Notre Dame Journal of Formal Logic
Volume VII, Number 2, April 1966

TWO SEPARATION THEOREMS FOR NATURAL DEDUCTION

HUGUES LEBLANC

Extending a result of mine in [7], I shall first establish that a Gentzen sequent of the sort

$$A_1, A_2, \ldots, A_n \longrightarrow B,$$

where $A_1, A_2, \ldots, A_n$ $(n \geqslant 0)$, and B are wffs of the first-order functional calculus (**FC**), is invariably provable - when intuitionistically valid - by means of the four structural rules **R**, **E**, **P**, and **C** in Table I below and the intelim rules of that table for such (and only such) of the seven operators '$\sim$', '$\supset$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in the sequent. I shall then establish that, save when in $\{\vee, \forall\}$, $\{\vee, \&, \forall\}$, $\{\vee, \forall, \exists\}$, or $\{\vee, \&, \forall, \exists\}$, a sequent of the selfsame sort is provable - when classically valid - by means of **R**, **E**, **P**, **C**, and the intelim rules of Table III below for such (and only such) of the seven operators '$\sim$', '$\supset$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in the sequent. The two results have interesting corollaries: one to the effect that a wff A of **FC**, when intuitionistically implied by a set S of wffs of **FC**, is invariably deducible from S by means of rule **GR'** in Table VI below and the intelim rules of that table for such (and only such) of the seven operators '$\sim$', '$\supset$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in a member of S or in A; another to the effect that A, when classically implied by S, is in all but four cases deducible from S by means of **GR'** and the intelim rules of Table VII below for such (and only such) of the seven operators in question as occur in a member of S or in A.

I

With all seven of '$\sim$', '$\supset$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' understood to serve as primitive signs of **FC**, let an expression of the sort

$$A_1, A_2, \ldots, A_n \longrightarrow B_1, B_2, \ldots, B_m,$$

where $A_1, A_2, \ldots, A_n$ $(n \geqslant 0)$, $B_1, B_2, \ldots,$ and B_m $(m \geqslant 0$ if $n = 0$, otherwise $m \geqslant 0)$ are wffs of **FC**, count as an L-sequent, and - when $m = 1$, as an N-sequent as well.[1] Let the sequent be said to be in α, where α is a (possibly empty) subset of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$, if every operator that occurs in

Received March 25, 1965

the sequent belongs to α and vice-versa. And let it be declared intuitionistically valid or, for short, I-valid [classically valid or, for short, C-valid] if: (i) in the case that $n = 0$, the wff $(\ldots (B_1 \vee B_2) \vee \ldots) \vee B_m$ is I-valid [C-valid], (ii) in the case that $m = 0$, the wff $((\ldots (A_1 \& A_2) \& \ldots) \& A_{n-1}) \supset \sim A_n$ is I-valid [C-valid], and (iii) in the case that $n > 0$ and $m > 0$, the wff $((\ldots (A_1 \& A_2) \& \ldots) \& A_n) \supset ((\ldots (B_1 \vee B_2) \vee \ldots) \vee B_m)$ is I-valid [C-valid].[2] Next, let a finite column of L-sequents [N-sequents] count as proof of an L-sequent [N-sequent] S by means of a set Φ of rules of inference if the column closes with S and every entry in the column follows from $p (p \geqslant 0)$ entries preceding the entry by application of a member of Φ.[3] Finally, let an L-sequent [N-sequent] S be declared provable by means of a set Φ of rules of inference if there is a proof of S by means of Φ.

It is readily shown that every N-sequent provable by means of the following rules of inference is I-valid, and hence that the said rules are intuitionistically sound:[4]

TABLE I

Structural rules

R(eiteration): $A \longrightarrow A$

E(xpansion): $\dfrac{K \longrightarrow A}{B, K \longrightarrow A}$

P(ermutation): $\dfrac{K, A, B, L \longrightarrow C}{K, B, A, L \longrightarrow C}$

C(ontraction): $\dfrac{A, A, K \longrightarrow B}{A, K \longrightarrow B}$

Intelim rules

For '$\sim$'

NE$_I$: $\dfrac{K \longrightarrow \sim A \text{ and } K \longrightarrow \sim\sim A}{K \longrightarrow A}$

NI: $\dfrac{K, A \longrightarrow B \text{ and } K, A \longrightarrow \sim B}{K \longrightarrow \sim A}$

For '$\supset$'

HE$_I$: $\dfrac{K \longrightarrow A \text{ and } K \longrightarrow A \supset B}{K \longrightarrow B}$

HI: $\dfrac{K, A \longrightarrow B}{K \longrightarrow A \supset B}$

For '&'

CE: $\dfrac{K \longrightarrow A \,\&\, B \text{ and } K, A, B \longrightarrow C}{K \longrightarrow C}$

CI: $\dfrac{K \longrightarrow A \text{ and } K \longrightarrow B}{K \longrightarrow A \,\&\, B}$

For '$\vee$'

DE:
$$\frac{(i)\ K \longrightarrow A \vee B,\ (ii)\ K, A \longrightarrow C, \text{ and } (iii)\ K, B \longrightarrow C}{K \longrightarrow C}$$

DI:
$$\frac{K \longrightarrow A \text{ or } K \longrightarrow B}{K \longrightarrow A \vee B}$$

For '$\equiv$'

$\mathsf{BE_I}$:
$$\frac{(i)\ K \longrightarrow A \text{ and } (ii)\ K \longrightarrow A \equiv B \text{ or } K \longrightarrow B \equiv A}{K \longrightarrow B}$$

BI:
$$\frac{K, A \longrightarrow A \text{ and } K, B \longrightarrow A}{K \longrightarrow A \equiv B}$$

For '$\forall$'

$\forall$E:
$$\frac{K \longrightarrow (\forall X)A \text{ and } K, A' \longrightarrow B}{K \longrightarrow B}$$

$\forall$I:
$$\frac{K \longrightarrow A}{K \longrightarrow (\forall X')A'}$$

For '$\exists$'

$\exists$E:
$$\frac{K \longrightarrow (\exists X')A' \text{ and } K, A \longrightarrow B}{K \longrightarrow B}$$

$\exists$I:
$$\frac{K \longrightarrow A'}{K \longrightarrow (\exists X)A}$$

Notes: (a) Throughout the above rules K and L are to be finite (and possibly empty) sequences of wffs separated by commas. (b) In $\forall$**E** and $\exists$**I** A' is to be like A except for exhibiting free occurrences of some individual variable X' (not necessarily distinct from X) wherever A exhibits free occurrences of X. (c) In $\forall$**I** and $\exists$**E** A' is to be like A except for exhibiting free occurrences of X' wherever A exhibits free occurrences of some individual variable X (not necessarily distinct from X'). (d) In $\forall$**I** X and X' -should they occur free in A- are not to occur free in any wff in K. (e) In $\exists$**E** X and X' -should they occur free in A- are not to occur free in any wff in K nor in B.

Leaving that matter of soundness to the reader, I shall restrict myself to proving -as earlier announced- the following so-called *separation theorem:*[5]

Theorem 1. *Let α be any subset of* $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$; *$S$ be any* N-*sequent in α; and Φ_α consist of rules* **R**, **E**, **P**, *and* **C** *in Table I and the intelim rules of that table for such (and only such) operators as belong to α. If S is* I-*valid, then S is provable by means of Φ_α.*

I shall make use, when proving Theorem 1, of a result of Gentzen's to the effect that every I-valid L-sequent of the sort

$$A_1, A_2, \ldots, A_n \longrightarrow$$

or the sort

$$A_1, A_2, \ldots, A_n \longrightarrow B$$

in which no individual variable occurs both bound and free, is provable by means of the following rules (and vice-versa):[6]

TABLE II

Structural rules

Reiteration (**R**): $A \longrightarrow A$

Expansion:

to the left (**El**)

$$\frac{K \longrightarrow L^*}{A, K \longrightarrow L^*}$$

to the right (**Er**)

$$\frac{K \longrightarrow}{K \longrightarrow A}$$

Permutation to the left (**Pl**):

$$\frac{K, A, B, K' \longrightarrow L^*}{K, B, A, K' \longrightarrow L^*}$$

Contraction to the left (**Cl**):

$$\frac{A, A, K \longrightarrow L^*}{A, K \longrightarrow L^*}$$

Introduction rules

For '$\sim$'

to the left (**NIl**)

$$\frac{K \longrightarrow A}{\sim A, K \longrightarrow}$$

to the right (**NIr**):

$$\frac{A, K \longrightarrow}{K \longrightarrow \sim A}$$

For '$\supset$'

to the left (**HIl**):

$$\frac{K \longrightarrow A \text{ and } B, K' \longrightarrow L^*}{A \supset B, K, K' \longrightarrow L^*}$$

to the right (**HIr**):

$$\frac{A, K \longrightarrow B}{K \longrightarrow A \supset B}$$

For '&'

to the left (**CIl**):

$$\frac{A, K \longrightarrow L^* \text{ or } B, K \longrightarrow L^*}{A \,\&\, B, K \longrightarrow L^*}$$

to the right (**CIr**):

$$\frac{K \longrightarrow A \text{ and } K \longrightarrow B}{K \longrightarrow A \,\&\, B}$$

For 'v'

to the left (**DIl**):

$$\frac{A, K \longrightarrow L^* \text{ and } B, K \longrightarrow L^*}{A \vee B, K \longrightarrow L^*}$$

to the right (**DIr**):

$$\frac{K \longrightarrow A \text{ or } K \longrightarrow B}{K \longrightarrow A \vee B}$$

For '$\equiv$'

to the left (**BIl**):

$$\frac{(i)\ K \longrightarrow A \text{ and } B, K' \longrightarrow L^* \text{ or } (ii)\ K \longrightarrow B \text{ and } A, K' \longrightarrow L^*}{A \equiv B, K, K' \longrightarrow L^*}$$

to the right (**BIr**):

$$\frac{A, K \longrightarrow B \text{ and } B, K \longrightarrow A}{K \longrightarrow A \equiv B}$$

For '**∀**'

to the left (**∀Il**):

$$\frac{A', K \longrightarrow L^*}{(\forall X)A, K \longrightarrow L^*}$$

to the right (**∀Ir**):

$$\frac{K \longrightarrow A'}{K \longrightarrow (\forall X)A}$$

For '**∃**'

to the left (**∃Il**):

$$\frac{A', K \longrightarrow L^*}{(\forall X)A, K \longrightarrow L^*}$$

to the right (**∃Ir**):

$$\frac{K \longrightarrow A'}{K \longrightarrow (\forall X)A}$$

Notes: (a) Throughout the above rules K and K' are to be finite (and possibly empty) sequences of wff separated by commas, and L^* is to be a sequence of at most one wff. (b) In the last four rules A' is to be as in Note (b) under Table I. (c) In **∀Ir** and **∃Il** X' is not to occur free in any wff in K, nor in L^*, nor — should X' be distinct from X — in A.

Gentzen's result can be put more sharply. Consider indeed a column of L-sequents that qualifies as a proof by means of rules from Table II of an L-sequent S of the sort $A_1, A_2, \ldots, A_n \longrightarrow$ or the sort $A_1, A_2, \ldots, A_n \longrightarrow B$; suppose S is in a subset α of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$; and suppose a given operator, call it O, does not belong to α. Any L-sequent that follows from one or two other L-sequents by application of a rule from Table II is sure - as the reader may verify - to exhibit O if the one L-sequent from which it follows or at least one of the two L-sequents from which it follows exhibits O. But if so, then any entry in the column under consideration that exhibits O is otiose, and hence can be lopped off. Hence:

Lemma 1. *Let α be any subset of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$, and S be any L-sequent in α of the sort $A_1, A_2, \ldots, A_n \longrightarrow$ or the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ in which no individual variable occurs both bound and free. If S is* I-*valid, then S is susceptible of a proof in which every entry follows from previous entries in the proof by application of a structural rule in* Table II *or an introduction rule of that table for a member of α.*

II

Suppose in proof of Theorem 1 that S is an I-valid N-sequent - and, hence, an I-valid L-sequent - in which no individual variable occurs both bound and free, and suppose S is in a subset α of $\{\supset, \&, \vee, \equiv, \forall, \exists\}$ (cases 1-64). Because of Lemma 1 S is susceptible of a proof in which every entry: (i) follows from previous entries in the proof by application of a structural rule in Table II or an introduction rule of that table for a member of α, and hence (ii) is of the sort $A_1, A_2, \ldots, A_n \longrightarrow B$.[7] But it is readily shown that: (a) if an L-sequent S_j of the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ follows from another L-sequent S_i of the same sort by application of a structural rule in Table II or a one-premiss rule of that table for a member of the aforementioned α, then S_j is sure to be provable by means of Φ_α if S_i is[8], and (b) if an L-sequent S_j of the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ follows from two other L-sequents S_h and S_i of the same sort by application of a two-premiss rule of Table II for a member of the aforementioned α, then S_j is sure to be provable by means of Φ_α if both S_h and S_i are. Hence every entry in the aforementioned proof of S is sure to be provable by means of Φ_α. Hence so is S.

To verify points (a)-(b),[9] suppose an L-sequent follows by application of **HII** from two L-sequents $K \longrightarrow A$ and $B, K' \longrightarrow C$, and hence reads $A \supset B, K, K' \longrightarrow C$; and suppose $K \longrightarrow A$ and $B, K' \longrightarrow C$ are both provable by means of Φ_α.[10] Then $A \supset B, K, K', B \longrightarrow C$, which follows from $B, K' \longrightarrow C$ by application of **E** and **P**, is sure to be provable by means of Φ_α. Hence so is $A \supset B, K, K' \longrightarrow B \supset C$, which follows from $A \supset B, K, K', B \longrightarrow C$ by application of **HI**. But $A \supset B, K, K' \longrightarrow A$, which follows from $K \longrightarrow A$ by application of **E** and **P**, is sure to be provable by means of Φ_α; $A \supset B, K, K' \longrightarrow A \supset B$ is provable by means of **R**, **E**, **P**; and hence $A \supset B, K, K' \longrightarrow B$, which follows from $A \supset B, K, K' \longrightarrow A$ and $A \supset B, K, K' \longrightarrow A \supset B$ by application of **HE$_\text{I}$**, is sure to be provable by means of Φ_α. Hence $A \supset B, K, K' \longrightarrow C$, which follows from $A \supset B, K, K' \longrightarrow B \supset C$ and $A \supset B, K, K' \longrightarrow B$ by application of **HE$_\text{I}$**, is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **CII** from another L-sequent $A, K \longrightarrow C$ or $B, K \longrightarrow C$, and hence reads $A \& B, K \longrightarrow C$; and suppose $A, K \longrightarrow C$ or $B, K \longrightarrow C$ is provable by means of Φ_α. Then $A \& B, K, A, B \longrightarrow C$, which follows from either one of $A, K \longrightarrow C$ and $B, K \longrightarrow C$ by application of **E** and **P**, is sure to be provable by means of Φ_α. But $A \& B, K \longrightarrow A \& B$ is provable by means of **R**, **E**, and **P**. Hence $A \& B, K \longrightarrow C$, which follows from $A \& B, K \longrightarrow A \& B$ and $A \& B, K, A, B \longrightarrow C$ by application of **CE**, is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **DII** from two L-sequents $A, K \longrightarrow C$ and $B, K \longrightarrow C$, and hence reads $A \vee B, K \longrightarrow C$; and suppose $A, K \longrightarrow C$ and $B, K \longrightarrow C$ are both provable by means of Φ_α. Then $A \vee B, K, A \longrightarrow C$ and $A \vee B, K, B \longrightarrow C$, which respectively follow from $A, K \longrightarrow C$ and $B, K \longrightarrow C$ by application of **E** and **P**, are sure to be provable by means of Φ_α. But $A \vee B, K \longrightarrow A \vee B$ is provable by means of **R**, **E**, and **P**. Hence $A \vee B, K \longrightarrow C$, which follows from $A \vee B, K, A \longrightarrow C$, $A \vee B, K, B \longrightarrow C$, and $A \vee B, K \longrightarrow A \vee B$ by application of **DE**, is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **BII** from two L-sequents $K \longrightarrow A$ and $B, K' \longrightarrow C$, and hence reads $A \equiv B, K, K' \longrightarrow C$; and suppose $K \longrightarrow A$ and $B, K' \longrightarrow C$ are both provable by means of Φ_α. Then $A \equiv B, K, K' \longrightarrow A$, which follows from $K \longrightarrow A$ by application of **E** and **P**, is sure to be provable by means of Φ_α. But $A \equiv B, K, K' \longrightarrow A \equiv B$ is provable by means of **R**, **E**, and **P**. Hence $A \equiv B, K, K' \longrightarrow B$, which follows from $A \equiv B, K, K' \longrightarrow A$ and $A \equiv B, K, K' \longrightarrow A \equiv B$ by application of **BE$_\text{I}$**, is sure to be provable by means of Φ_α. Hence so is $A \equiv B, K, K', C \longrightarrow B$, which follows from $A \equiv B, K, K' \longrightarrow B$ by application of **E** and **P**. But $A \equiv B, K, K', B \longrightarrow C$, which follows from $B, K' \longrightarrow C$ by application of **E** and **P**, is sure to be provable by means of Φ_α. Hence so is $A \equiv B, K, K' \longrightarrow B \equiv C$, which follows from $A \equiv B, K, K', B \longrightarrow C$ and $A \equiv B, K, K', C \longrightarrow B$ by application of **BI**. Hence so is $A \equiv B, K, K' \longrightarrow C$, which follows from $A \equiv B, K, K' \longrightarrow B \equiv C$ and $A \equiv B, K, K' \longrightarrow B$ by application of **BE**$_{\mathrm{I}}$. Suppose then the L-sequent follows by application of **BIl** from $K \longrightarrow B$ and $A, K' \longrightarrow C$. By a similar reasoning $A \equiv B, K, K' \longrightarrow C$ is sure to be provable by means of Φ_α if $K \longrightarrow B$ and $A, K' \longrightarrow C$ both are.

Or suppose an L-sequent follows by application of **∀Il** from another L-sequent $A', K \longrightarrow B$, where A' is as in Note (b) under Table I, and hence reads $(\forall X)A, K \longrightarrow B$; and suppose $A', K \longrightarrow B$ is provable by means of Φ_α. Then $(\forall X)A, K, A' \longrightarrow B$ is sure to be provable by means of Φ_α. But $(\forall X)A, K \longrightarrow (\forall X)A$ is provable by means of **R**, **E**, and **P**. Hence $(\forall X)A, K \longrightarrow B$, which follows from $(\forall X)A, K \longrightarrow (\forall X)A$ and $(\forall X)A, K, A' \longrightarrow B$ by application of **∀E**, is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **∀Ir** from another L-sequent $K \longrightarrow A'$, where A' is as in Note (b) under Table I and X' as in Note (c) under Table II, and hence reads $K \longrightarrow (\forall X)A$; and suppose $K \longrightarrow A'$ is provable by means of Φ_α. If X' is distinct from X (and hence does not occur free in A), then (i) A is sure to be like A' except for exhibiting free occurrences of X wherever A' exhibits free occurrences of X', and (ii) X is sure not to occur free in A'. Hence, whether or not X' is the same as $X, K \longrightarrow (\forall X)A$ follows from $K \longrightarrow A'$ by application of **∀I**. Hence $K \longrightarrow (\forall X)A$ is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **∃Il** from another L-sequent $A', K \longrightarrow B$, where A' and X' are as in the previous paragraph, and hence reads $(\exists X)A, K \longrightarrow B$; and suppose $A', K \longrightarrow B$ is provable by means of Φ_α. Then $(\exists X)A, K, A' \longrightarrow B$ is sure to be provable by means of Φ_α. But $(\exists X)A, K \longrightarrow (\exists X)A$ is provable by means of **R**, **E**, and **P**, and $(\exists X)A, K \longrightarrow B$ follows (again whether or not X' is the same as X) from $(\exists X)A, K \longrightarrow (\exists X)A$ and $(\exists X)A, K, A' \longrightarrow B$ by application of **∃E**. Hence $(\exists X)A, K \longrightarrow B$ is sure to be provable by means of Φ_α.

Take then S to be in a subset α of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$ to which belongs the connective '$\sim$' (Cases 65-128). In view of Lemma 1 S is susceptible of a proof in which every entry: (i) follows from previous entries in the proof by application of a structural rule in Table II or an introduction rule of that table for a member of α, and hence (ii) is of the sort $A_1, A_2, \ldots, A_n \longrightarrow$ or the sort $A_1, A_2, \ldots, A_n \longrightarrow B$.[11] Now let an L-sequent of the former sort have the N-sequent $A_1, A_2, \ldots, A_{n-1} \longrightarrow \sim A_n$ as its N-counterpart, and one of the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ have itself as its N-counterpart. It is readily shown that: (a) if an L-sequent S_j of the sort $A_1, A_2, \ldots, A_n \longrightarrow$ or the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ follows from another L-sequent S_i of either sort by application of a structural rule in Table II or a one-premiss rule in that table for a member of the aforementioned α, then the N-counterpart of S_j is sure to be provable by means of Φ_α if the N-counterpart of S_i is, and (b) if an L-sequent S_j of the sort $A_1, A_2, \ldots, A_n \longrightarrow$ or the sort $A_1, A_2, \ldots, A_n \longrightarrow B$ follows from two other L-sequents S_h and S_i of either sort by application of a two-premiss rule in Table II for a member of the aforementioned α, then the N-counterpart of S_j is sure to be provable by means of Φ_α if the N-counterpart of each one of S_h and S_i is. Hence the N-counterpart of every entry in the aforementioned proof of S is sure to be provable by means of Φ_α. Hence so is the N-counterpart of S. Hence so is S.

To verify points (a)-(b),[12] suppose an L-sequent follows by application of **Er** from another L-sequent $A_1, A_2, \ldots, A_n \longrightarrow$, and hence reads $A_1, A_2, \ldots, A_n \longrightarrow B$; and suppose the N-counterpart $A_1, A_2, \ldots, A_n \longrightarrow \sim A_n$ of $A_1, A_2, \ldots, A_n \longrightarrow$ is provable by means of Φ_α. Then $A_1, A_2, \ldots, A_n, B \longrightarrow \sim A_n$ is sure to be provable by means of Φ_α. But $A_1, A_2, \ldots, A_n, B \longrightarrow A_n$ is provable by means of **R, E,** and **P.** Hence $A_1, A_2, \ldots, A_n \longrightarrow \sim B$, which follows from $A_1, A_2, \ldots, A_n, B \longrightarrow A_n$ and $A_1, A_2, \ldots, A_n, B \longrightarrow \sim A_n$ by application of **NI**, is sure to be provable by means of Φ_α. Hence by the same reasoning so is $A_1, A_2, \ldots, A_n \longrightarrow \sim \sim B$. Hence so is the (N-counterpart of) $A_1, A_2, \ldots, A_n \longrightarrow B$, which follows from $A_1, A_2, \ldots, A_n \longrightarrow \sim B$ and $A_1, A_2, \ldots, A_n \longrightarrow \sim \sim B$ by application of $\mathbf{NE_I}$.

Or suppose an L-sequent follows by application of **NIl** from another L-sequent $\longrightarrow B$, and hence reads $\sim B \longrightarrow$; and suppose (the N-counterpart of) $\longrightarrow B$ is provable by means of Φ_α. Then $\sim B \longrightarrow B$ is sure to be provable by means of Φ_α. But $\sim B \longrightarrow \sim B$ is provable by means of **R.** Hence the N-counterpart $\longrightarrow \sim \sim B$ of $\sim B \longrightarrow$, which follows from $\sim B \longrightarrow B$ and $\sim B \longrightarrow \sim B$ by application of **NI**, is sure to be provable by means of Φ_α. Suppose then the L-sequent follows by application of **NIl** from $A_1, A_2, \ldots, A_n \longrightarrow B$, where $n > 0$, and hence reads $\sim B, A_1, A_2, \ldots, A_n \longrightarrow$; and suppose (the N-counterpart of) $A_1, A_2, \ldots, A_n \longrightarrow B$ is provable by means of Φ_α. Then $\sim B, A_1, A_2, \ldots, A_n \longrightarrow B$ is sure to be provable by means of Φ_α. But $\sim B, A_1, A_2, \ldots, A_n \longrightarrow \sim B$ is provable by means of **R, E,** and **P.** Hence the N-counterpart $\sim B, A_1, A_2, \ldots, A_{n-1} \longrightarrow \sim A_n$ of $\sim B, A_1, A_2, \ldots, A_n \longrightarrow$, which follows from $\sim B, A_1, A_2, \ldots, A_n \longrightarrow B$ and $\sim B, A_1, A_2, \ldots, A_n \longrightarrow \sim B$ by application of **NI**, is sure to be provable by means of Φ_α.

Or suppose an L-sequent follows by application of **NIr** from another L-sequent $B, A_1, A_2, \ldots, A_n \longrightarrow$, where $n > 0$, and hence reads $A_1, A_2, \ldots, A_n \longrightarrow \sim B$; and suppose the N-counterpart $B, A_1, A_2, \ldots, A_{n-1} \longrightarrow \sim A_n$ of $B, A_1, A_2, \ldots, A_n \longrightarrow$ is provable by means of Φ_α. Then $A_1, A_2, \ldots, A_n, B \longrightarrow \sim A_n$ is sure to be provable by means of Φ_α. But $A_1, A_2, \ldots, A_n, B \longrightarrow A_n$ is provable by means of **R, E,** and **P.** Hence (the N-counterpart of) $A_1, A_2, \ldots, A_n \longrightarrow \sim B$, which follows from $A_1, A_2, \ldots, A_n, B \longrightarrow A_n$ and $A_1, A_2, \ldots, A_n, B \longrightarrow \sim A_n$ by application of **NI**, is sure to be provable by by means of Φ_α.

Or suppose an L-sequent follows by application of **HIl** from two other L-sequents $\longrightarrow A$ and $B \longrightarrow$, and hence reads $A \supset B \longrightarrow$; and suppose the N-counterparts $\longrightarrow A$ and $\longrightarrow \sim B$ of $\longrightarrow A$ and $B \longrightarrow$ are both provable by means of Φ_α. Then $A \supset B \longrightarrow A$ and $A \supset B \longrightarrow \sim B$ are sure to be provable by means of Φ_α. But $A \supset B \longrightarrow A \supset B$ is provable by means of **R.** Hence $A \supset B \longrightarrow B$, which follows from $A \supset B \longrightarrow A$ and $A \supset B \longrightarrow A \supset B$ by application of $\mathbf{HE_I}$, is sure to be provable by means of Φ_α. Hence so is the

N-counterpart $\rightarrow \sim(A \supset B)$ of $A \supset B \rightarrow$, which follows from $A \supset B \rightarrow B$ and $A \supset B \rightarrow \sim B$ by application of **NI**. Suppose then the L-sequent follows by application of **HII** from two other L-sequents $A_1, A_2, \ldots, A_n \rightarrow A$, where $n > 0$, and $B \rightarrow$, and hence reads $A \supset B, A_1, A_2, \ldots, A_n \rightarrow$; and suppose the N counterparts $A_1, A_2, \ldots, A_n \rightarrow A$ and $\rightarrow \sim B$ of $A_1, A_2, \ldots, A_n \rightarrow A$ and $B \rightarrow$ are both provable by means of Φ_α. Then $A \supset B, A_1, A_2, \ldots, A_n \rightarrow A$ and $A \supset B, A_1, A_2, \ldots, A_n \rightarrow \sim B$ are sure to be provable by means of Φ_α. But $A \supset B, A_1, A_2, \ldots, A_n \rightarrow A \supset B$ is provable by means of **R**, **E**, and **P**. Hence $A \supset B, A_1, A_2, \ldots, A_n \rightarrow B$, which follows from $A \supset B, A_1, A_2, \ldots, A_n \rightarrow A$ and $A \supset B, A_1, A_2, \ldots, A_n \rightarrow A \supset B$ by application of **HE$_\mathbf{I}$**, is sure to be provable by means of Φ_α. Hence so is the N-counterpart $A \supset B, A_1, A_2, \ldots, A_{n-1} \rightarrow \sim A_n$ of $A \supset B, A_1, A_2, \ldots, A_n \rightarrow$, which follows from $A \supset B, A_1, A_2, \ldots, A_n \rightarrow B$ and $A \supset B, A_1, A_2, \ldots, A_n \rightarrow \sim B$ by application of **NI.**[13]

Or suppose an L-sequent follows by application of **CII** from another L-sequent $A \rightarrow$ or $B \rightarrow$, and hence reads $A \,\&\, B \rightarrow$; and suppose the N-counterpart $\rightarrow \sim A$ of $A \rightarrow$ or the N-counterpart $\rightarrow \sim B$ of $B \rightarrow$ is provable by means of Φ_α. Then $A \,\&\, B \rightarrow \sim A$ or $A \,\&\, B \rightarrow \sim B$ is sure to be provable by means of Φ_α. But $A \,\&\, B \rightarrow A \,\&\, B$ is provable by means of **R**; $A \,\&\, B, A, B \rightarrow A$ provable by means of **R**, **E**, and **P**; and $A \,\&\, B, A, B \rightarrow B$ provable by means of **R** and **E**. Hence $A \,\&\, B \rightarrow A$, which follows from $A \,\&\, B \rightarrow A \,\&\, B$ and $A \,\&\, B, A, B \rightarrow A$ by application of **CE**, and $A \,\&\, B \rightarrow B$, which follows from $A \,\&\, B \rightarrow A \,\&\, B$ and $A \,\&\, B, A, B \rightarrow B$ by application of **CE**, are sure to be provable by means of Φ_α. Hence so is the N-counterpart $\rightarrow \sim(A \,\&\, B)$ of $A \,\&\, B \rightarrow$, which follows from $A \,\&\, B \rightarrow A$ and $A \,\&\, B \rightarrow \sim A$ or from $A \,\&\, B \rightarrow B$ and $A \,\&\, B \rightarrow \sim B$ by application of **NI.**

Or suppose an L-sequent follows by application of **DII** from other two L-sequents $A \rightarrow$ and $B \rightarrow$, and hence reads $A \vee B \rightarrow$; and suppose the N-counterparts $\rightarrow \sim A$ and $\rightarrow \sim B$ of $A \rightarrow$ and $B \rightarrow$ are both provable by means of Φ_α. Then $A \vee B, A, A \vee B \rightarrow \sim A$ and $A \vee B, B, A \vee B \rightarrow \sim B$ are both provable by means of **R**, **E**, and **P**. Hence $A \vee B, A \rightarrow \sim(A \vee B)$, which follows from $A \vee B, A, A \vee B \rightarrow A$ and $A \vee B, A, A \vee B \rightarrow \sim A$ by application of **NI**, and $A \vee B, B \rightarrow \sim(A \vee B)$, which follows from $A \vee B, B, A \vee B \rightarrow B$ and $A \vee B, B, A \vee B \rightarrow \sim B$ by application of **NI**, are sure to be provable by means of Φ_α. But $A \vee B \rightarrow A \vee B$ is provable by means of **R.** Hence $A \vee B \rightarrow \sim(A \vee B)$, which follows from $A \vee B \rightarrow A \vee B$, $A \vee B, A \rightarrow \sim(A \vee B)$, and $A \vee B, B \rightarrow \sim(A \vee B)$ by application of **DE**, is sure to be provable by means of Φ_α. Hence so is the N-counterpart $\rightarrow \sim(A \vee B)$ of $A \vee B \rightarrow$, which follows from $A \vee B \rightarrow A \vee B$ and $A \vee B \rightarrow \sim(A \vee B)$ by application of **NI.**

Or suppose an L-sequent follows by application of **BII** from two other L-sequents $\rightarrow A$ and $B \rightarrow$, and hence reads $A \equiv B \rightarrow$; and suppose the N-counterparts $\rightarrow A$ and $\rightarrow \sim B$ of $\rightarrow A$ and $B \rightarrow$ are both provable by means of Φ_α. Then $A \equiv B \rightarrow A$ and $A \equiv B \rightarrow \sim B$ are sure to be provable by means of Φ_α. But $A \equiv B \rightarrow A \equiv B$ is provable by means of **R.** Hence $A \equiv B \rightarrow B$, which follows from $A \equiv B \rightarrow A \equiv B$ and $A \equiv B \rightarrow A$ by application of

BE$_{\mathbf{I}}$, is sure to be provable by means of Φ_α. Hence so is the N-counterpart $\rightarrow \sim(A \equiv B)$ of $A \equiv B \rightarrow$, which follows from $A \equiv B \rightarrow B$ and $A \equiv B \rightarrow \sim B$ by application of **NI.** Suppose then the L-sequent follows by application of **BII** from two other L-sequents $\rightarrow B$ and $A \rightarrow$, and hence reads again $A \equiv B \rightarrow$; and suppose the N-counterparts $\rightarrow B$ and $\rightarrow \sim A$ of $\rightarrow B$ and $A \rightarrow$ are both provable by means of Φ_α. Then $A \equiv B \rightarrow B$ and $A \equiv B \rightarrow \sim A$ are sure to be provable by means of Φ_α. But $A \equiv B \rightarrow A \equiv B$ is provable by means of **R.** Hence $A \equiv B \rightarrow A$, which follows from $A \equiv B \rightarrow A \equiv B$ and $A \equiv B \rightarrow B$ by application of **BE**$_{\mathbf{I}}$, is sure to be provable by means of Φ_α. Hence so is the N-counterpart $\rightarrow \sim(A \equiv B)$ of $A \equiv B \rightarrow$, which follows from $A \equiv B \rightarrow A$ and $A \equiv B \rightarrow \sim A$ by application of **NI.** Suppose then the L-sequent follows by application of **BII** from two other L-sequents $A_1, A_2, \ldots, A_n \rightarrow A$, where $n > 0$, and $B \rightarrow$, and hence reads $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow$; and suppose the N-counterparts $A_1, A_2, \ldots, A_n \rightarrow A$ and $\rightarrow \sim B$ of $A_1, A_2, \ldots, A_n \rightarrow A$ and $B \rightarrow$ are both provable by means of Φ_α. Then $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow A$ and $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow \sim B$ are sure to be provable by means of Φ_α. But $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow A \equiv B$ is provable by means of **R, E,** and **P.** Hence $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow B$, which follows from $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow A$ and $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow A \equiv B$ by application of **BE**$_{\mathbf{I}}$, is sure to be provable by means of Φ_α. Hence so is the N-counterpart $A \equiv B, A_1, A_2, \ldots, A_{n-1} \rightarrow \sim A_n$ of $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow$, which follows from $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow B$ and $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow \sim B$ by application of **NI.** Suppose then the L-sequent follows by application of **BII** from $A_1, A_2, \ldots, A_n \rightarrow B$, where $n > 0$, and $A \rightarrow$, and hence reads $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow$. By a similar reasoning the N-counterpart of $A \equiv B, A_1, A_2, \ldots, A_n \rightarrow$ is sure to be provable by means of Φ_α if the N-counterparts of $A_1, A_2, \ldots, A_n \rightarrow B$ and $A \rightarrow$ both are.

Or suppose an L-sequent follows by application of **∀II** from another L-sequent $A' \rightarrow$, where A' is as in Note (b) under Table I, and hence reads $(\forall X)A \rightarrow$; and suppose the N-counterpart $\rightarrow \sim A'$ of $A' \rightarrow$ is provable by means of Φ_α. Then $(\forall X)A \rightarrow \sim A'$ is sure to be provable by means of Φ_α. But $(\forall X)A \rightarrow (\forall X)A$ is provable by means of **R,** and $(\forall X)A, A' \rightarrow A'$ provable by means of **R** and **E.** Hence $(\forall X)A \rightarrow A'$, which follows from $(\forall X)A \rightarrow (\forall X)A$ and $(\forall X)A, A' \rightarrow A'$ by application of **∀E,** is provable by means of Φ_α. Hence so is the N-counterpart $\rightarrow \sim(\forall X)A$ of $(\forall X)A \rightarrow$, which follows from $(\forall X)A \rightarrow A'$ and $(\forall X)A \rightarrow \sim A'$ by application of **NI.**

Or suppose an L-sequent follows by application of **∃II** from another L-sequent $A' \rightarrow$, where A' is as in Note (b) under Table I and X' as in Note (c) under Table II, and hence reads $(\exists X)A \rightarrow$; and suppose the N-counterpart $\rightarrow \sim A'$ of $A' \rightarrow$ is provable by means of Φ_α. Then $(\exists X)A, A', (\exists X)A \rightarrow \sim A'$ is sure to be provable by means of Φ_α. But $(\exists X)A, A', (\exists X)A \rightarrow A'$ is provable by means of **R, E,** and **P.** Hence $(\exists X)A, A' \rightarrow \sim(\exists X)A$, which follows from $(\exists X)A, A', (\exists X)A \rightarrow A'$ and $(\exists X)A, A', (\exists X)A \rightarrow \sim A'$ by application of **NI,** is sure to be provable by means of Φ_α. But $(\exists X)A \rightarrow (\exists X)A$ is provable by means of **R.** Hence $(\exists X)A \rightarrow \sim(\exists X)A$, which follows (whether or not X' is the same as X) from $(\exists X)A \rightarrow (\exists X)A$ and $(\exists X)A, A' \rightarrow \sim(\exists X)A$ by application of **∃E,** is sure to be provable by means of Φ_α. Hence so is the N-counterpart $\longrightarrow \sim(\exists X)A$ of $(\exists X)A \longrightarrow$, which follows from $(\exists X)A \longrightarrow \sim(\exists X)A$ and $(\exists X)A \longrightarrow (\exists X)A$ by application of **NI.**

We have restricted ourselves so far to N-sequents in which no individual variable occurs both bound and free. Suppose, however, one or more individual variables, say, $X_1, X_2, \ldots$, and X_k, occur both bound and free in the N-sequent S of Theorem 1; suppose S' is like S except for exhibiting bound occurrences of k individual variables foreign to S at those and only those places where S exhibits bound occurrences of $X_1, X_2, \ldots$, and X_k, respectively; and suppose S is **I**-valid. Then S' is sure to be **I**-valid as well, and hence provable by means of Φ_α. But if S' is provable by means of Φ_α, then so is S, as a long -but elementary enough- argument will show. Hence S, if **I**-valid, is sure to be provable by means of Φ_α.

III

Now for **C**-valid N-sequents. It is readily shown that every N-sequent provable by means of the following rules of inference is **C**-valid, and hence that the said rules are classically sound:[14]

TABLE III

Structural rules: Same as in Table I

Intelim rules

For '&','v','∀', and '∃': Same as in Table I

For '∼'

NE:
$$\frac{K \longrightarrow \sim\sim A}{K \longrightarrow A}$$

NI: Same as in Table I

For '⊃'

HE:
$$\frac{K \longrightarrow A \supset B \text{ and } K \longrightarrow (A \supset C) \supset A}{K \longrightarrow B}$$

HI: Same as in Table I

For '≡'

BE:
$$\frac{K \longrightarrow A \text{ and } K \longrightarrow (B \equiv A) \equiv (B \equiv C)}{K \longrightarrow C}$$

BI: Same as in Table I

Leaving that matter to the reader, I shall prove in outline form the following separation theorem:

Theorem 2. Let α be any subset of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$ other than $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, and $\{\&, \vee, \forall, \exists\}$; S be any N-*sequent in α; and Ψ_α consist of* *rules* **R**, **E**, **P**, *and* **C** *in Table III and the intelim rules of that table for such (and only such) operators as belong to* α. *If S is* **C**-*valid, then S is provable by means of* Ψ_α.

I shall make use, when proving Theorem 1, of yet another result of Gentzen's:

Lemma 2. Let α *be any subset of* $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$, *and S be any L-sequent in* α *in which no individual variable occurs both bound and free. If S in* **C**-*valid, then S is susceptible of a proof in which every entry follows from previous entries in the proof by application of a structural rule in Table IV or an introduction rule of that table for a member of* α.[15]

TABLE IV

Structural rules

Reiteration (**R**): $A \longrightarrow A$

Expansion:

to the left (**El**)
$$\frac{K \longrightarrow L}{A, K \longrightarrow L}$$

to the right (**Er**)
$$\frac{K \longrightarrow L}{K \longrightarrow L, A}$$

Permutation:

to the left (**Pl**)
$$\frac{K, A, B, L \longrightarrow M}{K, B, A, L \longrightarrow M}$$

to the right (**Pr**)
$$\frac{K \longrightarrow L, A, B, M}{K \longrightarrow L, B, A, M}$$

Contraction:

to the left (**Cl**)
$$\frac{A, A, K \longrightarrow L}{A, K \longrightarrow L}$$

to the right (**Cr**)
$$\frac{K \longrightarrow L, A, A}{K \longrightarrow L, A}$$

Introduction rules

For '$\sim$':

to the left (**NIl**)
$$\frac{K \longrightarrow L, A}{\sim A, K \longrightarrow L}$$

to the right (**NIr**)
$$\frac{A, K \longrightarrow L}{K \longrightarrow L, \sim A}$$

For '$\supset$':

to the left (**HIl**)
$$\frac{K \longrightarrow M, A \text{ and } B, L \longrightarrow N}{A \supset B, K, L \longrightarrow M, N}$$

to the right (**HIr**)
$$\frac{A, K \longrightarrow L, B}{K \longrightarrow L, A \supset B}$$

For '&':

to the left (**CIl**)
$$\frac{A, K \longrightarrow L \text{ or } B, K \longrightarrow L}{A \,\&\, B, K \longrightarrow L}$$

to the right (**CIr**)
$$\frac{K \longrightarrow L, A \text{ and } K \longrightarrow L, B}{K \longrightarrow L, A \,\&\, B}$$

For 'v':

to the left (DIl)

$$\frac{A, K \rightarrow L \text{ and } B, K \rightarrow L}{A \vee B, K \rightarrow L}$$

to the right (DIr)

$$\frac{K \rightarrow L, A \text{ or } K \rightarrow L, B}{K \rightarrow L, A \vee B}$$

For '≡':

to the left (BIl)

$$\frac{(i)\ K \rightarrow M, A \text{ and } B, L \rightarrow N \text{ or } (ii)\ K \rightarrow M, B \text{ and } A, L \rightarrow N}{A \equiv B, K, L \rightarrow M, N}$$

to the right (BIr)

$$\frac{A, K \rightarrow L, B \text{ and } B, K \rightarrow L, A}{K \rightarrow L, A \equiv B}$$

For '∀'

to the left (∀Il)

$$\frac{A', K \rightarrow L}{(\forall X)A, K \rightarrow L}$$

to the right (∀Ir)

$$\frac{K \rightarrow L, A'}{K \rightarrow L, (\forall X)A}$$

For '∃'

to the left (∃Il)

$$\frac{A', K \rightarrow L}{(\exists X)A, K \rightarrow L}$$

to the right (∃Il)

$$\frac{K \rightarrow L, A'}{K \rightarrow L, (\exists X)A}$$

Notes: (a) Throughout the above rules, K, L, M, and N are to be finite (and possibly empty) sequences of wffs separated by commas. (b) In the last four rules A' is to be as in Note (b) under Table I. (c) In ∀Ir and ∃Il X' is not to occur free in any wff in K, nor in any wff in L, nor – should X' be distinct from X – in A.

I shall also make use of the following result, due to R. H. Thomason and myself:[16]

Lemma 4. *Let* α *be any subset of* $\{\&, \forall, \exists\}$; *let* S *be any* L*-sequent in* α *in which no individual variable occurs both bound and free; and let* χ_α *consist of the structural rules in Table IV and the introduction rules of that table for such (and only such) operators as belong to* α. *Then* S *is provable by means of* χ_α *if and only if* S *is provable by means of* χ_α *minus rule* **Cr** *(Contraction to the right).*

IV

Suppose in proof of Theorem 2 that S is a C-valid N-sequent, and hence a C-valid L-sequent; suppose, as we indeed may without loss of generality, that no individual variable occurs both bound and free in S; and suppose S is in a subset α of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$ to which belongs at least one of '~', 'v', '⊃', and '≡'. Because of Lemma 2 S is susceptible of a proof in which every entry follows from previous entries in the proof by application of a structural rule in Table IV or an introduction rule of that table for a member of α, and -save when '$\sim$' belongs to α - is of the sort $A_1, A_2, \ldots, A_n \rightarrow B_1, B_2, \ldots, B_m$, where $m \geq 1$. Now (i) in the case that '$\sim$' belongs to α, let an L-sequent of the sort $A_1, A_2, \ldots, A_n \rightarrow$ have the N-sequent $A_1, A_2, \ldots, A_{n-1} \rightarrow \sim A_n$ as its N-counterpart, one of the sort $A_1, A_2, \ldots, A_n \rightarrow B$ have itself as its N-counterpart, and one of the sort $A_1, A_2, \ldots, A_n \rightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, have the N-sequent $A_1, A_2, \ldots, A_n, \sim B_1, \sim B_2, \ldots, \sim B_{m-1} \rightarrow B_m$ as its N-counterpart; (ii) in the case that '$\sim$' does not belong to α, but 'v' does, let an L-sequent of the sort $A_1, A_2, \ldots, A_n \rightarrow B$ have itself as its N-counterpart, and one of the sort $A_1, A_2, \ldots, A_n \rightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, have the N-sequent $A_1, A_2, \ldots, A_n \rightarrow (\ldots(B_1 \vee B_2) \vee \ldots) \vee B_m$ as its N-counterpart; (iii) in the case that neither one of '$\sim$' and 'v' belongs to α, but '$\supset$' does, let an L-sequent of the sort $A_1, A_2, \ldots, A_n \rightarrow B$ have itself as its N-counterpart, and one of the sort $A_1, A_2, \ldots, A_n \rightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, have the N-sequent $A_1, A_2, \ldots, A_n, B_1 \supset B_m, B_2 \supset B_m, \ldots, B_{m-1} \supset B_m \rightarrow B_m$ as its N-counterpart; and (iv) in the case that none of '$\sim$', 'v', and '$\supset$' belongs to α, but '$\equiv$' does, let an L-sequent of the sort $A_1, A_2, \ldots, A_n \rightarrow B$ have itself as its N-counterpart, and one of the sort $A_1, A_2, \ldots, A_n \rightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, have the N-sequent $A_1, A_2, \ldots, A_n, B_1 \equiv B_m, B_2 \equiv B_m, \ldots, B_{m-1} \equiv B_m \rightarrow B_m$ as its N-counterpart. It can be shown - readily in some cases, by dint of hard labor in others - that: (a) if an L-sequent S_j follows from another L-sequent S_i by application of a structural rule in Table IV or a one-premiss rule of that table for a member of the aforementioned α, then the N-counterpart of S_j is sure to be provable by means of Ψ_α if the N-counterpart of S_i is, and (b) if an L-sequent S_j follows from two other L-sequents S_h and S_i by application of a two-premiss rule of Table IV for a member of the aforementioned α, then the N-counterpart of S_j is sure to be provable by means of Ψ_α if the N-counterpart of each one of S_h and S_i is. But if so, then S is sure to be provable by means of Ψ_α.

The argument accounts for 116 out of the 124 cases covered by Theorem 2.

Suppose then S to be in a subset of α of $\{\&, \forall, \exists\}$. In view of Lemmas 2 and 3 S is susceptible of a proof in which every entry follows from previous entries in the proof by application of one of rules **R**, **El**, **Er**, **Pl**, **Pr**, and **Cl** in Table IV or an introduction rule of that table for a member of α. But if so, then S is susceptible of a proof in which every entry follows from previous entries in the proof by application of one of rules **R**, **E**, **P**, and **C** in Table I or an introduction rule of Table V for a member of α.

TABLE V

For '&':

to the left	to the right
$\dfrac{A, K \rightarrow C \text{ or } B, K \rightarrow C}{A \& B \rightarrow C}$	Like **Cl** in Table I

For '$\forall$':

to the left	to the right
$\dfrac{A', K \longrightarrow B}{(\forall X)A, K \longrightarrow B}$	Like **∀I** in Table I

For '$\exists$':

to the left	to the right
$\dfrac{A', K \longrightarrow B}{(\exists X)A, K \longrightarrow B}$	Like **∃I** in Table I

Notes: (a) In the introduction rule for '$\forall$' to the left A' is to be as in Note (b) under Table I. (b) In the introduction rule for '$\exists$' to the left X is to be any individual variable that does not occur free in any wff in K nor in B.

But any L-sequent that is provable by means of the said rules is sure - as the reader may verify - to be I-valid. Hence S is sure to be I-valid. Hence in view of Theorem 1 S is sure to be provable by means of Ψ_α.

The argument accounts for the remaining 8 of the 124 cases covered by Theorem 2.[17]

In view of Theorem 1, Theorem 2 also holds true of such N-sequents in $\{\vee,\forall\}$, $\{\&,\vee,\forall\}$, $\{\vee,\forall,\exists\}$, or $\{\&,\vee,\forall,\exists\}$ as happen to be both **C**-valid and I-valid. It does not hold true, however, of those which, though **C**-valid, are not I-valid as well.[18] And this for a simple reason: any N-sequent provable by means of **R, E, P, NI, HI, CE, CI, DE, DI, BI, ∀E, ∀I, ∃E,** or **∃I** is sure - as we remarked earlier - to be I-valid. The argument of pages 169-171 is thus bound to falter for the four subsets $\{\vee,\forall\}$, $\{\&,\vee,\forall\}$, $\{\vee,\forall,\exists\}$, and $\{\&,\vee,\forall,\exists\}$ of $\{\supset,\&,\vee,\equiv,\forall,\exists\}$. It does go through, however, if the following rule, to be called **∀I**$_\vee$, is enlisted as an extra intelim rule for '$\forall$':

$$\frac{K \longrightarrow A \vee B}{K \longrightarrow (\forall X')A' \vee B}$$

where A' is like A except for exhibiting free occurrences of X' wherever A exhibits free occurrences of some individual variable X (not necessarily distinct from X'), and X' and X -should they occur free in A- are not to occur free in any wff in K nor in B.

Hence:

Theorem 3. Let α be any one of the four subsets $\{\vee,\forall\}$, $\{\&,\vee,\forall\}$, $\{\vee,\forall,\exists\}$, and $\{\&,\vee,\forall,\exists\}$ of $\{\sim,\supset,\&,\vee,\equiv,\forall,\exists\}$; S be any **C***-valid* N-*sequent in α; and Ψ_α consist of rules* **R, E, P,** *and* **C** *in Table III and the intelim rules of Table III for such and only such operators as belong to α.*

(a) If S is I*-valid, then S is provable by means of Ψ_α.*

(b) If S is not I*-valid, then S is provable by means of Ψ_α and rule* **∀I**$_\vee$

Since any N-sequent that is provable by means of **∀I**$_\vee$ is provable by means of: (i) **DE, DI, ∀E, ∀I, NE,** and **NI,** (ii) **DE, DI, ∀E, ∀I, HE,** and **HI,** and (iii) **DE, DI, ∀E, ∀I, BE,** and **BI,** we also have:

Theorem 4. *Let α, S, and Ψ_α be as in Theorem 3.*
(*a*) *If S is* I-*valid, then S is provable by means of Ψ_α.*
(*b*) *If S is not* I-*valid, then S is provable by means of Ψ_α plus rules* **NE** *and* **NI**, *or rules* **HE** *and* **HI**, *or rules* **BE** *and* **BI**.

When **R** is generalized to read:

$$\textbf{GR}: \ K, A, L \longrightarrow A,$$

the remaining three structural rules of Table I are expendable. Proof that the first two, **E** and **P**, are, will be found in [8]. Suppose then there is a proof of $A, A, K \longrightarrow B$ by means of **GR** and zero or more intelim rules from Table I [Table III], and suppose some entry in the proof of $A, A, K \longrightarrow B$ does not open with A, A, K. The result of lopping off that entry is sure -as the reader may verify- to constitute a proof of $A, A, K \longrightarrow B$ by means of **GR** and the intelim rules in question. Suppose then the proof of $A, A, K \longrightarrow B$ runs:

$$\begin{array}{l} A, A, K, L_1 \longrightarrow B_1 \\ A, A, K, L_2 \longrightarrow B_2 \\ \cdot \\ \cdot \\ \cdot \\ A, A, K, L_{p-1} \longrightarrow B_{p-1} \\ A, A, K \longrightarrow B_p \ \ (= B), \end{array}$$

where for each i from 1 to p-1 L_i consists of zero or more wffs separated by commas. The parallel column

$$\begin{array}{l} A, K, L_1 \longrightarrow B_1 \\ A, K, L_2 \longrightarrow B_2 \\ \cdot \\ \cdot \\ \cdot \\ A, K, L_{p-1} \longrightarrow B_{p-1} \\ A, K \longrightarrow B_p \ \ (= B) \end{array}$$

is sure - as the reader may verify - to constitute a proof of $A, K \longrightarrow B$ by means of **GR** and the intelim rules in question.

Hence the following corollaries of Theorems 1-3:

Theorem 5. *Let α be any subset of* $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$; *let S be any* N-*sequent in α; and let Φ'_α consist of rule* **GR** *and the intelim rules of Table I for such (and only such) operators as belong to α. If S is* I-*valid, then S is provable by means of Φ'_α.*

Theorem 6. *Let α be any subset of* $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$ *other than* $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, *and* $\{\&, \vee, \forall, \exists\}$; *let S be any* N-*sequent in α; and let Ψ'_α consist of rule* **GR** *and the intelim rules of Table III for such (and only such) operators as belong to α. If S is* **C**-*valid, then S is provable by means of Ψ'_α.*

Theorem 7. *Let α be any one of the four subsets* $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, *and* $\{\&, \vee, \forall, \exists\}$ *of* $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$; *let S be any* **C**-*valid* N-*sequent in α; and let Ψ'_α be as in Theorem 6.*
(*a*) *If S is* I-*valid, then S is provable by means of Ψ'_α.*
(*b*) *If S is not* I-*valid, then S is provable by means of Ψ'_α and rule* **$\forall$I$_\vee$**.

Now consider the following two sets of rules for conducting so-called *natural deductions* in Heyting's first-order functional calculus (Table VI) and in the classical first-order functional calculus (Table VII):

TABLE VI

Generalized Reiteration: From a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce any one of $A_1, A_2, \ldots,$ and A_n as a conclusion (**GR'**)

Intelim rules for '$\sim$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions $\sim B$ and $\sim\sim B$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce B as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ of premisses one may deduce each one of two conclusions C and $\sim C$, then from the subset $\{A_1, A_2, \ldots, A_n\}$ of the original set $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ one may deduce $\sim B$ as a conclusion.

Intelim rules for '$\supset$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions B and $B \supset C$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce C as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ of premisses one may deduce a conclusion C, then from the subset $\{A_1, A_2, \ldots, A_n\}$ of the original set $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ one may deduce $B \supset C$ as a conclusion.

Intelim rules for '&': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion B & C and from the superset $\{A_1, A_2, \ldots, A_n\} \cup \{B, C\}$ of $\{A_1, A_2, \ldots, A_n\}$ deduce a conclusion D, then from the original set $\{A_1, A_2, \ldots, A_n\}$ one may deduce D as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions B and C, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce B & C as a conclusion.

Intelim rules for '$\vee$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion $B \vee C$ and from each one of the two supersets $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ and $\{A_1, A_2, \ldots, A_n\} \cup \{C\}$ of $\{A_1, A_2, \ldots, A_n\}$ deduce a conclusion D, then from the original set $\{A_1, A_2, \ldots, A_n\}$ one may deduce D as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion B, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce either one of $B \vee C$ and $C \vee B$ as a conclusion.

Intelim rules for '$\equiv$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions B and $B \equiv C$ or B and $C \equiv B$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce C as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ of premisses one may deduce a conclusion C and from the set $\{A_1, A_2, \ldots, A_n\} \cup \{C\}$ deduce B as a conclusion, then from the subset $\{A_1, A_2, \ldots, A_n\}$ of the two sets $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ and $\{A_1, A_2, \ldots, A_n\} \cup \{C\}$ one may deduce $B \equiv C$ as a conclusion.

Intelim rules for '$\forall$': Let B' be like B except for exhibiting free occurrences of an individual variable X' wherever A exhibits free occurrences of an individual variable X.

(a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion $(\forall X)B$ and from the superset $\{A_1, A_2, \ldots, A_n\} \cup \{B'\}$ of $\{A_1, A_2, \ldots, A_n\}$ deduce a conclusion C, then from the original set $\{A_1, A_2, \ldots, A_n\}$ one may deduce C as a conclusion;
(b) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion B, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce $(\forall X')B'$ as a conclusion, so long as X' and X - should they occur free in B - do not occur free in any one of $A_1, A_2, \ldots,$ and A_n.

Intelim rules for '$\exists$': Let B' be as in the previous rules.
(a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion $(\exists X')B'$ and from the superset $\{A_1, A_2, \ldots, A_n\} \cup \{B\}$ of $\{A_1, A_2, \ldots, A_n\}$ deduce a conclusion C, then from the original set $\{A_1, A_2, \ldots, A_n\}$ one may deduce C as a conclusion, so long as X' and X - should they occur free in B - do not occur free in any one of $A_1, A_2, \ldots, A_n$, and B;
(b) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion B', then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce $(\exists X)B$ as a conclusion.

TABLE VII

GR': Same as in Table VI.

Intelim rules for '&', 'v', '$\forall$', and '$\exists$': Same as in Table VI

Intelim rules for '$\sim$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion $\sim\sim B$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce B as a conclusion;
(b) As in Table VI

Intelim rules for '$\supset$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions $B \supset C$ and $(B \supset D) \supset B$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce C as a conclusion;
(b) As in Table VI.

Intelim rules for '$\equiv$': (a) If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce each one of two conclusions B and $(C \equiv B) \equiv (C \equiv D)$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce D as a conclusion.
(b) As in Table VI.

It is readily shown that if an N-sequent $A_1, A_2, \ldots, A_n \longrightarrow B$ in α is provable by means of rule **GR** and the intelim rules of Table I [Table III] for such (and only such) operators as belong to α, then B is deducible from $\{A_1, A_2, \ldots, A_n\}$ by means of rule **GR'** in Table VI [Table VII] and the intelim rules of Table VI [Table VII] for such and only such operators as belong to α.[19] Suppose then that B is held to be I-implied [C-implied] by $\{A_1, A_2, \ldots, A_n\}$ if $A_1, A_2, \ldots, A_n \longrightarrow B$ is I-valid [C-valid]. We will have the following two corollaries of Theorems 5-6 (and, hence, of Theorems 1-2):

Theorem 8. *Let* $A_1, A_2, \ldots, A_n$ $(n \geq 0)$, *and* B *be wffs of* **FC**, *and* α *consist of those among the operators* '$\sim$', '$\supset$', '&', 'v', '$\equiv$', '$\forall$', *and* '$\exists$' *that occur in* $A_1, A_2, \ldots, A_n$, *and* B. *If* B *is* I-*implied by* $\{A_1, A_2, \ldots, A_n\}$, *then* B *is* *deducible from* $\{A_1, A_2, \ldots, A_n\}$ *by means of rule* **GR'** *in Table VI and the intelim rules of that table for such (and only such) operators as belong to* α.

Theorem 9. *Let* $A_1, A_2, \ldots, A_n, B$, *and* α *be as in Theorem* 8. *If* α *is other than* $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, *and* $\{\&, \vee, \forall, \exists\}$, *and* B *is* **C**-*implied by* $\{A_1, A_2, \ldots, A_n\}$, *then* B *is deducible from* $\{A_1, A_2, \ldots, A_n\}$ *by means of rule* **GR'** *in Table VII and the intelim rules of that table for such (and only such) operators as belong to* α.

With a wff A held to be: (i) I-implied [**C**-implied] by an infinite set S of wffs if A is I-implied [**C**-implied] by a finite subset of S,[20] and (ii) deducible from S by means of **GR'** and zero or more intelim rules from Table VI [Table VII] if A is deducible from a finite subset of S by means of those rules, Theorems 7-8 readily generalize into:

Theorem 10. *Let* A *be a wff of* **FC**, S *be a set of wffs of* **FC**, *and* α *consist of those among the operators* '$\sim$', '$\supset$', '&', '$\vee$', '$\equiv$', '$\forall$', *and* '$\exists$' *which occur in a member of* S *or in* A. *If* A *is* I-*implied by* S, *then* A *is deducible from* S *by means of rule* **GR'** *in Table VI and the intelim rules of that table for such (and only such) operators as belong to* α.

Theorem 11. *Let* A, S, *and* α *be as in Theorem* 10. *If* A *is* **C**-*implied by* S *and* α *is other than* $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, *and* $\{\&, \vee, \forall, \exists\}$, *then* A *is deducible from* S *by means of rule* **GR'** *in Table VII and the intelim rules of that table for such (and only such) operators as belong to* α.

A like-minded corollary of Theorem 7 (and, hence, of Theorem 3) holds true, with rule $\forall I'_\vee$ understood to be the following analogue of $\forall I_\vee$: "Let B' be like B except for exhibiting free occurrences of X' wherever B exhibits free occurrences of some individual variable X (not necessarily distinct from X'). If from a set $\{A_1, A_2, \ldots, A_n\}$ of premisses one may deduce a conclusion $B \vee C$, then from the same set $\{A_1, A_2, \ldots, A_n\}$ one may deduce $(\forall X')B' \vee C$ as a conclusion, so long as X and X' -should they occur free in B- do not occur free in any one of $A_1, A_2, \ldots, A_n$, and C."[21]

Theorem 12. *Let* A, S, *and* α *be as in Theorem* 10; *Let* A *be* **C**-*implied by* S; *and let* α *be one of* $\{\vee, \forall\}$, $\{\&, \vee, \forall\}$, $\{\vee, \forall, \exists\}$, *and* $\{\&, \vee, \forall, \exists\}$.
(*a*) *If* A *is* I-*implied by* S, *then* A *is deducible from* S *by means of rule* **GR'** *in Table VII and the intelim rules of that table for such (and only such) operators as belong to* α.
(*b*) *If* A *is not* I-*implied by* S, *then* A *is deducible from* S *by means of rule* **GR'** *in Table VII, the intelim rules of that table for such (and only such) operators as belong to* α, *and rule* $\forall I_\vee$.[22]

NOTES

1. The letter 'N', which along with the letter 'L' comes from Gentzen [5], is short for 'natural' in the phrase 'natural deduction'. The relationship between the rules of Tables I and III and the familiar ones for conducting so-called *natural deductions* is studied in the fifth section of this paper.

2. In the absence of a handy criterion of intuitionist validity, the reader may take a wff of **FC** to be I-valid if A is provable in Heyting's first-order functional calculus.

3. When $p = 0$, the member of Φ in question is more commonly called an *axiom schema.* When $p = 1$, I shall occasionally refer to the member of Φ in question as a *one-premiss rule*; when $p = 2$, as a *two-premiss rule.*

4. Except for **NE**$_\mathrm{I}$, **CE**, **BE**$_\mathrm{I}$, **BI**, and **∀E**, the rules in question all turn up in Gentzen [6]. **CE** was suggested to me by Nuel D. Belnap, Jr. **NE**$_\mathrm{I}$ is due to Paul Bernays, who offered it in [2] as an alternative to the more familiar rule:

$$\frac{K \longrightarrow A \text{ and } K \longrightarrow \sim A}{K \longrightarrow B}$$

The appellation 'intelim', coined by F. B. Fitch in [4], is short of course for 'introduction-elimination'. The 'I' in '**NI**', '**HI**', '**CI**', and so on, is short for 'Introduction'; and the '**E**' in '**NE**$_\mathrm{I}$', '**HE**$_\mathrm{I}$', '**CE**', and so on, short for 'Elimination'. I write '**NE**$_\mathrm{I}$', '**HE**$_\mathrm{I}$', and '**BE**$_\mathrm{I}$', with the subscript 'I' short for 'intuitionist', to distinguish the three rules from their counterparts in Table III.

5. I borrow the phrase 'separation theorem' from Curry [3].

6. See Gentzen [5]. Gentzen treats $A \equiv B$ as short for $(A \supset B)$ & $(B \supset A)$, and hence does without introduction rules for '≡'. That the result quoted in the text still holds true when '≡' serves as a primitive connective and rules **BIl** and **BIr** in Table II serve as introduction rules for '≡', has independently been noted by Bernays (see Bernays [2]). The '**l**' in '**El**', '**Pl**', '**Cl**', and so on, is short for 'to the left'; the '**r**' in '**Er**', '**Pr**', '**Cr**', and so on, is short for 'to the right'.

7. Note for proof of (ii) that '∼' does not belong to any of the subsets of $\{\sim, \supset, \&, \vee, \equiv, \forall, \exists\}$ under consideration here. Hence no entry of the proof of S can be had by application of **NIl**. Hence none can be of the sort $A_1, A_2, \ldots, A_n \longrightarrow$. But no L-sequent of the sort $A_1, A_2, \ldots, A_n \longrightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, can be had by application of a rule from Table II. Hence (ii).

8. Φ_α, the reader may recall, consists of **R**, **E**, **P**, **C**, and the intelim rules of Table I for such (and only such) operators as belong to α.

9. That (a) holds true when S_j follows from S_i by application of **El**, **Pl**, **Cl** (in view of (ii) **Er** does not enter into account here), **HIr**, **DIr**, or **∃Ir**, and (b) holds true when S_j follows from S_h and S_i by application of **CIr** or **BIr**, needs no proof.

10. Note here that '⊃' is sure to belong to α, and hence **HE**$_\mathrm{I}$ and **HI** - two rules we make use of a few lines hence - sure to belong to Φ_α. Like remarks apply throughout this section.

11. Recall for proof of (ii) that no L-sequent of the sort $A_1, A_2, \ldots, A_n \longrightarrow B_1, B_2, \ldots, B_m$, where $m > 1$, can be had by application of a rule from Table II.

12. That (a) holds true when S_j follows from S_i by application of **El**, **Pl**, **Cl**, **HIr**, **DIr**, **∀Ir**, or **∃Ir**, and (b) holds true when S_j follows from S_h and S_i by application of **CIr** or **BIr**, needs no proof.

13. The case where the L-sequent in question follows from $A_1, A_2, \ldots, A_n \longrightarrow A$, where $n \geq 0$, and $B, A_{n+1}, A_{n+2}, \ldots, A_{n+k} \longrightarrow C$, where $k > 0$, has already been treated; and so - by implication - has been the one where the L-sequent follows from $A_1, A_2, \ldots, A_n \longrightarrow A$, where $n \geq 0$, and $B, A_{n+1}, A_{n+2}, \ldots, A_{n+k} \longrightarrow$, where $k > 0$, since the N-counterparts of $B, A_{n+1}, A_{n+2}, \ldots, A_{n+k} \longrightarrow$ and $A \supset B, A_1, A_2, \ldots, A_{n+k} \longrightarrow$ are $B, A_{n+1}, A_{n+2}, \ldots, A_{n+k-1} \longrightarrow \sim A_{n+k}$ and $A \supset B, A_1, A_2, \ldots, A_{n+k-1} \longrightarrow \sim A_{n+k}$. Like remarks apply throughout the balance of the section.

14. Rule HE was suggested to me by Stig Kanger. Rule **BE** is due to Paul Bernays, who offered it in [2] as an alternative to the introduction rule for '≡' that I used in [7], to wit:

$$\frac{K \longrightarrow A \text{ and either } K \longrightarrow (B \equiv A) \equiv (B \equiv C) \text{ or } K \longrightarrow (B \equiv C) \equiv (B \equiv A)}{K \longrightarrow C}$$

15. Gentzen again has no rules for '≡', but his original result easily generalizes into Lemma 2. Incidentally, it follows from results of S. Maehara and M. Ohnishi (see Umezawa [10]) that *so long as rules* **NIr**, **HIr**, **BIr**, *and* **∀Ir** *in Table IV are weakened to read like their counterparts in Table II,* every L-sequent provable by means of the rules of Table IV is I-valid, and vice-versa.

16. See Leblanc and Thomason [9].

17. For a different proof of Theorem 2 for the 32 cases where α is a subset of $\{\sim, \supset, \&, \vee, \equiv\}$, and an algorithm for proving S (when **C**-valid) in each one of those 32 cases, see Leblanc [7].

18. Cases in point are '$(\forall x)(f(x) \vee p) \longrightarrow (\forall x)f(x) \vee p$', '$(\forall x)(f(x) \vee (p \& p)) \longrightarrow (\forall x)f(x) \vee (p \& p)$', '$(\forall x)(f(x) \vee (\exists x)p) \longrightarrow (\forall x)f(x) \vee (\exists x)p$', and '$(\forall x)(f(x) \vee (\exists x)(p \& p)) \longrightarrow (\forall x)f(x) \vee (\exists x)(p \& p)$'.

19. See for proof Leblanc [8].

20. A may also be said to be **C**-implied by S if $S \quad \{\sim A\}$ is not simultaneously satisfiable. That A is **C**-implied by S if and only if A is **C**-implied by a finite subset of S then follows from the so-called *compactness theorem* (to the effect that a set of wffs of **FC** is simultaneously satisfiable if and only if every finite subset of the set is).

21. A congener of rule $\mathbf{\forall I}_{\vee}^{\prime}$ appears in Fitch [4].

22. My thanks go to Paul Bernays, Michael D. Resnik, and Richmond H. Thomason, who read a first draft of this paper.

BIBLIOGRAPHY

[1] Bernays, P., "Betrachtungen zum Sequenzen-Kalkul," in *Contributions to Logic and Methodology in Honor of J. M. Bocheński,* Amsterdam (1965).

[2] Bernays, P., Review of H. Leblanc's "Etudes sur les règles d'inférence dites règles de Gentzen," *The Journal of Symbolic Logic,* vol. 27 (1962), pp. 248-249.

[3] Curry, H. B., *Foundations of Mathematical Logic,* New York (1963).

[4] Fitch, F. B., *Symbolic Logic, An Introduction,* New York (1952).

[5] Gentzen, G., "Untersuchungen über das logische Schliessen," *Mathematische Zeitschrift,* vol. 39 (1934), pp. 176-210, 403-431.

[6] Gentzen, G., "Die Widerspruchsfreiheit der reinen Zahlentheorie," *Mathematische Annalen,* vol. 112 (1936), pp. 493-565.

[7] Leblanc, H., "Proof routines for the propositional calculus," *Notre Dame Journal of Formal Logic,* vol. 4 (1963), pp. 81-104.

[8] Leblanc, H., *Techniques of Deductive Inference,* Englewood Cliffs, N.J. (1966).

[9] Leblanc, H., and Thomason, R. H., "A demarcation line between intuitionist logic and classical logic," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* forthcoming.

[10] Umezawa, T., "On intermediate propositional logics," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 20-36.

Bryn Mawr College
Bryn Mawr, Pennsylvania