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A NOTE ON REGRESSIVE ISOLS

JOSEPH BARBACK

1. **Introduction**. We wish to compare two well-known binary relations for recursive equivalence types on the collection Λ_R of all regressive isols. We shall assume that the reader is familiar with the basic terminology associated with the two notions regressive set and regressive isol. These are defined and studied in [1] and [2]. ε will denote the set of all non-negative integers (*numbers*) and if α is a set (of numbers) then $\bar{\alpha}$ will denote its cardinality. We write Ω for the collection of all RETs and Λ for the collection of all isols. For any function f(from a subset of ε into ε), δf will denote the domain of f.

The two relations for RETs which we are concerned with are $\preccurlyeq$ and $\leq^*$; the first is introduced in [4, page 99] and the second is introduced and studied in [2]. Their definitions can be given in terms of two relations between sets which we shall now recall. Let α and β be two sets of numbers. Then $\alpha \simeq \beta$ if there is a one-to-one partial recursive function f such that $\alpha \subset \delta f$ and $f(\alpha) = \beta$. And $\alpha \leq^* \beta$ if either α is finite and $\bar{\alpha} \leq \bar{\beta}$ or α is infinite and there is a partial recursive function g such that $\alpha \subset \delta g$, g is one-to-one on α and $g(\alpha) = \beta$. The following two properties will be used:

$$(1) \quad \begin{cases} \text{Let } \alpha \text{ be a regressive set and } \beta \simeq \alpha. \\ \text{Then } \beta \text{ is also a regressive set.} \end{cases}$$

$$(2) \quad \begin{cases} \text{For regressive sets } \alpha \text{ and } \beta, \\ \alpha \subset \beta \implies \alpha \leq^* \beta. \end{cases}$$

Property (1) follows from [1, Proposition 3] and (2) is [2, Corollary 1 of P 12]. We now define the two relations $\preccurlyeq$ and $\leq^*$ for recursive equivalence types. Let $A, B \;\varepsilon\; \Omega$. Then

$$A \preccurlyeq B \text{ if } (\exists \alpha)(\exists \beta)[\alpha \,\varepsilon\, A \text{ and } \beta \,\varepsilon\, B \text{ and } \alpha \text{ is } \simeq \text{ to a subset of } \beta],$$

$$A \leq^* B \text{ if } (\exists \alpha)(\exists \beta)[\alpha \,\varepsilon\, A \text{ and } \beta \,\varepsilon\, B \text{ and } \alpha \leq^* \beta].$$

It is known that each of these relations partially orders Λ. In addition, it can be shown that each of the existential quantifiers which occur in their definition can be replaced by a universal quantifier.

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2. **The main result.** In [2], J. C. E. Dekker introduced and studied an extension of the function minimum $(x,y) : \varepsilon^2 \longrightarrow \varepsilon$ to a function $\min(X,Y) : \Lambda_R^2 \longrightarrow \Lambda_R$. This extension and the $\leq^*$ relation are closely related, for one has that for $A,B \in \Lambda_R$, $\min(A,B) = \min(B,A)$, and $\min(A,B) = A$ if and only if $A \leq^* B$. Let us say that A and B are $\leq^*$ comparable if either $A \leq^* B$ or $B \leq^* A$. In [2] it is proven that there exist (infinite) regressive isols which are not $\leq^*$ comparable. One consequence of this fact is that there are regressive isols A and B such that $\min(A,B) \notin (A,B)$. The following two additional properties of the min function will be useful:

(3) Let $A,B,W \in \Lambda_R$. Then
$W \leq^* A$ and $W \leq^* B \implies W \leq^* \min(A,B)$.

(4) Let $A,B \in \Lambda_R$. Then
$\min(A-1, B-1) = \min(A,B)-1$, provided $A,B \geq 1$.

Property (3) follows from [2, Theorem T4(a)] and (4) can be easily proven. The principal result of this paper is the following theorem.

Theorem. *Let $A,B \in \Lambda_R$. Then $A \preccurlyeq B \implies A \leq^* B$, yet not conversely.*

Before we proceed to its proof we will first state and prove two lemmas.

Lemma 1. (Dekker) *Let $A,B \in \Lambda_R$. Then $A \preccurlyeq B \implies A \leq^* B$.*

Proof. Let $\alpha \in A$ and $\beta' \subset \beta \in B$ with $\alpha \simeq \beta'$. Then $\beta' \in A$ and by (1) each of the sets β' and β is regressive. Since $\beta' \subset \beta$, it follows from (2) that $\beta' \leq^* \beta$. This means that $A \leq^* B$ and completes the proof.

Lemma 2. *Let $A \in \Omega$. Then*
(a) *A is an isol $\iff A + 1 \leq^* A$ is false,*
(b) *A is not an isol $\iff A + 1 \leq^* A$.*

Proof. Since (a) is equivalent to (b), it suffices to prove (b). For this purpose, let us first assume that A is not an isol. Then by [4, Theorem 34] it follows that $A + 1 = A$ and hence also that $A + 1 \leq^* A$. Assume now that $A + 1 \leq^* A$. Then there exist a set $\alpha \in A$ and a number $u \notin \alpha$ such that $\alpha + (u) \leq^* \alpha$. Let f denote a partial recursive function such that $\alpha + (u) \subset \delta f$, f is one-to-one on $\alpha + (u)$ and $f(\alpha + (u)) = \alpha$. Consider the set

$$\pi = (f(u), f^2(u), f^3(u), \ldots).$$

Clearly π is an re subset of α. In addition, since f is one-to-one on $\alpha + (u)$ and $u \notin \alpha$ it follows that π is an infinite set. Hence α contains an infinite re subset and therefore A is not an isol.

Proof of the Theorem. In view of Lemma 1, in order to complete the proof of the Theorem it remains only to show that there exist regressive isols A and B with $A \leq^* B$ yet not $A \preccurlyeq B$. Let S and T denote any two (infinite) regressive isols which are incomparable relative to the $\leq^*$ relation. Set $A = \min(S,T)$. Then A is a regressive isol such that

$$A \leq^* S, \quad A \leq^* T \quad \text{and} \quad A \notin (S,T).$$

We claim,

(5) $$\text{not both} \quad A \preccurlyeq S \quad \text{and} \quad A \preccurlyeq T.$$

To prove this fact let us suppose otherwise, namely that $A \preccurlyeq S$ and $A \preccurlyeq T$. Then from the two facts $A \preccurlyeq S$ and $A \neq S$, it follows that $A \preccurlyeq S\text{-}1$. By Lemma 1 this yields that $A \leq^* S\text{-}1$. Similarly one also has that $A \leq^* T\text{-}1$, and therefore by (3),

$$A \leq^* \min(S\text{-}1, T\text{-}1).$$

According to (4) and the definition of A, this relation implies that $A \leq^* A\text{-}1$, or equivalently, by Lemma 2, that $A\text{-}1$ and hence also A is not an isol. This contradiction establishes (5). If we set $B = S$ if $A \preccurlyeq S$ is false and set $B = T$ otherwise, then A and B will have the desired properties, $A, B \,\varepsilon\, \Lambda_R$ and $A \leq^* B$ yet not $A \preccurlyeq B$. This completes the proof of the Theorem.

Remark. It is proven in [2] that there exist cosimple regressive isols which are incomparable relative to the $\leq^*$ relation. Moreover, the minimum of two cosimple regressive isols is also a cosimple regressive isol. In view of the last parts of the previous proof it follows that one can also obtain the following result:

There exist cosimple regressive isols A and B with $A \leq^* B$ yet not $A \preccurlyeq B$.

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University of Pittsburgh
Pittsburgh, Pennsylvania