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A GROUP-THEORETIC CHARACTERIZATION OF THE ORDINARY AND ISOTROPIC EUCLIDEAN PLANES*

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I. INTRODUCTION. It is known that every geometry determines a unique group, namely, the group of transformations under which the geometry remains invariant.

The converse problem is concerned with the determination of a geometry corresponding to an abstract group. The problem of characterizing a geometric space group-theoretically is solved by defining an abstract group in such a way that it determines uniquely the geometric space in question and such that its structure corresponds to that of the transformation group of the space.

Group-theoretic characterization of a geometric space is based on the line reflection as its fundamental concept. G. Hessenberg [5] and J. Hjelmslev [6, 7] first brought out the significance of the property—known as the theorem of the three reflections—that the product of the reflections in three lines concurrent in a point is again a reflection in a line through the same point. Hjelmslev used methods based on reflections systematically and studied the foundations of geometry in this light.

G. Thomsen [13] described the Euclidean plane group-theoretically and developed effective methods of proof based on reflection calculus. In his work Thomsen also indicated the extension of these methods to higher dimensional Euclidean spaces and to the projective and non-Euclidean planes.

A. Schmidt [10] assumed the theorem of the three reflections as an axiom. He was the first to formulate group-theoretic axioms for the plane absolute geometry, including the elliptic plane, giving dominance to the importance of the line reflections as generating the other motions. F. Bachmann [2] later reduced Schmidt's axioms and developed methods of considering metric planes for the Euclidean, hyperbolic and elliptic cases.

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More recently work has been done by E. Sperner and H. Karzel in an effort to characterize geometric planes group-theoretically. As an example, Sperner [11] has given a group-theoretic proof of the theorem of Desargues, using an axiom system based on Bachmann's. With a view to making these methods universally applicable, i.e., to higher dimensions, he has studied general affine spaces in the light of the problem of coordinatization [12].

A modification of Sperner's axioms was employed by Karzel [8] to characterize the classes of elliptic, regular pole-free, and "Lotkern" (every line is incident with the pencil of its perpendiculars) geometries. Karzel has also characterized the finite Desarguesian geometries, including those coordinatized over fields of characteristic 2 [9].

The aim of this paper is to characterize by group-theoretic axioms the ordinary and isotropic Euclidean planes, the latter being not totally isotropic but containing isotropic vectors. Planes coordinatized over fields of characteristic $\neq 2$ are considered here. Such a plane has associated with it a quadratic form which is invariant under rigid motions, and this quadratic form corresponds to a symmetric matrix over the associated field.

Motivated by geometric considerations, certain existence and uniqueness axioms are assumed for the elements of a group generated by involutions. Group-theoretic definitions of incidence and orthogonality relations and of motions as inner automorphisms enable us to construct a 2-dimensional vector space over a field, such that the structure of its group of linear transformations is determined by the group which has been described axiomatically. Coordinatization of the plane over the field gives a matrix representation of the transformation group and makes possible the determination of the required quadratic form.

II. FUNDAMENTAL PROPERTIES OF THE GROUP

We start with a group $\mathcal{G}$ generated by a set of involutions called lines and define points, translations and rotations as group elements satisfying certain conditions. Incidence, parallelism and orthogonality relations are similarly defined. Four existence axioms and two uniqueness axioms are assumed for the elements of $\mathcal{G}$, and these are sufficient to ascertain the existence of an invariant abelian subgroup of translations (THEOREM 1).

A. Definitions, axioms and fundamental relations. Let $\mathcal{G}$ be a group generated by a set $\mathcal{S}$ of involutions called *lines*. Let $\mathcal{S}$ consist of full conjugacy classes. The usual notation for a multiplicative group will be used, i.e., gh denotes the product of g times h, g^{-1} the inverse of g, and 1 the identity element of $\mathcal{G}$.

A *point* P is a product of two distinct commuting lines: $P = gh = hg$. P is called the point of intersection of g and h.

Two distinct lines, g and h, are said to be *orthogonal* if $gh = hg$, denoted by $g \perp h$.

A point P and a line g are *incident* with each other if $Pg = gP$, denoted by $P|g$.

Points P and Q are *collinear* if there is a line g such that $P|g$ and $Q|g$.

A set of lines is said to be *concurrent* if there is a point P which is incident with every line in the set.

Two lines, g and h, are called *parallel*, in symbols $g||h$, if either $g = h$ or if g and h are incident with no common point.

Negation of any relation is denoted by "/", e.g., the symbols "$\neq$", "$\not|$", "$\not\perp$", "$\not||$" denote the negation of the corresponding relations "$=$", "$|$", "$\perp$", "$||$".

If $P|g$ and $Q|g$, we use the abbreviated symbol $P,Q\,|g$. The symbols $P\,|g,h$, $g||h,k$, etc., are defined similarly.

Other notation used in this paper includes

"$\rightarrow\leftarrow$" - contradiction
"$\exists$" - there exist(s)
"$\Longrightarrow$" - implies
"$\Longleftrightarrow$" - implies and is implied by
"$\ni'$" - such that.

The symbols "$\cup$", "ε", etc., have the usual set-theoretic connotation.

$\bar{x}(y) = xyx^{-1}$ is the *transform* of y by x. It is an immediate consequence of the above definitions and of the fact that $\mathcal{S}$ is invariant under all inner automorphisms that transformation by any group element x carries lines into lines, points into points, and preserves orthogonality, incidence and parallelism. In particular, transformation by a line g is called *reflection in* g, and transformation by a point P is called *reflection in* P. Since $\bar{g}(P) = P$ if and only if $P|g$ and $\bar{P}(g) = g$ if and only if $P|g$, it is clear that reflection in g fixes only points incident with g and reflection in P fixes only lines through P. Also, reflection in g leaves a line $h \neq g$ invariant (but not pointwise) if and only if $h \perp g$.

The point M is called the *midpoint* of points P and Q if $\bar{M}(P) = Q$.

We require that the elements of $\mathcal{G}$ satisfy the following axioms:

A_1. *There exist at least three non-collinear points.*

A_2. *Given two distinct points, P and Q, there is at most one line g such that $P,Q\,|g$.*

A_3. *The product of three concurrent lines is a line.*

A_4. *Given a point P and a line g, there is at most one line h such that $P|h$ and $h||g$.*

A_5. *The product of three points is a point.*

A_6. *Given two collinear points P and Q, there exists a midpoint, M, of P and Q.*

Using A_2 we can now prove

Proposition 1. *If two distinct lines have a common perpendicular, the lines are parallel.*

Proof. Let $P = gh = hg$ and $Q = gj = jg$ and $h \neq j$. If there exists a point R such that $R\,|h,j$, then $\bar{g}(R)|h,j$, so $\bar{g}(R) = R$. Hence $R|g,h \Longrightarrow R = P = gh$, and $R|g,j \Longrightarrow R = Q = gj$, so $h = j$ which is a contradiction. Therefore $h\,||j$.

From A_3, the property of the three reflections, it follows that a perpendicular can be erected at a given point on a line, i.e.,

Proposition 2. *If $P|g$, $\exists$ a line h such that $P|h$ and $h \perp g$.*

Proof. Let $P = jk = kj$. Since $P|g,j,k$, by A_3, $h = gjk$ is a line. Then $(gh)^2 = (jk)^2 = 1 \Longrightarrow g \perp h$. It is clear that $P|h$, since P commutes with g,j and k.

The fourth axiom—the uniqueness of parallels—is the axiom which, in effect, keeps the geometry 2-dimensional. Using A_4 we prove

Proposition 3. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

Proof. Let $g || h$ and $k \perp g$. Suppose $k \not\perp h$. Let $P = gk = kg$. Since $g || h$ by A_4, $k \not|| h$, say $Q | k,h$. Let j be the perpendicular to k through Q. Then k is a common perpendicular to g and j, so $g || j$. But h is the unique parallel to g through Q, hence $j = h$, and $k \perp h$, $\rightarrow\leftarrow$. Therefore $k \perp h$.

Two further definitions are now made, and some properties concerning them are derived.

A *translation* is a product of two points.

Four points P, Q, R, S form a *parallelogram* if $PQRS = 1$. This definition will be justified, but we observe first that the vector axiom or "small theorem of Desargues" follows immediately from the definition, i.e.,

Proposition 4. *If P, Q, R, S form a parallelogram and S, R, U, T form a parallelogram, then P, Q, U, T form a parallelogram.*

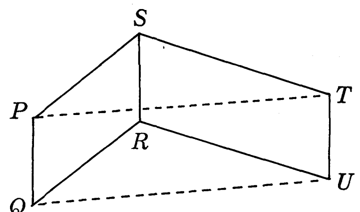

Figure 1

Proof. $PQRS = 1$ and $SRUT = 1$, so $PQRSSRUT = PQUT = 1$.

The definition of a parallelogram is justified by considering that the "opposite sides," if they exist, are parallel, i.e.:

Given $PQRS = 1$, suppose lines g, h, j, k exist such that $P|g,j$ and $Q|j,h$ and $R|h,k$ and $S|k,g$.

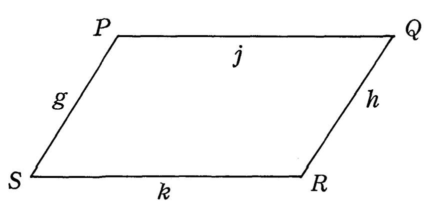

Figure 2

Suppose $j \not\parallel k$, say there exists a point T such that $T|j,k$. Let $U = TPQ$. $Uj = TPQj = jTPQ = jU$. Similarly, $Uk = TPQk = TSRk = kTSR = kU$, since $PQ = SR$. By A_5 U is a point, so $U|j,k \Longrightarrow U = T$. Therefore, $T = TPQ$ implies that $P = Q$ and $R = S$, i.e., the parallelogram is degenerate. Therefore, the joins of the points forming a non-degenerate parallelogram are parallel.

B. The translation group and its properties.

It is now possible to prove

THEOREM 1. *The set $\mathcal{T}$ of all translations is a normal abelian subgroup of $\mathcal{G}$.*

Proof. $PP = 1$ for any point P, so $1 \varepsilon \mathcal{T}$. $PQ \cdot RS = P \cdot QRS \varepsilon \mathcal{T}$ since QRS is a point. $(PQ)^{-1} = QP \varepsilon \mathcal{T}$. If P,Q,R are points, $\bar{P}(QR) = RQ$, since $PQRP^{-1}QR = (PQR)^2 = 1$. This follows from the fact that PQR, being a point, must be an involution. Therefore, a translation is transformed by a point into its inverse. This implies that $\mathcal{T}$ is abelian, since

$$PQ \cdot RS \cdot (PQ)^{-1} \cdot (RS)^{-1} = PQRSQPSR = P \cdot Q(RS)Q^{-1} \cdot PSR = (PSR)^2 = 1$$

Finally $\mathcal{T}$ is normal in $\mathcal{G}$: for any $g \varepsilon \mathcal{G}$, $gPQg^{-1} = \bar{g}(P)\bar{g}(Q) = RS \,\varepsilon\, \mathcal{T}$, since the transform of a point is a point.

In order to eliminate the possibility that a translation have order 2, it must be shown that for any two distinct points P and Q, $PQ \neq QP$. First, suppose there exists a line k such that $P,Q|k$. By Proposition 2 there exist lines g and h such that $P = gk = kg$ and $Q = hk = kh$. Then k is a common perpendicular to g and h, so $g||h$. Therefore, $PQ = gkkh = gh \neq hg = QP$; otherwise gh would be the point of intersection of g and h.

Next consider the general case. Given P and Q, two distinct points, let $P = gj = jg$ and assume $Q \not| g,j$; otherwise the preceding argument suffices. Furthermore let us suppose—this point will be taken up later—that there are lines h and k passing through Q and perpendicular to j and g respectively. Since g is a common perpendicular to j and k, $j||k$. But $j||k$ and $h \perp j$ implies that $h \perp k$. Then $PQP = gjhkgj = ghjgkj = ghgjkj = \bar{g}(h)\bar{j}(k)$. But $Q \not| \bar{g}(h)$, since $Q|h,\bar{g}(h)$ implies $\bar{g}(Q)|h,\bar{g}(h)$. Hence $\bar{g}(Q) = Q$, $\rightarrow\leftarrow$. Therefore $\bar{g}(h)\bar{j}(k) \neq Q$, i.e. $PQP \neq Q$. This proves

Proposition 5. *For any two distinct points P and Q, $PQ \neq QP$.*

Corollary 1. *There exists no translation of order 2.*

Corollary 2. *A non-trivial translation has no fixed points.*

Proof. Let $\tau = PQ$ be a translation and R an arbitrary point. If $\bar{\tau}(R) = R$, then $PQRQP = R$, i.e., $PQR = RPQ$. But $PQR = RQP$ since points are involutions. Then $RPQ = RQP$ which implies that $PQ = QP$ $\rightarrow\leftarrow$. Therefore τ has no fixed points.

Before completing the proof of Proposition 5, we proceed to

Proposition 6. *For any translation τ and any line h, $\bar{\tau}(h)||h$.*

Proof. If $\overline{\tau}(h) = h$, $h \| h$ by definition. Consider the case where $\overline{\tau}(h) \neq h$. Suppose $\exists$ a point R such that $R | h, \overline{\tau}(h)$. It follows that $\overline{\tau}(R) | \overline{\tau}(h), \overline{\tau}^2(h)$. Let $\tau = PQ$. PQR is a point S, so $PQ = SR = \tau$. Therefore $\overline{\tau}(h) = \overline{SR}(h) = \overline{S}(h)$ since $R | h$, and $\overline{\tau}^2(h) = \overline{SR}(ShS) = \overline{S}(RShSR) = \overline{S}(ShS) = h$, since $R | \overline{\tau}(h)$, i.e., $R | \overline{S}(h) \Longrightarrow RShSR = ShS$. Therefore $R, \overline{\tau}(R) | h, \overline{\tau}(h)$ implies that $R = \overline{\tau}(R)$, $\rightarrow\leftarrow$. Hence $\overline{\tau}(h) \| h$.

If P and Q are collinear the assertion follows from the first part of the proof of Proposition 5. For P and Q not collinear the proof is contingent upon the existence of a perpendicular from a point to a line.

The existence of such a perpendicular is now established as a consequence of $\mathbf{A}_6$ and of Propositions 5 and 6 for the collinear case. It in turn establishes the validity of these propositions in the non-collinear case.

We first note that by $\mathbf{A}_6$ if two points P and Q are incident to the same line g, there is a point M such that $\overline{M}(P) = Q$. Since transformation by M takes P into Q and Q into P and transformation preserves incidence, it follows that $\overline{M}(g) = g$ and hence $M | g$. Furthermore $M \neq P$; otherwise $\overline{M}(P) = Q$ implies $P = Q$, which is impossible if P and Q are distinct points. Similarly, $M = Q$.

Proposition 7. *Given any point P and line g, there exists a unique line h such that $P | h$ and $h \perp g$.*

Proof. If $P | g$,h exists by Proposition 2. If $P \not| g$, let $P = jk = kj$. Not both j and k are parallel to g, say, there exists a point Q such that $Q | g,j$. $Q | g \Longrightarrow \exists\ h'$ such that $Q | h'$ and $h' \perp g$. $P,Q | j \Longrightarrow \exists$ a point M on $j \ni' \overline{M}(P) = Q$. Let $\tau = MQ$. Then $\overline{\tau}(Q) = P$. Since M and Q are collinear $h' \| \overline{\tau}(h')$. This, together with the fact that $h' \perp g$, implies that $\overline{\tau}(h') \perp g$. But $Q | h' \Longrightarrow \overline{\tau}(Q) | \overline{\tau}(h')$, i.e. $P | \overline{\tau}(h')$. Therefore $\overline{\tau}(h')$ is the desired line h.

Uniqueness: If $P | g$ and $P | h$ and $gh = hg$, then $P = gh$ because of the uniqueness of the join. Hence if h^* is another line satisfying the assertion of the theorem, $P = gh^*$ also, so $h = h^*$. If $P \not| g$ and h and h^* are two perpendiculars from P to g, then $gh = hg$, $gh^* = h^*g$ and $P | h,h^* \Longrightarrow \overline{g}(P) | h,h^*$ also. But $\overline{g}(P) \neq P$, so $h = h^*$.

Corollary. *Given any point P and line g there exists a line h such that $P | h$ and $h \| g$.*

Proof. For $P | g$, $h = g$. If $P \not| g$, $\exists\ j \ni' P | j$ and $j \perp g$. Also $\exists\ h \ni'\ P | h$ and $h \perp j$. Then $h \| g$ since j is a common perpendicular.

Proposition 8. *For any two points P and Q, there is a translation τ taking P into Q.*

Proof. If $P = Q$, let $\tau = 1$. If $P \neq Q$ and P and Q are collinear, let $\tau = MP$, where M is the midpoint of P and Q. If $P \neq Q$ and P and Q are not collinear, then $P = gh$ and $Q \not| g,h$. Let j be the perpendicular from Q to g, say $R = gj = jg$.

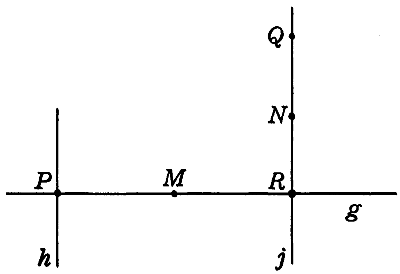

Figure 3

There exist points M and N such that $\overline{M}(P) = R$ and $\overline{N}(R) = Q$. Then $\overline{\tau}(P) = Q$ for $\tau = NM$.

Proposition 9. *On every line there are at least three points.*

Proof. For any line g $\exists$ by A_1 at least one point P such that $P \not| g$. There exists a line h such that $P|h$ and $h \perp g$, i.e., $gh = hg = R$. Hence $R|g$.

Again by A_1 $\exists$ a point $Q \ni' Q \not| h$. Either $Q|g$ or $\exists$ a line $k \ni' Q|k$ and $k \perp g$.

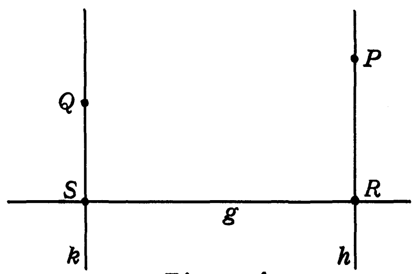

Figure 4

In the latter case $gk = kg = S \neq R$ (otherwise $k = h$ and $Q|h$, $\rightarrow\leftarrow$). Hence there are two distinct points on g. Applying A_6 we obtain a point M, the midpoint, as a third distinct point on g.

III. CONSTRUCTION OF VECTORS, SCALAR MAPPINGS AND THE TRANSFORMATION GROUP

In this section we define vectors as translations and scalars as vector mappings. We then obtain a group $\hat{\mathcal{G}}$ of linear transformations on the set of vectors (THEOREM 2). $\hat{\mathcal{G}}$ is a homomorphic image of $\mathcal{G}$ and is expressed in terms of an abelian subgroup of rotations defined in $\hat{\mathcal{G}}$ (THEOREM 3).

A. Vectors. The *vector* $\overrightarrow{PQ}$ is defined to be the translation RP for which $RPR = Q$. We observe first that every vector $\overrightarrow{RS}$ may be produced from an arbitrary fixed point P. For, $\overrightarrow{RS} = TR$ where $TRT = S$. Let TRP be the point U and STU be the point Q. Then $QU = ST = TR = UP = \overrightarrow{PQ}$ since $UPU = Q$.

Furthermore, since no translation except the identity has any fixed points, we know that the midpoint is always unique. Thus the vector $\overrightarrow{PQ}$ is well-defined and we are justified in interpreting the abelian group of translations as an additive group of vectors where the sum $\overrightarrow{PQ} + \overrightarrow{RS}$ corresponds to the product $TRUP$ if $\overrightarrow{PQ} = UP$ and $\overrightarrow{RS} = TR$. The usual notation for an additive abelian group will be used.

Under this definition we obtain the usual vector sum $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$:

Let $\overrightarrow{PQ} = UP$, where $UPU = Q$, and $\overrightarrow{QR} = TQ$, where $TQT = R$. Then $\overrightarrow{PQ} + \overrightarrow{QR} = TQ \cdot UP = SP$, where $S = TQU$. But $SPS = (TQU)P(UQT) = (TQ)Q(QT) = TQT = R$ so that $\overrightarrow{PR} = SP = \overrightarrow{PQ} + \overrightarrow{QR}$.

It also follows that $-\overrightarrow{PQ} = \overrightarrow{QP}$, since $\overrightarrow{PQ} = RP$ and $RPR = Q \Longrightarrow RQR = P$, so $\overrightarrow{QP} = RQ = PR = (RP)^{-1} = -\overrightarrow{PQ}$.

Proposition 10. *Given point P not incident with line g, h the perpendicular from P to g, meeting g in point M, then $\overrightarrow{PM} = -\overrightarrow{P'M}$ where P' denotes $\bar{g}(P)$.*

Proof. $P \not| g$, $P | h$, $gh = hg = M$, $\bar{g}(P) = P'$. $P' = gPg = ghPhg = MPM$. Let $\overrightarrow{PM} = RP$; $RPR = M$ and $\overrightarrow{P'M} = QP'$; $QP'Q = M$, so $-\overrightarrow{P'M} = P'Q = QM$. It suffices to show that $RP = P'Q$.

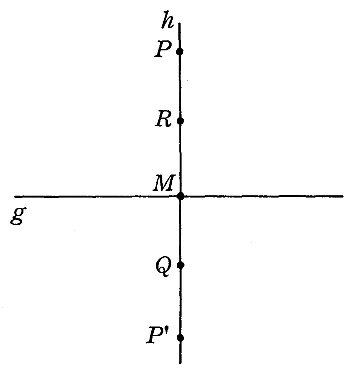

Figure 5

$RPR = M \Longrightarrow MRPRM = M \Longrightarrow (MRM)(MPM)(MRM) = M \Longrightarrow (MRM)P'(MRM) = M \Longrightarrow MRM = Q$. Hence $RP = MR = QM = P'Q$.

A vector $\mathbf{a}$ is said to be *regular* if there exists a line g and points P and Q incident with g such that $\overrightarrow{PQ} = \mathbf{a}$.

Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are *collinear* if there exists a line g and points P', Q', R', S' on g such that $\overrightarrow{P'Q'} = \overrightarrow{PQ}$ and $\overrightarrow{R'S'} = \overrightarrow{RS}$.

We note that if a regular non-zero vector is produced from two distinct points the lines along which the vector is produced are parallel. For:

Let $\overrightarrow{PQ} = \overrightarrow{RS}$ with $P, Q | g$ and $S, R | h$. $\overrightarrow{PQ} = TP \Longrightarrow T | g$ and $\overrightarrow{RS} = UR \Longrightarrow U | h$. Suppose $\exists$ a point V such that $V | g, h$ and $g \neq h$. Let $W = VTP = VUR$. Then $V, T, P | g \Longrightarrow W | g$ and $V, U, R | h \Longrightarrow W | h$. Hence $W = V$, so $TP = 1$, $\rightarrow\leftarrow$, since $\overrightarrow{PQ} \neq \mathbf{0}$. Therefore $g \| h$.

On the other hand we note that if $PQ = RS \neq 1$ and $P, Q | g$ and $R | h$ and $g \| h$, then $S | h$ also. For: let $P | j$, $j \perp g$. Then $j \perp h$; let $M = jh = hj$. Similarly, let $Q | k$, $k \perp g$, so $k \perp h$; let $N = kh = hk$.

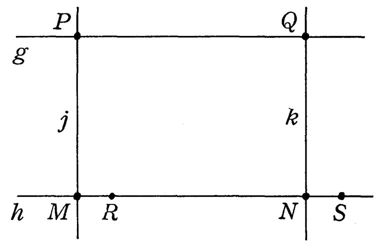

Figure 6

Then $PQNM = jggkkhhj = 1$, and by hypothesis $RSQP = 1$. Hence $RSNM = 1$ by Proposition 4, so $S = RMN$. Therefore $S|h$, since $R,M,N|h$.

Proposition 11. *The collinearity relation is an equivalence relation on the set of all regular vectors.*

Proof. (a) $\mathfrak{a}$ is collinear with $\mathfrak{a} \Longleftrightarrow \mathfrak{a}$ is a regular vector, by the definition of regularity.

(b) Symmetry is clear from the definition of collinearity.

(c) Transitivity: let $\mathfrak{a} = \overrightarrow{PQ}$, $\mathfrak{b} = \overrightarrow{RS}$, $\mathfrak{c} = \overrightarrow{TU}$. $\mathfrak{a}$ collinear with $\mathfrak{b} \Longrightarrow \exists$ a line g and points P', Q', R', S' on g such that $\overrightarrow{PQ} = \overrightarrow{P'Q'}$ and $\overrightarrow{RS} = \overrightarrow{R'S'}$. $\mathfrak{b}$ collinear with $\mathfrak{c} \Longrightarrow \exists$ a line h and points R'', S'', T', U' on h such that $\overrightarrow{RS} = \overrightarrow{R''S''}$ and $\overrightarrow{TU} = \overrightarrow{T'U'}$. Therefore $g||h$ since $\overrightarrow{R'S'} = \overrightarrow{R''S''}$.

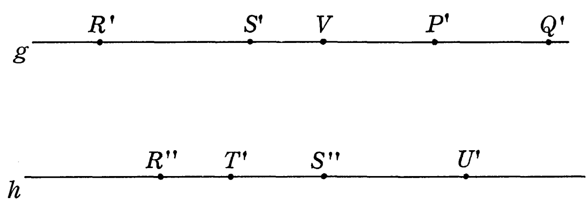

Figure 7

Now we have $g||h$ and $R'|g$ and $T', U'|h$. Producing the vector $\overrightarrow{T'U'}$ from point R', we have a point $V \ni' \overrightarrow{R'V} = \overrightarrow{T'U'}$. Let us say $\overrightarrow{T'U'} = MT'$, where $MT'M = U'$ (so $M|h$) and $\overrightarrow{R'V} = WR'$, where $WR'W = V$. Hence $WR' = MT'$. Then $M,T'|h$ and $R'|g \Longrightarrow W|g$. Therefore $V|g$ also. We now have P', Q', R', $V|g$ and $\mathfrak{a} = \overrightarrow{P'Q'} = \overrightarrow{PQ}$ and $\overrightarrow{R'V} = \overrightarrow{T'U'} = \overrightarrow{TU} = \mathfrak{c}$. Therefore $\mathfrak{a}$ is collinear with $\mathfrak{c}$.

B. Scalars.

A scalar is a vector mapping λ with the properties:

(1) $\lambda\mathfrak{a}$ is a vector.

(2) $\lambda(\mathfrak{a} + \mathfrak{b}) = \lambda\mathfrak{a} + \lambda\mathfrak{b}$.

(3) If $\lambda \neq 0$, λ is one-to-one.

(4) If $\mathfrak{a}$ is regular, then $\lambda\mathfrak{a}$ and $\mathfrak{a}$ are collinear.

We define addition and multiplication of scalars as follows:

$$(\lambda + \mu)\mathrm{a} = \lambda\mathrm{a} + \mu\mathrm{a}$$

$$(\lambda\mu)\mathrm{a} = \lambda(\mu\mathrm{a})$$

Under these definitions the set of all scalars is a ring with unit.

Further, assuming that there are at least three lines through a point, then, given $\mathrm{a} \neq 0$, there exist regular vectors b and c such that $\mathrm{c} = \mathrm{a} + \mathrm{b}$, and no two of the vectors $\mathrm{a},\mathrm{b},\mathrm{c}$ are collinear.

Thus for $\lambda \neq 0$, $\lambda\mathrm{a} = \lambda\mathrm{c} - \lambda\mathrm{b} \neq 0$, since λ is a one-to-one mapping.

Let $\lambda \neq 0$, $\mu \neq 0$; then for $\mathrm{a} \neq 0$ we have $\mu\mathrm{a} \neq 0$ and $\lambda(\mu\mathrm{a}) \neq 0$. Hence $\lambda\mu \neq 0$. Thus it follows that the ring of scalars has no proper divisors of zero.

C. Linear transformations.

A group of linear transformations on the set of vectors is obtained as follows:

For any $x \varepsilon \mathcal{G}$, define the vector mapping

$$\hat{x}: \overrightarrow{PQ} \longrightarrow \overrightarrow{\bar{x}(P)\bar{x}(Q)}$$

Since $\widehat{xy} = \hat{x}\hat{y}$, then the mapping

$$\phi: x \longrightarrow \hat{x}$$

is a homomorphism of $\mathcal{G}$. It is now shown that the image group $\hat{\mathcal{G}}$ is a group of linear transformations on the set of all vectors.

THEOREM 2. *For any $x \varepsilon \mathcal{G}$, the mapping $\hat{x} : \overrightarrow{PQ} \longrightarrow \overrightarrow{\bar{x}(P)\bar{x}(Q)}$ is a linear transformation.*

Proof.

(1) Let $\overrightarrow{PQ} = TP$, where $TPT = Q$ and $\overrightarrow{RS} = UR$, where $URU = S$. If $\mathrm{a} = \overrightarrow{PQ} = \overrightarrow{RS}$, then $TP = UR$. Then $\bar{x}(T)\bar{x}(P)\bar{x}(T) = \bar{x}(Q) \Rightarrow \overrightarrow{\bar{x}(P)\bar{x}(Q)} = \bar{x}(T)\bar{x}(P)$ and $\bar{x}(U)\bar{x}(R)\bar{x}(U) = \bar{x}(S) \Rightarrow \overrightarrow{\bar{x}(R)\bar{x}(S)} = \bar{x}(U)\bar{x}(R)$. But $TP = UR \Rightarrow \bar{x}(T)\bar{x}(P) = \bar{x}(U)\bar{x}(R)$. Therefore $\hat{x}(\overrightarrow{PQ}) = \hat{x}(\overrightarrow{RS})$, i.e., $\hat{x}$ is well-defined.

(2) If a and b are any two vectors, let $\mathrm{a} = \overrightarrow{PQ}$ and produce b from point Q, i.e., $\mathrm{b} = \overrightarrow{QR}$, so $\mathrm{a} + \mathrm{b} = \overrightarrow{PR}$. Then $\hat{x}(\mathrm{a} + \mathrm{b}) = \hat{x}(\overrightarrow{PR}) = \overrightarrow{\bar{x}(P)\bar{x}(R)} = \overrightarrow{\bar{x}(P)\bar{x}(Q)} + \overrightarrow{\bar{x}(Q)\bar{x}(R)} = \hat{x}(\overrightarrow{PQ}) + \hat{x}(\overrightarrow{QR}) = \hat{x}(\mathrm{a}) + \hat{x}(\mathrm{b})$.

(3) It remains for us to show $\hat{x}(\lambda\mathrm{a}) = \lambda\hat{x}(\mathrm{a})$. It suffices to consider only $\hat{x} = \hat{g}$, where g is a line, since the elements $\hat{g}$ generate $\hat{\mathcal{G}}$. Let P be any point on g and produce a from P, say, $\mathrm{a} = \overrightarrow{PQ}$. Let h be the perpendicular from Q to g, and $R = gh = hg$.

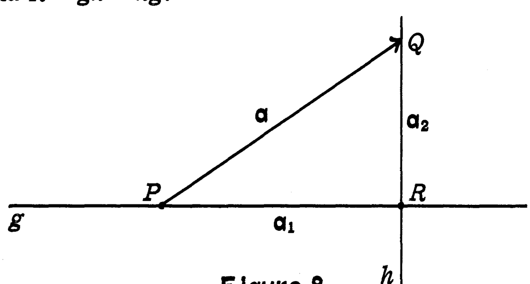

Figure 8

Then for $\mathfrak{a}_1 = \overrightarrow{PR}$ and $\mathfrak{a}_2 = \overrightarrow{RQ}$, $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$, and $\lambda\mathfrak{a} = \lambda\mathfrak{a}_1 + \lambda\mathfrak{a}_2$. But $\mathfrak{a}_1$, $\lambda\mathfrak{a}_1$ are collinear along line g so $\hat{g}$ fixes $\mathfrak{a}_1$ and $\lambda\mathfrak{a}_1$. Also, $\mathfrak{a}_2$ and $\lambda\mathfrak{a}_2$ are collinear along a line perpendicular to g, so by Proposition 10, $\hat{g}$ carries $\mathfrak{a}_2$ and $\lambda\mathfrak{a}_2$ into $-\mathfrak{a}_2$ and $-\lambda\mathfrak{a}_2$ respectively. Hence

$$\hat{g}(\mathfrak{a}) = \hat{g}(\mathfrak{a}_1) + \hat{g}(\mathfrak{a}_2) = \mathfrak{a}_1 - \mathfrak{a}_2$$

implies

$$\lambda\hat{g}(\mathfrak{a}) = \lambda\mathfrak{a}_1 - \lambda\mathfrak{a}_2,$$

and

$$\hat{g}(\lambda\mathfrak{a}) = \hat{g}(\lambda\mathfrak{a}_1) + \hat{g}(\lambda\mathfrak{a}_2) = \lambda\mathfrak{a}_1 - \lambda\mathfrak{a}_2.$$

D. Rotations. It is desirable to express $\hat{\mathcal{G}}$ in a more convenient form. To this end we define a rotation in $\mathcal{G}$.

If O is a point and g and h are lines incident with O, then the product $r = gh$ is called a *rotation about O*.

We note the following property of a rotation:

If r is a rotation about O and g is any line through O, then reflection in g transforms r into its inverse. For:

Let $r = jk$. Then $gjk = h$, a line through O, so $jk = gh$. $\bar{g}(r) = \bar{g}(jk) = \bar{g}(gh) = hg = (gh)^{-1} = (jk)^{-1} = r^{-1}$.

Proposition 12. *The rotations about a fixed point O form an abelian group $\mathcal{R}$.*

Proof. Let $r_1 = gh$ and $r_2 = jk$, where g, h, j, k are all lines through O. $r_1r_2 = gh\cdot jk = g\cdot hjk = gf\,\varepsilon\,\mathcal{R}$, since $f = hjk$ is a line through O. $r_1^{-1} = (gh)^{-1} = hg\,\varepsilon\,\mathcal{R}$. $g\cdot g = 1$ for any line g through O. Thus $\mathcal{R}$ is a group.

If f is any line through O, then $r_1^{-1} = \bar{f}(r_1)$ and $r_2^{-1} = \bar{f}(r_2)$, so that $r_1^{-1}r_2^{-1} = fghffjkf = \bar{f}(r_1r_2) = (r_1r_2)^{-1} = r_2^{-1}r_1^{-1}$. Therefore $r_1r_2 = r_2r_1$, and $\mathcal{R}$ is abelian.

THEOREM 3. *If $\mathcal{R}$ is the group of rotations about a fixed point O and g is a fixed line through O, then $\hat{\mathcal{G}} = \hat{\mathcal{R}} \cup \hat{g}\hat{\mathcal{R}}$.*

Proof. We show first that $\mathcal{T} \subseteq ker\ \phi$ where $\phi : x \longrightarrow \hat{x}$, $x\,\varepsilon\,\mathcal{G}$. Let $\tau\,\varepsilon\,\mathcal{T}$ and let $\overrightarrow{PQ} = RP$ be an arbitrary vector, i.e., $RPR = Q$. Then

$$\bar{\tau}(R)\bar{\tau}(P)\bar{\tau}(R) = \bar{\tau}(Q)$$

implies

$$\hat{\tau}(\overrightarrow{PQ}) = \overrightarrow{\bar{\tau}(P)\bar{\tau}(Q)} = \bar{\tau}(R)\bar{\tau}(P) = \bar{\tau}(RP).$$

But $\bar{\tau}(RP) = RP$ since $\mathcal{T}$ is abelian. Hence $\hat{\tau}(\overrightarrow{PQ}) = \overrightarrow{PQ}$ and $\hat{\tau} = \hat{1}$.

Suppose $\hat{h}\,\varepsilon\,\hat{\mathcal{G}}$ and $O \not| h$, where h is a line in $\mathcal{G}$. Let j be the perpendicular from O to h, $P = jh = hj$. Let $h^* || h$, $O | h^*$, and let M be the midpoint of O and P.

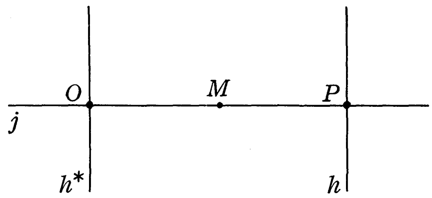

Figure 9

Choose $\tau = MP$. $\overline{\tau}(P) = O$ and $\overline{\tau}(h) \| h \Rightarrow \overline{\tau}(h) = h^*$. Thus $\widehat{h}^* = \widehat{\tau h \tau^{-1}}$ $\Rightarrow \widehat{h}^* = \widehat{\tau}\widehat{h}\widehat{\tau^{-1}} = \widehat{h}$. Therefore the images of lines through O can be chosen as generators of $\widehat{\mathcal{G}}$.

Finally, if k is an arbitrary line through O, $r = gk \,\varepsilon\, \mathcal{R}$, and $k = g \cdot r \,\varepsilon\, \widehat{g}\widehat{\mathcal{R}}$. Hence $\widehat{\mathcal{G}} = \widehat{\mathcal{R}} \cup \widehat{g}\widehat{\mathcal{R}}$.

IV. CONSTRUCTION OF THE PLANE

In this section we construct over a field a 2-dimensional linear space which is invariant under the group of linear transformations, $\widehat{\mathcal{G}}$ (THEOREM 4 and COROLLARY).

A. Properties of rotations; scalar field. The following property is first noted:

If r is a rotation about a point O and P is any point of a line g through O, then $\overline{r^{-1}}(P) = \overline{g}(\overline{r}(P))$.

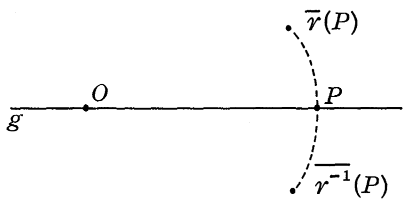

Figure 10

Let $r = hj$. $O | g,h,j \Rightarrow rg = hjg = k$, a line through O; hence $r = hj = kg$. Then we have $\overline{r}(P) = rPr^{-1} = kgPgk = kPk$ and $\overline{r^{-1}}(P) = r^{-1}Pr = gkPkg = \overline{g}(kPk) = \overline{g}(\overline{r}(P))$.

Addition of linear transformations is defined as follows: Given $\widehat{x}, \widehat{y} \,\varepsilon\, \widehat{\mathcal{G}}$, $\mathfrak{a}$ a vector,

$$(\widehat{x} + \widehat{y})\mathfrak{a} = \widehat{x}(\mathfrak{a}) + \widehat{y}(\mathfrak{a}).$$

Proposition 13. *In the transformation group, $\widehat{\mathcal{G}}$, the sum of a rotation and its inverse is a scalar mapping.*

Proof. Let $r \,\varepsilon\, \mathcal{R}$, the group of rotations about point O, and let $\widehat{r}$ be its image in $\widehat{\mathcal{R}}$. Let $\mathfrak{x}$ be a regular vector, say $\mathfrak{x} = \overrightarrow{OX}$, where O and X are incident with some line g. (It suffices to consider only regular vectors, since every vector is the sum of two regular vectors.)

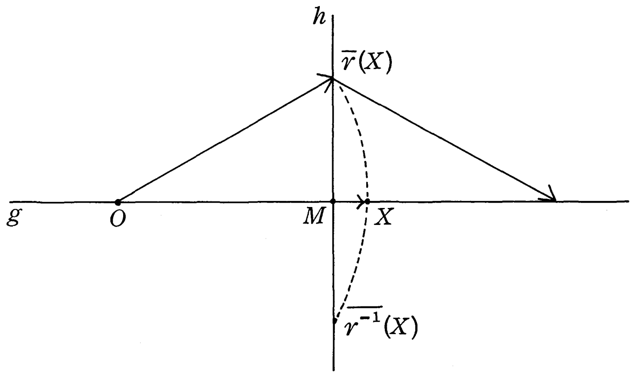

Figure 11

Let h be the perpendicular from $\bar{r}(X)$ to g, and $M = gh = hg$. Then

$$
\begin{aligned}
(\hat{r} + \hat{r}^{-1})\overrightarrow{OX} &= \hat{r}(\overrightarrow{OX}) + \hat{r}^{-1}(\overrightarrow{OX}) \\
&= \overrightarrow{\bar{r}(O)\bar{r}(X)} + \overrightarrow{\overline{r^{-1}}(O)\overline{r^{-1}}(X)} \\
&= \overrightarrow{O\bar{r}(X)} + \overrightarrow{O\overline{r^{-1}}(X)}, \text{ since } \bar{r}(O) = \overline{r^{-1}}(O) = O, \\
&= \overrightarrow{OM} + \overrightarrow{M\bar{r}(X)} + \overrightarrow{OM} + \overrightarrow{M\overline{r^{-1}}(X)} \\
&= 2\overrightarrow{OM} + \overrightarrow{M\bar{r}(X)} + \overrightarrow{M\bar{g}(\bar{r}(X))}, \text{ by the preceding remark,} \\
&= 2\overrightarrow{OM} + \overrightarrow{M\bar{r}(X)} - \overrightarrow{M\bar{r}(X)}, \text{ by Proposition 10.} \\
&= 2\overrightarrow{OM}.
\end{aligned}
$$

Hence, $\hat{r} + \hat{r}^{-1}$ maps a regular vector onto a collinear vector, and since $\hat{r}$ and $\hat{r}^{-1}$ are homomorphic mappings, so is $\hat{r} + \hat{r}^{-1}$. Therefore $\hat{r} + \hat{r}^{-1}$ is a scalar mapping, provided that the characteristic of the ring of scalars is not 2. This is no restriction however, since if $2\overrightarrow{PQ} = \mathbf{0}$ for an arbitrary vector $\overrightarrow{PQ} = RP$, then $(RP)^2 = 1$, contrary to Proposition 5.

The scalars of the form $\lambda = \hat{r} + \hat{r}^{-1}$ together with identity generate by additions and multiplications an integral domain. We denote its quotient field by $\mathfrak{J}$.

B. Construction of the linear space. We define $\mathfrak{L}$ to be the linear space generated by the vectors $\mathbf{a}$, $\hat{r}_1(\mathbf{a})$, $\hat{r}_2(\mathbf{a})$, $\widehat{r_1 r_2}(\mathbf{a})$ over the field $\mathfrak{J}$, where $\mathbf{a}$ is an arbitrary regular vector and $\hat{r}_1$ and $\hat{r}_2$ are arbitrary rotations in $\hat{\mathfrak{R}}$. We know that

$$\hat{r}_i + \hat{r}_i^{-1} = \lambda \in \mathfrak{J},\ i = 1, 2.$$

Hence

$$\hat{r}_i^{\,2} = \lambda_i \hat{r}_i - \hat{1},$$

which implies that $\mathfrak{L}$ is invariant under $\hat{r}_1$ and $\hat{r}_2$.

Let g be the line of $\overrightarrow{OP} = \mathbf{a}$, so

$$\hat{g}^2 = \hat{1}$$

and

$$\hat{g}(\mathrm{a}) = \mathrm{a}.$$

Also, by the remark made at the beginning of Section IV, A,

$$\hat{g}\hat{r}_i\hat{g}^{-1} = \hat{r}_i^{-1}.$$

Hence

$$\hat{g}\hat{r}_i(\mathrm{a}) = \hat{g}\hat{r}_i\hat{g}^{-1}\hat{g}(\mathrm{a}) = \hat{r}_i^{-1}\hat{g}(\mathrm{a}) = \hat{r}_i^{-1}(\mathrm{a}) = \lambda_i\mathrm{a} - \hat{r}_i(\mathrm{a}).$$

Similarly,

$\hat{g}\widehat{r_1r_2}(\mathrm{a}) = \hat{g}\widehat{r_1r_2}\hat{g}^{-1}\hat{g}(\mathrm{a}) = (\widehat{r_1r_2})^{-1}\hat{g}(\mathrm{a}) = (\widehat{r_1r_2})^{-1}(\mathrm{a}) = \hat{r}_1^{-1}\hat{r}_2^{-1}(\mathrm{a}) = \hat{r}_1^{-1}(\lambda_2\mathrm{a} - \hat{r}_2(\mathrm{a})) = -\hat{r}_2\hat{r}_1^{-1}(\mathrm{a}) + \lambda_2\hat{r}_1^{-1}(\mathrm{a}) = \hat{r}_2(\hat{r}_1 - \lambda_1)(\mathrm{a}) + \lambda_2(\lambda_1 - \hat{r}_1)(\mathrm{a}) = \widehat{r_1r_2}(\mathrm{a}) - \lambda_1\hat{r}_2(\mathrm{a}) - \lambda_2\hat{r}_1(\mathrm{a}) + \lambda_1\lambda_2\mathrm{a}.$

Therefore $\mathcal{L}$ is also invariant under $\hat{g}$.

We now consider the dimension of $\mathcal{L}$ over $\mathcal{F}$, denoted by $dim_{\mathcal{F}}\ \mathcal{L}$, where dimension is defined as usual in the theory of vector spaces over a field. Since $\hat{g}^2 = \hat{1}$, we can express $\mathcal{L}$ as the direct sum

$$\mathcal{L} = \mathcal{L}_1 \dotplus \mathcal{L}_{-1}$$

where $\hat{g}(\mathrm{x}) = \mathrm{x}$ if $\mathrm{x} \varepsilon \mathcal{L}_1$ and $\hat{g}(\mathrm{y}) = -\mathrm{y}$ if $\mathrm{y} \varepsilon \mathcal{L}_{-1}$.

For any vector $\mathrm{x} \varepsilon \mathcal{L}$, $\hat{g}(\mathrm{x}) = \mathrm{x}$ if and only if $\mathrm{x} = \lambda\mathrm{a}$ for some $\lambda \varepsilon \mathcal{F}$, since $\bar{g}$ fixes only points of g and hence $\hat{g}$ fixes only vectors collinear with a. Therefore $\dim_{\mathcal{F}} \mathcal{L}_1 = 1$.

If $\hat{g}(\mathrm{y}) = -\mathrm{y}$, then $-\hat{g}(\mathrm{y}) = \mathrm{y}$. For $P|g$, let h be the perpendicular to g at P, i.e., $P = gh = hg$. It will be shown that $\hat{h} = -\hat{g}$. Thus $\hat{h}(\mathrm{y}) = \mathrm{y}$. But $\hat{h}$ also has only a 1-dimensional fixed space, so that $dim_{\mathcal{F}}\ \mathcal{L}_{-1} \leq 1$. Therefore $dim_{\mathcal{F}}\ \mathcal{L} \leq 2$.

It will be shown (Proposition 14) that if P is any point then $\hat{P}$ carries an arbitrary vector x into $-\mathrm{x}$. From this it follows that if $P = gh = hg$, then

$$\hat{g}(\hat{g} + \hat{h})\mathrm{x} = \hat{g}^2(\mathrm{x}) + \widehat{gh}(\mathrm{x}) = \mathrm{x} + \hat{P}(\mathrm{x}) = 0.$$

Therefore $\hat{h} = -\hat{g}$.

Proposition 14. $\hat{P}(\mathrm{x}) = -\mathrm{x}$ *for any point P and vector* x.

Proof. Producing x from P, we have $\mathrm{x} = \overrightarrow{PQ}$. Let $P = gh = hg$, and let j be the perpendicular from Q to g. For $R = gj = jg$, $\mathrm{x} = \overrightarrow{PR} + \overrightarrow{RQ}$.

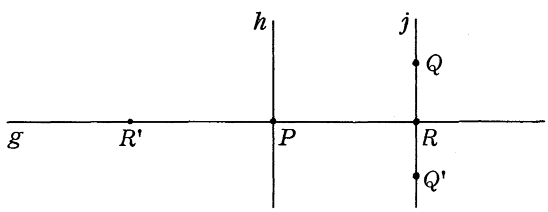

Figure 12

Consider the action of $\hat{P}$ on $\overrightarrow{PR}$ and $\overrightarrow{RQ}$. Let $R' = \bar{h}(R)$. Then $\hat{P}(\overrightarrow{PR}) = \overrightarrow{P\bar{P}(R)} = \overrightarrow{P\bar{h}\bar{g}(R)} = \overrightarrow{P\bar{h}(R)} = \overrightarrow{PR'}$. By Proposition 10, $\overrightarrow{PR'} = -\overrightarrow{R'P} = \overrightarrow{RP}$, so $\hat{P}(\overrightarrow{PR}) = -\overrightarrow{PR}$.

Next let $Q' = \bar{g}(Q)$. On the basis of the argument used in the proof of Theorem 3, $\hat{h} = \hat{j}$. Hence

$$\hat{P}(\overrightarrow{RQ}) = \hat{g}\hat{h}(\overrightarrow{RQ}) = \hat{g}\hat{j}(\overrightarrow{RQ}) = \hat{g}(\overrightarrow{RQ}) = \overrightarrow{R\bar{g}(Q)} = \overrightarrow{RQ'} = -\overrightarrow{RQ}.$$

Thus $\hat{P}(\mathbf{x}) = -\overrightarrow{PR} - \overrightarrow{RQ} = -\mathbf{x}$. This completes the proof that the dimension of $\mathcal{L}$ over $\mathcal{F}$ is at most 2.

It is noted that if $\hat{r}(\mathbf{x}) = \lambda\mathbf{x}$, $\lambda \varepsilon \mathcal{F}$, for any regular vector $\mathbf{x}$, then $\hat{r} = \pm\hat{1}$, for:

Let h be a line along which $\mathbf{x}$ is produced; $\hat{h}(\mathbf{x}) = \mathbf{x}$. Then

$$\widehat{rh^{-1}}(\mathbf{x}) = \hat{r}(\mathbf{x}) = \lambda\mathbf{x} \Longrightarrow$$

$$\widehat{hrh^{-1}}(\mathbf{x}) = \hat{h}(\lambda\mathbf{x}) = \lambda\hat{h}(\mathbf{x}) = \lambda\mathbf{x} \Longrightarrow$$

$$\hat{r}^{-1}(\mathbf{x}) = \lambda\mathbf{x} = \hat{r}(\mathbf{x}) \Longrightarrow$$

$$\hat{r}^2(\mathbf{x}) = \mathbf{x},\ i.e.,\ \hat{r} = \pm\hat{1}.$$

If the basis for $\mathcal{L}$ is chosen so that at least one of the rotations $\hat{r}_1, \hat{r}_2$ is not equal to $\pm\hat{1}$, then $\mathcal{L}$ has dimension not less than 2. (If there exist no non-trivial rotations, we have a special case which requires a slightly different construction.)

This completes the proof of

THEOREM 4. *$\mathcal{L}$ is a 2-dimensional linear space over $\mathcal{F}$.*

COROLLARY. *$\mathcal{L}$ is invariant under $\hat{\mathcal{G}} = \hat{\mathcal{R}} \cup \hat{g}\hat{\mathcal{R}}$.*

Proof. $\mathcal{L}$ has the basis $\mathbf{a}$, $\hat{r}_1(\mathbf{a})$, where $\hat{r}_1 \neq \pm\hat{1}$. $\mathcal{L}$ is invariant under $\hat{r}_2$, hence under any arbitrary rotation in $\hat{\mathcal{R}}$. $\mathcal{L}$ is also invariant under $\hat{g}$, so the invariance is extended to the entire group $\hat{\mathcal{G}} = \hat{\mathcal{R}} \cup \hat{g}\hat{\mathcal{R}}$.

V. CHARACTERIZATION OF THE PLANE BY A QUADRATIC FORM

We can now coordinatize the linear space $\mathcal{L}$ over the field $\mathcal{F}$. The coordinatization enables us to represent the elements of $\hat{\mathcal{G}}$ by matrices, and we then obtain a quadratic form invariant under $\hat{\mathcal{G}}$ (THEOREM 5). Thus we can conclude that the geometric structure corresponds to the class of equivalent quadratic forms so obtained.

A. Coordinatization and matrix representation. $\mathcal{L}$ has the basis $\mathbf{a} = \overrightarrow{OP}$ and $\hat{r}_1(\mathbf{a}) = \overrightarrow{OQ}$, where $O,P|g$ and $\hat{r}_1$ is not a scalar on $\mathbf{a}$, i.e., $\hat{r}_1 \neq \pm\hat{1}$, and $\hat{r}_1 + \hat{r}_1^{-1} = \lambda_1$.

Let $\overrightarrow{OX(\alpha,\beta)} = \alpha\mathbf{a} + \beta\hat{r}_1(\mathbf{a})$ for $\alpha,\beta \varepsilon \mathcal{F}$.

We define Π to be the set of all points $X(\alpha,\beta)$ together with the lines joining them when these lines exist.

Π contains the three fundamental points $X(0,0) = O, X(1,0) = P$ and $X(0,1) = Q$, and is invariant under rotations about O and reflections in lines through O, if we define for $\hat{h} \varepsilon \hat{\mathcal{G}}$,

$$\hat{h}X(\alpha,\beta) = X(\hat{h}\alpha,\hat{h}\beta)$$

where $\overrightarrow{OX(\hat{h}\alpha,\hat{h}\beta)}$ is defined to be $\hat{h}(\overrightarrow{OX(\alpha,\beta)}) = \hat{h}(\alpha\mathfrak{a}) + \hat{h}(\beta\hat{r}_1(\mathfrak{a}))$.

We shall represent a vector $\overrightarrow{OX(\alpha,\beta)}$ in $\mathcal{L}$ as a column vector, i.e., by the 2×1-matrix $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, and derive a representation for a linear transforma-- tion as a 2×2-matrix $\begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{bmatrix}$, where transformation of a vector is indicated by multiplication on the left.

Let $\begin{bmatrix}1\\ 0\end{bmatrix} \leftrightarrow \overrightarrow{OP}$

and $\begin{bmatrix}0\\ 1\end{bmatrix} \leftrightarrow \overrightarrow{OQ}$.

We know that

$$\hat{r}_1 \text{ carries } \begin{bmatrix}1\\ 0\end{bmatrix} \text{ into } \begin{bmatrix}0\\ 1\end{bmatrix}.$$

Suppose that

$$\hat{r}_1 \text{ carries } \begin{bmatrix}0\\ 1\end{bmatrix} \text{ into } \begin{bmatrix}\alpha\\ \beta\end{bmatrix},$$

so that

$$\hat{r}_1^2 \text{ carries } \begin{bmatrix}1\\ 0\end{bmatrix} \text{ into } \begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Then since $\hat{r}_1 + \hat{r}_1^{-1} = \lambda_1 \Longrightarrow \hat{r}_1^2 = \lambda_1\hat{r}_1 - \hat{1}$, we have $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}0\\ \lambda_1\end{bmatrix} - \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}-1\\ \lambda_1\end{bmatrix}$

Hence $\begin{bmatrix}0 & -1\\ 1 & \lambda_1\end{bmatrix} \leftrightarrow \hat{r}_1$.

We denote this matrix by $A(\hat{r}_1)$ and note that its determinant, *det* $A(\hat{r}_1)$, is equal to 1. Since $\hat{r}(\mathfrak{a})$ would serve as a basis vector for $\mathcal{L}$ for any non-trivial rotation $\hat{r}$ and the determinant is invariant under change of base, then *det* $A(\hat{r}) = 1$ for any rotation matrix, $A(\hat{r})$.

We know that

$$\hat{g} \text{ carries } \begin{bmatrix}1\\ 0\end{bmatrix} \text{ into } \begin{bmatrix}1\\ 0\end{bmatrix}$$

and from the argument used to show the invariance of $\mathcal{L}$ under $\hat{g}$, $\hat{g}(\hat{r}_1(\mathfrak{a})) = -\hat{r}_1(\mathfrak{a}) + \lambda_1\mathfrak{a}$.

Hence $\hat{g}$ carries $\begin{bmatrix}0\\ 1\end{bmatrix}$ into $\begin{bmatrix}0\\ -1\end{bmatrix} + \begin{bmatrix}\lambda_1\\ 0\end{bmatrix} = \begin{bmatrix}\lambda_1\\ -1\end{bmatrix}$.

Thus $A(\hat{g}) = \begin{bmatrix} 1 & \lambda_1 \\ 0 & -1 \end{bmatrix} \leftrightarrow \hat{g}$.

It has been seen that $\mathcal{L}$, which is generated by $\mathfrak{a}$ and $\hat{r}_1(\mathfrak{a})$, is invariant under any arbitrary rotation $\hat{r}$. In particular

$$\hat{r}(\mathfrak{a}) = \kappa_1 \mathfrak{a} + \kappa_2 \hat{r}_1(\mathfrak{a}), \quad \kappa_i \varepsilon \mathcal{I}.$$

Hence $$A(\hat{r}) = \kappa_1 I + \kappa_2 A(\hat{r}_1), \text{ i.e.,}$$

$$A(\hat{r}) = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_1 \end{bmatrix} + \begin{bmatrix} 0 & -\kappa_2 \\ \kappa_2 & \lambda_1 \kappa_2 \end{bmatrix} = \begin{bmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 + \lambda_1 \kappa_2 \end{bmatrix}.$$

Thus $$A(\hat{r}) = \begin{bmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 + \lambda_1 \kappa_2 \end{bmatrix} \leftrightarrow \hat{r},$$

where $\det A(\hat{r}) = 1 \Longrightarrow \kappa_1^2 + \lambda_1 \kappa_1 \kappa_2 + \kappa_2^2 = 1$.

Therefore we have a matrix representation for every element of $\hat{\mathcal{G}}$.

B. The quadratic form. If $\mathcal{Q} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is the matrix corresponding to the quadratic form $\alpha x^2 + 2\beta xy + \gamma y^2 = X^t \mathcal{Q} X$ where X corresponds to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and X^t to its transpose $[x\ y]$, and if A is a matrix corresponding to an element of $\hat{\mathcal{G}}$, then $\mathcal{Q}$ is invariant under transformation by A if

$$(AX)^t \mathcal{Q}(AX) = X^t A^t \mathcal{Q} AX = X^t \mathcal{Q} X,$$

i.e., we require that $A^t \mathcal{Q} A = \mathcal{Q}$ for an arbitrary transformation matrix A. Thus we must find $\mathcal{Q}$ such that $A^t \mathcal{Q} = \mathcal{Q} A^{-1}$ for a rotation matrix A.

$$A^t = \begin{bmatrix} \kappa_1 & \kappa_2 \\ -\kappa_2 & \kappa_1 + \lambda_1 \kappa_2 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} \kappa_1 + \lambda_1 \kappa_2 & \kappa_2 \\ -\kappa_2 & \kappa_1 \end{bmatrix}$$

so we must find α, β, γ in $\mathcal{I}$ such that, for κ_1, κ_2 satisfying $\kappa_1^2 + \lambda_1 \kappa_1 \kappa_2 + \kappa_2^2 = 1$,

$$\begin{bmatrix} \kappa_1 & \kappa_2 \\ -\kappa_2 & \kappa_1 + \lambda_1 \kappa_2 \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \cdot \begin{bmatrix} \kappa_1 + \lambda_1 \kappa_2 & \kappa_2 \\ -\kappa_2 & \kappa_1 \end{bmatrix}$$

which implies that

(1) $\alpha \kappa_1 + \beta \kappa_2 = \alpha \kappa_1 + \alpha \lambda_1 \kappa_2 - \beta \kappa_2$

(2) $\beta \kappa_1 + \gamma \kappa_2 = \alpha \kappa_2 + \beta \kappa_1$

(3) $-\alpha \kappa_2 + \beta \kappa_1 + \beta \lambda_1 \kappa_2 = \beta \kappa_1 + \beta \lambda_1 \kappa_2 - \gamma \kappa_2$

(4) $-\beta \kappa_2 + \gamma \kappa_1 + \gamma \lambda_1 \kappa_2 = \beta \kappa_2 + \gamma \kappa_1$

We obtain from either (2) or (3) that $\alpha = \gamma$, and from either (1) or (4) that $2\beta = \alpha \lambda_1$. Hence $\alpha = 2$, $\beta = \lambda_1$ is a solution and

$$\mathcal{Q} = \begin{bmatrix} 2 & \lambda_1 \\ \lambda_1 & 2 \end{bmatrix}$$

is invariant under $\hat{\mathcal{R}}$.

$\mathcal{Q}$ is also invariant under $\hat{g}$, since

$$\begin{bmatrix} 1 & 0 \\ \lambda_1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & \lambda_1 \\ \lambda_1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \lambda_1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & \lambda_1 \\ \lambda_1 & 2 \end{bmatrix}.$$

Therefore $\mathcal{Q}$ is invariant under $\hat{\mathcal{G}}$.

Given a non-zero vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathcal{L}$, we say that X is *isotropic* if there exists a symmetric matrix $\mathcal{Q}$ invariant under linear transformations such that $X^t \mathcal{Q} X = 0$.

Hence, for $\mathcal{Q} = \begin{bmatrix} 2 & \lambda_1 \\ \lambda_1 & 2 \end{bmatrix}$,

$\mathcal{L}$ has isotropic vectors if and only if $X^t \mathcal{Q} X = 2(x^2 + \lambda_1 xy + y^2)$ has nontrivial zeros. Π in this case is called an *isotropic* or *pseudo-Euclidean* plane. If $\mathcal{L}$ has no isotropic vectors, Π is a Euclidean plane.

The quadratic form determined here is, in fact, unique up to a constant factor. For, suppose $\lambda_1 \neq 0$. Then, for arbitrary β,

$$2\beta = \alpha\lambda_1 \Rightarrow \alpha = 2\beta\lambda_1^{-1},$$

and since $\alpha = 0$ if and only if $\beta = 0$, $\mathcal{Q}$ becomes

$$\mathcal{Q} = \begin{bmatrix} 2\beta\lambda_1^{-1} & \beta \\ \beta & 2\beta\lambda_1^{-1} \end{bmatrix} = \beta\lambda_1^{-1} \begin{bmatrix} 2 & \lambda_1 \\ \lambda_1 & 2 \end{bmatrix},$$

and $X^t \mathcal{Q} X = 2\beta\lambda_1^{-1}x^2 + 2\beta xy + 2\beta\lambda_1^{-1}y^2$, which differs from the previously obtained form only by the constant factor $\beta\lambda_1^{-1}$.

On the other hand, if $\lambda_1 = 0$, (1) implies that $\beta = 0$, so for arbitrary α, $\mathcal{Q}$ takes the form

$$\mathcal{Q} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This time $X^t \mathcal{Q} X = \alpha(x^2 + y^2)$, which is again unique up to a constant factor.

Hence there is essentially only one invariant quadratic form defined on the space $\mathcal{L}$ under this coordinatization, since the quadratic forms differing only by a constant factor are all equivalent with respect to the structure of the plane.

This proves

THEOREM 5. *On $\mathcal{L}$ there exists a quadratic form, dependent on the choice of basis and unique up to a constant factor, which is invariant under transformation of $\mathcal{L}$ by elements of $\hat{\mathcal{G}}$. $\mathcal{L}$ has isotropic vectors if and only if the quadratic form represents zero nontrivially.*

C. A minimal model. Under the hypothesis that $\hat{\mathcal{R}}$ consists only of the trivial rotations it is impossible to generate a 2-dimensional space by applying a rotation $\hat{r}$ to any vector. In this case we can choose any point $O = gh = hg$ as the origin and any points P and Q, both different from O, such that $P|g$ and $Q|h$, as the other two fundamental points. Then $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are a basis, and since g and h are the only lines through O, $\hat{\mathcal{G}}$ is the Klein-Four group consisting of elements $\hat{1},\hat{g},\hat{h},\hat{O}$.

Coordinatization, matrix representation and determination of a quadratic form can be carried out as in the general case. The matrix $\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, corresponding to $\hat{g}$, is in fact a symmetric matrix invariant under transformation by elements of $\hat{\mathcal{G}}$. This time, however, $\mathcal{Q}$ is not unique; the entry -1 may be replaced by any other element of $\mathcal{F}$.

The vectors from 0 to points on g,h respectively form non-zero vector spaces $\mathcal{V}(g)$, $\mathcal{V}(h)$ over the field $\mathcal{F}$ of characteristic $\neq 2$. The space $\mathcal{V}$ of all vectors is the direct sum of $\mathcal{V}(g)$ and $\mathcal{V}(h)$, and $\hat{g}$ and $\hat{h}$ transform $\mathcal{V}$ such that

$$\hat{g}(\mathbf{a} + \mathbf{b}) = \mathbf{a} - \mathbf{b}$$

$$\hat{h}(\mathbf{a} + \mathbf{b}) = -\mathbf{a} + \mathbf{b}$$

for $\mathbf{a} \varepsilon \mathcal{V}(g)$ and $\mathbf{b} \varepsilon \mathcal{V}(h)$. Conversely, if $\mathcal{A}$ and $\mathcal{B}$ are two non-trivial 2-divisible abelian groups, then the group $\mathcal{G}$ generated by $g,h,\mathcal{A},\mathcal{B}$ with defining relations:

$$\begin{aligned} g^2 &= h^2 = 1 \\ gh &= hg \\ ab &= ba \\ ga &= ag \\ ha &= a^{-1}h \\ hb &= bh \\ gb &= b^{-1}g \end{aligned}$$

for $a \varepsilon \mathcal{A}$, $b \varepsilon \mathcal{B}$, and preserving the group relations among elements of $\mathcal{A}$, and similarly preserving relations among elements of $\mathcal{B}$, is a group with normal subgroup $\mathcal{T} = \mathcal{A} \times \mathcal{B}$. $\mathcal{G}/\mathcal{T}$ is the Klein-Four group.

$\mathcal{G}$ has generator system $g\mathcal{B} \cup h\mathcal{A}$ invariant under inner automorphisms and consisting of involutions. If these generators are used as "lines", $\mathcal{G}$ satisfies A_1 - A_6. "Points" are obtained as the set $gh\mathcal{T}$. $\mathcal{T}$ is the translation subgroup and there are only two lines through each point.

VI. INDEPENDENCE OF THE AXIOMS

It has been shown that a group $\mathcal{G}$ generated by a set of involutions $\mathcal{S}$, which is invariant under inner automorphisms on $\mathcal{G}$, determines a Euclidean or "pseudo-Euclidean" plane under the definitions of Section II and the hypotheses of the following axioms:

A_1. There exist at least three non-collinear points.
A_2. Given two distinct points, P and Q, there is at most one line g such that $P,Q|g$.
A_3. The product of three concurrent lines is a line.

A$_4$. Given a point P and a line g, there is at most one line h such that $P|h$ and $h||g$.

A$_5$. The product of three points is a point.

A$_6$. Given two collinear points P and Q there exists a midpoint, M, of P and Q.

In this chapter the independence of these six axioms will be shown. Hence they are necessary as well as sufficient (THEOREM 6).

(1) Independence of A$_1$. Let a, b, c be the involutoric elements of $\mathcal{S}_3$, the symmetric group on three elements. Let $\mathcal{C}_2$ be the cyclic group of two elements generated by g, with $g^2 = e$. Define $\mathcal{G}$ to be the direct product $\mathcal{G} = C_2 \times \mathcal{S}_3$, generated by the *lines* (e,a), (e,b), (e,c) and $(g,1)$ where 1 is the identity element of $\mathcal{S}_3$. The *points* of $\mathcal{G}$ are (g,a), (g,b) and (g,c).

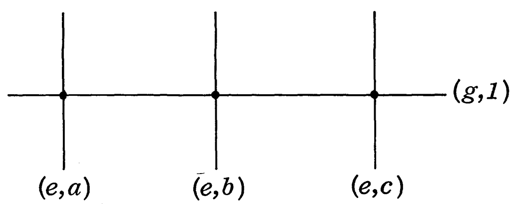

Figure 13

$\mathcal{G}$ satisfies the required definitions and **A**$_2$ - **A**$_6$.

Line $(g,1)$ is conjugate only to itself, hence is fixed under every line reflection. Every other line contains only one point, but $(g,1)$, which is orthogonal to every other line, is incident with every point. Hence there are not three non-collinear points. Therefore, **A**$_1$ is independent of the other five axioms.

(2) Independence of A$_2$. Hjelmslev [7, first part] gives the following axiom system:

I. *There exist points and there exist point sets called lines. There exist transformations called motions. Every motion is a correspondence which associates to every line and its points a line and its points, reversibly. Every motion has an inverse; the set of all motions is a group. Two figures which correspond under a motion are congruent.*

II. *Besides the identity motion there is a unique motion leaving a line pointwise fixed; this is the reflection in that line. Every line is the axis of such a reflection. Every point not on the axis is not fixed.*

III. *For two lines $a \neq b$, $b \perp a$ if b is fixed under reflection in a. Through every point there is a unique line b such that b is perpendicular to a given line a; a and b have a unique common point.*

IV. *If $A,B|g$ they have a reflection axis m such that A corresponds to B reversibly under reflection in m while g is fixed; $g \perp m$; m meets g in point M, which is the midpoint of A and B.*

V. *Two congruent sequences of points, $ABC\ldots\ldots$ and $AB'C'\ldots\ldots$, in one or two lines, with common point A, can always correspond under a reflection.*

As one example of this system Hjelmslev gives the following:

Adjoin to the real numbers $\mathbf{R}$ an element ε such that $\varepsilon^2 = 0$.

The affine plane obtained in the usual way with coordinates of the form $a + b\,\varepsilon$, for a, $b \in \mathbf{R}$, satisfies **I - V**.

It will be shown that this extended system satisfies all our axioms except $\mathbf{A}_2$.

Since there is a 1-1 correspondence between points and lines and their associated reflections, the same notation is used for both.

By Hjelmslev's axiom **I** the set of lines consists of full conjugacy classes. By axiom **III** a point is a product of two commuting lines. $\mathbf{A}_6$ follows immediately from axiom **IV.** $\mathbf{A}_3$ is a theorem in Hjelmselv's paper, as is $\mathbf{A}_5$ in the case of three collinear points.

The other desired properties, i.e., $\mathbf{A}_1$, $\mathbf{A}_4$ and $\mathbf{A}_5$ in the general case, are obtained by considering the example of the plane defined over the extended real number system.

Let $\mathbf{R}^* = \mathbf{R} \cup \{\varepsilon\}$, where $\varepsilon^2 = 0$ and operations involving ε and elements of $\mathbf{R}$ are performed as usual in a commutative ring. Thus $\mathbf{R}^*$ is a commutative ring with unit, and every element $a + b\varepsilon$, $a \neq 0$, has an inverse in $\mathbf{R}^*$: $(a + b\varepsilon)^{-1} = a^{-1} - a^{-2}b\varepsilon$.

As usual we define:

Point: $P = (x,y)$, $x,y \in \mathbf{R}^*$.

Line: $g = [u,v,w]$, u, v, $w \in \mathbf{R}^*$ and $u^2 + v^2 \neq 0$; the latter condition is equivalent to saying that either u or v is invertible.

Also, for $m \neq 0$, $[mu,mv,mw] = [u,v,w]$.

Incidence: $P|g$ if $ux + vy + w = 0$.

Orthogonality: Two lines g and g' are orthogonal if $uu' + vv' = 0$.

Parallelism: Two lines are parallel if they have the same slope, where slope has the usual definition.

If we represent points by the column matrix

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

the group of orthogonal transformations consists of all matrices

$$\begin{bmatrix} \dfrac{v^2 - u^2}{v^2 + u^2} & e\,\dfrac{2uv}{v^2 + u^2} & a \\[2ex] -\dfrac{2uv}{v^2 + u^2} & e\,\dfrac{v^2 - u^2}{v^2 + u^2} & b \\[2ex] 0 & 0 & 1 \end{bmatrix}$$

with $u,v,a,b,e \in \mathbf{R}^*$, $u^2 + v^2 \neq 0$, $e = \pm 1$.

The generators of this group are the line reflections. Reflection in an arbitrary line $[u,v,w]$ is represented by

$$\begin{bmatrix} \frac{v^2-u^2}{v^2+u^2} & -\frac{2uv}{v^2+u^2} & -\frac{2uw}{v^2+u^2} \\ \frac{2uv}{v^2+u^2} & -\frac{v^2-u^2}{v^2+u^2} & -\frac{2vw}{v^2+u^2} \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection in an arbitrary point (a,b) is represented by

$$\begin{bmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{bmatrix}$$

and is the product of two commuting line reflections, namely, $[1,\ 1,\ -(a+b)]$ and $[1,\ -1,\ b-a]$.

Since points are defined over $\mathbf{R}^*$, which contains $\mathbf{R}$, $\mathbf{A}_1$ is valid.

The definition of parallelism implies that two parallel lines have no common points, and parallelism is transitive. Hence $\mathbf{A}_4$ is valid.

We now verify $\mathbf{A}_5$ for three non-collinear points. Let A, B, C be three non-collinear points. Suppose there is a line g such that $A,B|g$; then $C\not| g$. By axiom III there are lines a and b which are perpendicular to g at points A and B respectively, i.e., $A = ag = ga$ and $B = bg = gb$. Let c be the perpendicular from C to g and h the perpendicular to c at C.

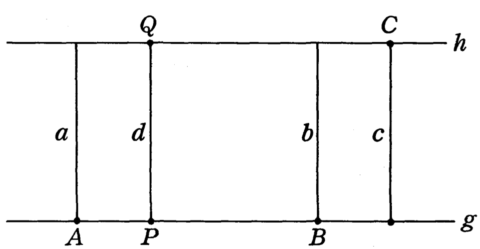

Figure 14

By a theorem in Hjelmslev's paper,

$$a,\ b,\ c \perp g \implies abc = d,$$

where d is also a line perpendicular to g.

Hence

$$ABC = aggbch = dh,$$

so we need only show that $dh = hd$, i.e., that $h \perp d$.

Let $dg = gd = P$. We know that $h \| g$, for if they had a common point there would be two perpendiculars from that point to the line c, contrary to axiom **III**.

Since $h \| g$ and parallels are unique, $d \nparallel h$. Let $Q | d,h$. If $h \not\perp d$ let j be the perpendicular to d through Q. Then

$$Q | h,j \quad \text{and} \quad g \| h \Rightarrow j \nparallel g,$$

so there is a point $S | g,j$, which implies that the perpendicular from S to d is not unique, contrary to axiom **III**. Therefore $ABC = dh = hd$, i.e., ABC is a point.

If there is no line joining any two of the points A, B and C there are still enough lines over $\mathbf{R}^*$ to verify $\mathbf{A}_5$. Let $A = (a,a')$, $B = (b,b')$, $C = (c,c')$, and consider the lines $y = a'$ and $x = b$. Let $F = (b,a')$, so F and B are both on line x = b.

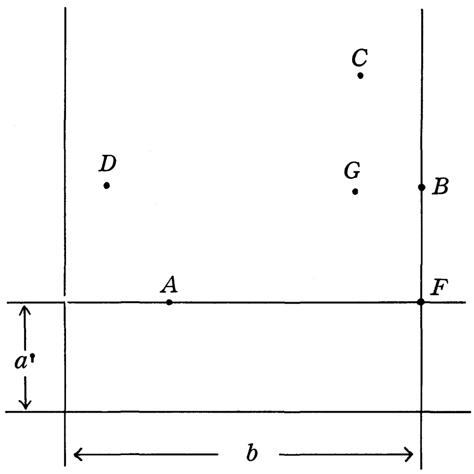

Figure 15

Then by the preceding argument, $FBC = G$ is a point. Hence $BC = FG$, and since both A and F lie on line $y = a'$, we have

$$ABC = AFG = D \text{ is a point}$$

Therefore $\mathbf{A}_5$ is valid in all cases.

However, two points in this plane do not necessarily have a unique join, e.g., points $(0,0)$ and $(\varepsilon,0)$ are both incident with the lines $[0,1,0]$ and $[\varepsilon,1,0]$.

Thus $\mathbf{A}_2$ is independent of the other five axioms.

(3) Independence of $\mathbf{A}_3$. The near-field, **NF(9)**, of nine elements is constructed by adjoining to **GF(3)**, the finite field of three elements, an element t which satisfies the equation $x^2 = 2$ over **GF(3)**. **NF(9)** is an additive abelian group, with a multiplication satisfying the following properties:

i. $(ab)c = a(bc)$
ii. $(a + b)c = ac + bc$
iii. $ab = -ba$, for all $a,b \notin$ **GF (3)** where $a \neq \pm b$.
iv. For every $a \varepsilon$ **NF (9)** $\exists\, a^{-1} \varepsilon$ **NF (9)** $\ni'\, aa^{-1} = a^{-1}a = 1$.

The projective plane over **NF (9)** is defined as usual:

Point: $P = (x_0,x_1,x_2) = (mx_0,mx_1,mx_2)$ for all $m \neq 0$.
Line: $g = [u_0,u_1,u_2] = [mu_0,mu_1,mu_2]$ for all $m \neq 0$.
Incidence: $P|g$ if and only if $x_0u_0 + x_1u_1 + x_2u_2 = 0$.

The embedded affine plane is obtained by deleting one line, say $x_2 = 0$, so that points may be represented in the form $(x_0,x_1,1)$.

Reflection in the line $x = a$ has the representation

$$\text{(i)} \qquad \begin{bmatrix} -1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and similarly reflection in line $y = b$ has the representation

$$\text{(ii)} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{bmatrix}.$$

These lines (i) and (ii) comprise two orthogonal pencils, because of the commutativity of the matrices.

A line $y = mx + a$ has the reflection matrix

$$\text{(iii)} \qquad \begin{bmatrix} 0 & m^{-1} & -m^{-1}a \\ m & 0 & a \\ 0 & 0 & 1 \end{bmatrix}$$

All line reflections are involutions and leave their respective reflection axes pointwise fixed.

If $\mathcal{G}$ is the group generated by all line reflections (i) - (iii), it is easily verified that the set of generators consists of full conjugacy classes.

Every matrix (i) commutes with every matrix (ii), and the product is the matrix

$$\text{(iv)} \qquad \begin{bmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{bmatrix}$$

which is the reflection in point $(a,b,1)$. Conversely, every point reflection is the product of two such line reflections.

Reflection in a point P is an involution and leaves P linewise fixed.

The product of two commuting lines is always a point in the case of matrices of type (iii) also:

In order for the reflections in the lines $y = mx + a$ and $y = kx + b$ to commute, we must have

$$m^{-1}k = k^{-1}m$$
$$m^{-1}b - m^{-1}a = k^{-1}a - k^{-1}b$$
$$-mk^{-1}b + a = -km^{-1}a + b$$

from which we obtain $k = -m$. Hence the product is the matrix.

$$\text{(iv)} \qquad \begin{bmatrix} -1 & 0 & \pm m(b-a) \\ 0 & -1 & (a+b) \\ 0 & 0 & 1 \end{bmatrix}$$

which is the reflection in the point $(\mp m(a-b), 2a+2b, 1)$.

It is clear that A_1, A_2 and A_4 are all valid in $\mathcal{G}$.

Also the product of three matrices of type (iv) is again a matrix of the same type, since this involves only the commutativity of addition. Hence A_5 is also valid.

Given 2 points $P = (a,b,1)$ and $Q = (c,d,1)$, let $M = (-a-c, -b-d, 1)$. Then $MPM = Q$, if we let these same letters represent the point reflections also. Thus A_6 holds in $\mathcal{G}$.

Finally we show that A_3 does not hold in $\mathcal{G}$. The lines $y = mx$, $y = kx$ and $y = rx$ are all incident with the point $(0,0,1)$. But the product of the corresponding matrices is

$$\begin{bmatrix} 0 & m^{-1}kr^{-1} & 0 \\ mk^{-1}r, & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is not a line reflection unless

$$(mk^{-1}r)^{-1} = m^{-1}kr^{-1}$$

i.e.,

$$r^{-1}km^{-1} = m^{-1}kr^{-1}$$

But multiplication in **NF(9)** is not in general commutative, e.g.;

Let $\quad r = t,\;$ where $t^2 = 2,$

$$k = t + 1,$$
$$m = 1.$$

Therefore A_3 is independent of the other five axioms.

(4) Independence of A_4. Let E_2 be the real Euclidean plane and let $\mathcal{O}_2$ be the group of rigid motions on E_2.

$\mathcal{O}_2$ is generated by the set of all line reflections, and this set is invariant under inner automorphisms of $\mathcal{O}_2$. Every point reflection is a product of two commuting line reflections, and conversely. $\mathcal{O}_2$ satisfies A_1 - A_6.

Consider the direct product

$$\mathfrak{O}_2 \times \mathfrak{O}_2 = \{(\sigma,\tau) \mid \sigma,\tau \in \mathfrak{O}_2\}.$$

Let $\mathcal{G}$ be the subgroup of $\mathfrak{O}_2 \times \mathfrak{O}_2$ which is generated by all elements of the form (g,g') such that g and g' are reflections in parallel lines in E_2. This set of generators is invariant under inner automorphisms, since—identifying points and lines of E_2 with their reflections in $\mathfrak{O}_2$—

$$(g,g') \cdot (h,h') \cdot (g,g')^{-1} = (ghg^{-1}, g'h'g'^{-1})$$

and

$$g \,||\, g',\ h \,||\, h' \Rightarrow \bar{g}(h) \,||\, \bar{g}'(h').$$

Let $E = E_2 \dotplus E_2$ and let $\mathbf{R}$ represent the real numbers.

Point of E: $P = (P_1,P_2)$, where P_1, P_2 are points of E_2, i.e., if $P_1 = (a_1,a_2)$ and $P_2 = (a_3,a_4)$, then $P = (a_1,a_2,a_3,a_4)$, $a_i \in \mathbf{R}$, $i = 1,2,3,4$.

For P in E and (σ,τ) in $\mathcal{G}$, we define

$$(\sigma,\tau)(P) = (\sigma,\tau)(P_1,P_2) = (\sigma(P_1), \tau(P_2)).$$

Line of E: L is the point set

$$L = \{P \mid P = (a_1+ku_1, a_2+ku_2, a_3+ku_1, a_4+ku_2)\}_{k \in \mathbf{R}} \quad ,$$

for fixed a_i in $\mathbf{R}$, and $\mathbf{u} = \{u_1,u_2\}$ a 2-dimensional vector over $\mathbf{R}$.

In E_2 the lines in the direction of the vector $\mathbf{u}$ produced from points (a_1,a_2) and (a_3,a_4) respectively are parallel; their reflections are g and g' which are parallel in $\mathfrak{O}_2$. Hence $(g,g') \in \mathcal{G}$ is the line reflection in L. Conversely, every generator (g,g') of $\mathcal{G}$ is a reflection in some line of E.

Orthogonality in E: If L_1 is a line in the direction of $\mathbf{u} = \{u_1,u_2\}$, and L_2 is a line in the direction of $\mathbf{v} = \{v_1,v_2\}$, then we say $L_1 \perp L_2$ in E if and only if $\mathbf{u} \perp \mathbf{v}$ in E_2, i.e., if and only if $u_1 v_1 + u_2 v_2 = 0$.

It is clear that a line reflection (g,g') in L is an involution leaving points of L (and only these) fixed and also leaving fixed only those lines which are orthogonal to L.

If $P = (P_1,P_2)$ is a point in E, the reflection in P is defined by (P_1,P_2) where the P_i denote points in E_2 and hence also their reflections in $\mathfrak{O}_2$. In $\mathfrak{O}_2$, $P_1 = gh = hg$ and $P_2 = g'h' = h'g'$ where g,g',h,h' are reflections in lines parallel to the coordinate axes and $g \,||\, g'$ and $h \,||\, h'$. Hence

$$(P_1,P_2) = (g,g')(h,h') = (gh,g'h') = (hg,h'g') = (h,h')(g,g')$$

where (g,g') and (h,h') are generators of $\mathcal{G}$. Conversely, every such pair of commuting generators determines a point reflection in $\mathcal{G}$.

It is clear that (P_1,P_2) is an involution leaving P line-wise fixed.

We see from the definition of point and line reflection in $\mathcal{G}$ that $P|L$ if and only if (P_1,P_2) commutes with (g,g').

Axioms $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathcal{G}$ follow immediately from the corresponding properties in $\mathfrak{O}_2$.

If $P|L_1,L_2,L_3$ then $P=(P_1,P_2)$ commutes with the corresponding line reflections (g,g'), (h,h') and (j,j') in $\mathcal{G}$. So P_1 commutes with g,h,j in $\mathcal{O}_2$ and the product ghj is a line k in $\mathcal{O}_2$, such that P_1 commutes with k; similarly for P_2 and $k'=g'h'j'$. Therefore (P_1,P_2) commutes with $(k,k')=(ghj,g'h'j')$, which verifies $\mathbf{A}_3$ in $\mathcal{G}$.

$(P_1,P_2)(Q_1,Q_2)(R_1,R_2)=(S_1,S_2)$ is a point in $\mathcal{G}$ since $P_iQ_iR_i$ is a point in $\mathcal{O}_2$ for $i=1,2$. Hence $\mathbf{A}_5$ holds in $\mathcal{G}$.

Given (P_1,P_2) and (Q_1,Q_2) there exist $M_i \in \mathcal{O}_2$ such that $M_iP_iM_i^{-1}=Q_i$. Hence

$$(M_1,M_2)(P_1,P_2)(M_1,M_2)=(Q_1,Q_2)$$

so $\mathbf{A}_6$ is also valid in $\mathcal{G}$.

Finally, $\mathbf{A}_4$ does not hold in $\mathcal{G}$:

Let $P=(1,1,0,0)$, and $L=\{Q|Q=(k,0,k,0)\}_{k\in\mathbf{R}}$, i.e., L is the line through $(0,0,0,0)$ in the direction of vector $\{1,0\}$.

Consider the lines L_1 and L_2, both through point P, in the directions $\{0,1\}$ and $\{1,0\}$ respectively, i.e.,

$$L_1=\{Q|Q=(1,1+k,0,k)\}_{k\in\mathbf{R}}$$

and

$$L_2=\{Q|Q=(1+k,1,k,0)\}_{k\in\mathbf{R}}.$$

Then $P|L_1,L_2$, and L_1 and L_2 are both parallel to L, for

$$L_1 \not\parallel L \Rightarrow (k,0,k,0)=(1,1+m,0,m), \text{ for some } k,m\in\mathbf{R}, \Rightarrow 1=0,$$

and

$$L_2 \not\parallel L \Rightarrow (k,0,k,0)=(1+m,1,m,0), \text{ for some } k,m\in\mathbf{R}, \Rightarrow 1=0.$$

Therefore $\mathbf{A}_4$ is independent of the other five axioms.

(5) Independence of $\mathbf{A}_5$. The real elliptic plane can be considered as the surface of the unit-sphere in Euclidean 3-space, E_3. Lines are great circles, i.e., a line is the intersection of a plane through the origin with the unit-sphere. Similarly, every point is the intersection of a line through the origin with the unit-sphere, and every line through the origin is the intersection of two orthogonal planes through the origin.

Hence we represent a *line* by the plane

$$ax+by+cz=0$$

and a *point* by the intersection of two such planes which are orthogonal.

In the following discussion, points are denoted by $P,Q,\dots$, lines by $g,h,\dots$ and planes by $\pi,\rho,\dots$ The same notation is used for the corresponding reflections in lines and planes.

In E_3 every reflection in a plane is an involution and leaves the plane line-wise fixed. Similarly, every reflection in a line is an involution and fixes the line plane-wise.

Let $\mathcal{G}$ be the group generated by the reflections in planes through the origin. As noted by Thomsen [13] the following relations hold:

(i) $(\pi\rho)^2 = 1$ if and only if $\pi \perp \rho$.

(ii) $\rho\pi\rho = \sigma$ is a plane if and only if ρ is the angle-bisecting plane of π and σ where π and σ intersect.

(iii) $(\pi\rho\sigma)^2 = 1$ if and only if either $\pi\rho\sigma$ is a plane—in this case π,ρ,σ all pass through the same line—or $\pi\rho\sigma$ is a point—in this case π,ρ,σ are pairwise orthogonal.

From (i) and (ii) we obtain that $\mathcal{G}$ is generated by full conjugacy classes and that points satisfy the required definition.

Since the poles on the unit-sphere occur as the intersection of the same two orthogonal planes through the origin, we identify poles in $\mathcal{G}$.

That A_1 holds is immediate. Since poles are identified with each other in $\mathcal{G}$, A_2 is also valid. To verify A_3 we must have: the product of three planes through the same line is a plane. This follows from relation (iii) above. A_4 holds since there are no parallels. To verify A_6 we consider that two lines through the origin determine a plane and have an angle bisector in that plane. That angle-bisector, considered as a point, is the midpoint of the two given lines considered as points. The three lines are coplanar; hence as points they are collinear.

However, A_5 does not hold in $\mathcal{G}$, since the real elliptic plane contains polar triangles. The three sides of a polar triangle correspond to three pairwise perpendicular planes through the origin. By relation (iii) their product is a point in E_3, hence not a point in $\mathcal{G}$. Therefore A_5 is independent of the other five axioms.

(6) Independence of A_6. Consider the Cartesian plane consisting of the *lattice points*

$P = (x,y)$ where x, y are integers, or $x = \frac{a}{2}$, and $y = \frac{b}{2}$, where a, b are integers and $a,b \equiv 1 \pmod 2$,

and *lines*

$$\begin{array}{ll} x = a & \text{for any integer } a \\ y = a & \text{for any integer } a \\ y = \pm x + a & \text{for any integer } a\,. \end{array}$$

Let $\mathcal{G}$ be the group of transformations generated by

$$\begin{bmatrix} -1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2a \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -a \\ 1 & 0 & a \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & a \\ -1 & 0 & a \\ 0 & 0 & 1 \end{bmatrix},$$

for an arbitrary integer a. These are the reflections in the above lines, respectively.

Reflection in point $P = (a,b)$ is represented by

$$\begin{bmatrix} -1 & 0 & 2a \\ 0 & -1 & 2b \\ 0 & 0 & 1 \end{bmatrix}.$$

If a and b are integral, this reflection is the product of the reflections in lines $x = a$ and $y = b$, and this product is commutative. If $(a,b) = \left(\frac{c}{2}, \frac{d}{2}\right)$ where $c,d \equiv 1 \pmod 2$, then the reflection is the product of the reflections in lines $y = x + f$ and $y = -x + g$, where $f = \frac{d-c}{2}$ and $g = \frac{d+c}{2}$.

It is easily verified that $\mathcal{G}$ is generated by full conjugacy classes and that axioms $A_1 - A_5$ are satisfied.

However, A_6 does not hold, e.g., the points $(0,0)$ and $(0,1)$ have no midpoint.

Therefore A_6 is independent of $A_1 - A_5$.

This completes the proof of

THEOREM 6. *The axioms $A_1 - A_6$ are independent of each other and therefore are necessary for this characterization of the ordinary and isotropic Euclidean planes.*

One final property of this set of axioms will be noted here.

Proposition 15. *If $\mathcal{G}$ is a finite group generated by an invariant set of involutions, axioms $A_1 - A_5$ suffice, since A_6 is a consequence of the first five axioms.*

Proof. Given $P_1, P_2 | g$, we wish to find a point M such that $\overline{M}(P_1) = P_2$. Let $P_1, \ldots, P_n$ be the points on g.

$$\begin{aligned} \overline{P}_1\,(P_1) &= P_1. \\ \overline{P}_2\,(P_1) &= P_3 \neq P_1. \\ \overline{P}_3\,(P_1) &= P_4 \end{aligned}$$

If $P_4 = P_2$ then P_3 is the required point M. If not, then continuing in the same fashion we have

$$\overline{P}_{i-1}(P_1) = P_i, \quad \text{for } 1 < i \leq n.$$

If any $P_i = P_2$, then P_{i-1} is M as required.
If not, then since g contains only n points, a repetition must occur, i.e.,

$$\overline{P}_k(P_1) = P_j, \text{ for } 3 \leq j < k.$$

Then

$$\begin{aligned} &\overline{P}_k(P_1) = \overline{P}_{j-1}(P_1) \Rightarrow \\ &\overline{P_{j-1}P_k}(P_1) = P_1 \quad \Rightarrow \\ &P_{j-1}P_k = 1, \end{aligned}$$

since only the identity translation has fixed points. Therefore $P_k = P_{j-1}$, i.e., no repetition is possible, so reflecting P_1 in points of g always gives rise to new points; hence we must have

$$\overline{P}_n(P_1) = P_2,$$

so P_n is the midpoint of P_1 and P_2.

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