From this normal schema one can readily get an **M**-graph which incorporates the content of the existential closure of the given open schema, viz.

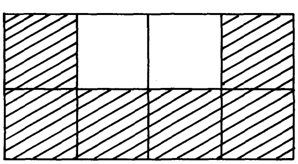


. This **M**-graph is constructed by putting an occurrence of the same circled-numeral in every bin which corresponds to a disjunct of the full disjunctive normal schema to signify the nonemptiness of one or more of the specific-predicates represented by those bins. To get an **M**-graph for the universal closure of that same open schema, simply shade all those bins, if any, which do not correspond to any of the disjuncts of the equivalent full disjunctive normal schema. Thus

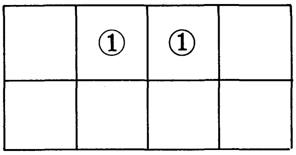


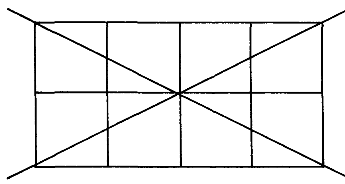
is an **M**-graph for the universal closure of the given uniform open schema. To get an **M**-graph for the closure of a consistent uniform open schema, *it should be emphasized that the detour through full disjunctive normal schemata is often unnecessary.* Logical acumen alone usually enables one to see *immediately* which bins require shading or which need circled-numeraling as the case may be. But it is comforting to know that the detour, which hinges on nothing more difficult than truth-tables, is always available should acumen happen to fail.

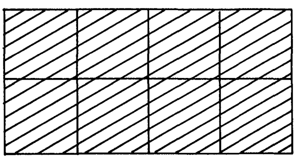
To get the **M**-graph for the negation of the universal or existential closure of a uniform open schema, one simply replaces shading with a circled-numeral, or vice versa as the case may be, in the **M**-graph for the universal or existential closure. The result is an **M**-graph which embodies the necessary and sufficient condition for the falsity of the unnegated closure. Thus the **M**-graph for '$\sim(\exists z) \,.\, Fz \supset (Gz \equiv Hz)$' is



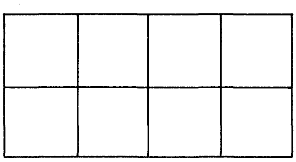
whereas the **M**-graph for '$\sim(z) \,.\, Fz \supset (Gz \equiv Hz)$'

is 

. Suppose, however, that a uniform open schema is inconsistent, e.g. '*FxGxHx* ≡ ∼ (*FxGxHx*)'. Clearly the **M**-graph for its existential closure must be an inconsistent **M**-graph which is obtained by drawing a cross through an **M**-graph thus  . But the universal closure of an inconsistent uniform open schema, although always false in all nonempty domains of individuals, is always true in the empty domain of individuals. Its **M**-graph, then, is a completely shaded-out **M**-graph. For example, the **M**-graph for '(*x*) . *FxGxHx* ≡ ∼ (*FxGxHx*)' is



. To get the **M**-graph for the negation of the universal closure of an inconsistent uniform open schema, one proceeds exactly as above, i.e. by supplanting shading with a circled-numeral. But how does one graph the negation of the existential closure of such a schema, i.e. how does one graph the negation of a wff which has an inconsistent **M**-graph? Since the negated wff will be valid, one needs an **M**-graph which depicts every world-state. Quite fortunately the vacuous **M**-graph, i.e. an **M**-graph with no shading and no circled-numerals, has this very property. Thus the **M**-graph for '∼(∃*x*) . *FxGxHx* ≡ ∼ (*FxGxHx*)'

is  .

So far, then, I have shown how one could mechanically graph the existential and the universal closures of uniform open schemata, and the negations of such closures. And, when discussing Venn-procedures in syllogistics, I suggested how to graph a conjunction of graphable formulas. To obtain such an **M**-graph, merely superimpose the **M**-graphs for all the conjuncts. I also indicated how to draw a **WSP** for a disjunction of graphable formulas, viz. by stringing out the **M**-graphs for each disjunct and separating them by 'v's.

Again closely following Quine, by a *uniform closed* (quantificational) *schema,* I shall understand a truth function of closures of uniform open schemata.[6] I shall call the closures of uniform open schemata *quantificational units*. Truth-tabular methods suffice to show that any uniform closed schema is equivalent to some "normal" uniform closed schema, i.e. to a schema which is a disjunction of conjunctions each conjunct of which is either a quantificational unit or the negation of a quantificational unit. Moreover, truth-tables effectively yield these equivalent normal schemata. And, from the preceding paragraph it should be obvious how to get mechanically a **WSP** for any normal schema. It follows, then, that all uniform closed schemata are world-state-picturable. And from this last statement it follows immediately that all arguments of uniform closed quantification theory are world-state-diagrammable. **WSD**s, then, constitute a decision procedure for validity for uniform closed quantification theory.

In addition, uniform quantification theory may be said to be complete in this very intuitive sense: given any **WSP**, there is a uniform closed schema whose content is represented by that **WSP**. To prove this last assertion, one simply shows how, given an arbitrary **M**-graph, to get a uniform closed schema whose content is represented by that **M**-graph. Then, to get a uniform closed schema whose content is that represented by an arbitrary **WSP**, first find a uniform closed schema for each **M**-graph of the **WSP** and then merely write a disjunction of those schemata. Now a **WSP** depicts none, some, or all of the possible world-states determined by the specific-predicates. Further, given any set of the possible world-states, there is a **WSP** that depicts all and only those world-states. Thus one can look upon the semantical import of a uniform closed schema as expressing that the world is in one of certain world-states, viz. those depicted by its **WSP**. (Again, although there are many **WSP**s for a schema, they all depict the same world-states. Thus one may speak as if there were only one such **WSP**.) The semantical completeness of uniform closed quantification theory, then, comes to this: given k distinct predicates, they determine 2^{2^k} possible world-states; given any set of those world-states, there is a uniform closed schema which says that the world is in one of those states.

If effective procedures for reducing monadic schemata to equivalent uniform schemata are incorporated, the foregoing decision procedure may be applied to any argument of monadic quantification theory. One such elegant procedure is that of Quine which turns on interchange of equivalents of only four simple types.[10]

There remains only the problem of dealing with arguments from no premisses. Such an argument is valid if and only if its conclusion is valid. Now a **WSP** for a valid schema depicts all possible world-states. But a vacuous **M**-graph contains the content of a **WSP** if and only if the latter depicts all possible world-states. Hence, to get a **WSD** for an argument from no premisses, one could construct a conclusion **WSP** as usual and then add a vacuous **M**-graph as a premiss **WSP**. There are, however, shortcuts for accomplishing the same thing. If the conclusion of an argument from no premisses has the form $\ulcorner B \supset C \urcorner$, that argument is valid if and only if the argument, more easily handled by **WSD**s, from B to C is valid. And, should its conclusion have the form $\ulcorner B \equiv C \urcorner$, such an argument is valid if and only if the two arguments from B to C and from C to B are both valid.

Those who find diagrammatic methods more perspicuous than algebraic ones will perhaps find the methods outlined in this paper to be a useful pedagogical device as well as a practical decision procedure for relatively simple arguments. And those who do not may at least find it interesting to learn that Venn-diagrammatic techniques are not so restricted in scope as is commonly believed.

NOTES

1. Gardner, *Logic Machines and Diagrams,* p. 43.
2. Quine, *Methods of Logic,* p. 81.
3. *Ibid.,* p. 81.
4. *Ibid.,* p. 81 f.
5. *Ibid.,* p. 82.
6. In this paper uniform quantification theory is understood in a sense slightly different from Quine's. It is here considered to be the study of the class of wffs that can be obtained from Quine's uniform closed quantificational schemata by alphabetic change of bound individual variables. Cf. Quine, *op. cit.,* p. 91.
7. There are, of course, many well-known decision procedures for monadic quantification theory (Cf. Church, *Introduction to Mathematical Logic,* sec. 46) and, *a fortiori,* for uniform quantification theory. A particularly elegant decision procedure for the latter is that of Quine, *op. cit.,* pp. 107-113. The justification for putting forth the decision procedure of this paper is that, unlike other known decision procedures for uniform quantification theory, it is *diagrammatic* and related in obvious ways to Venn procedures.
8. Quine, *op. cit.,* p. 74.
9. Actually, once a **WSD** had been constructed, one could use truth-tables to determine the validity of the diagrammed argument. By associating distinct sentence letters with distinct marked bins, one could read off directly from the **WSD** a truth function which is a tautology if and only if the argument is valid. (Reduction of the validity of a monadic schema to the tautologousness of a certain truth function of existential closures of specific predicates is essentially Herbrand's decision procedure for monadic schemata. Cf. Herbrand, *Recherches sur la Théorie de la Démonstration,* pp. 52-54.) Thus, in a sense, the methods of WSDs are merely a diagrammatic test of tautologousness, and the same can be said of Venn procedures in general.
10. Quine, *op. cit.,* pp. 193-194.

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