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A RECONSTRUCTION OF FORMAL LOGIC*

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Introductory Notes and Summary. The purpose of this paper is to propose a reconstruction of classical so-called 1st order logic designed to avoid some undesirable features of the traditional formulations of that calculus. No final commitment to the classical viewpoint, as opposed to intuitionism or other tendencies, is thereby intended.

In §1 the semantical motivation for departing from the usual type of formalism in 1st order logic is made clear in a discussion of traditional systems. In the new System L, described in subsequent sections, the expressions that may occur as lines in derivations have no free individual variables, and the theorems are valid in every domain, including the empty one. The semantical equivalents of the theorems of traditional formulations of 1st order logic are derivable in the new system from an *assumption form*, standing for a *premise*, or *axiom in a theory*, to the effect that the universe is not empty.

In some fruitful exchanges of information and comments with Professor Lambert of the University of West Virginia and with Professor Hailperin of Lehigh University, which took place after an earlier version of this paper had already been accepted by this *Journal*, the author learned of other systems with features like, or similar to those of the System L described above. Accordingly, §3 of this article, which originally was devoted solely to a comparison of the System L with the principles of quantification in Quine's *Mathematical Logic*, has been revised to include reference to those systems.

§1. *Free Variables and Existence*. The use, in formalized languages, of expressions with free variables as lines in derivations has been often criticized. The authors of *Principia Mathematica* themselves, in the introduction to the second edition of their work [18], made a proposal for the elimination of that feature from their system, being obviously unsatisfied with their previous "primitive idea" of "ambiguous assertion." A different

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method for dispensing, in logic, with the use of free variables in the lines of derivations was proposed by Fitch [5] and adopted, in substance, by Quine in [12]. Church observes that "it has been urged with some force that the device of asserting propositional forms constitutes an unnecessary duplication of ways of expressing the same thing [what is expressed by means of universal quantifiers], and ought to be eliminated from a formalized language," but he none the less considers the device a useful one ([4], p. 46).

In connection with Church's remark, it may be noted that, as a rule, well-formed formulas with free variables (i.e. symbols which, by the rules of the game, may be made the objects of substitution and/or be generalized upon[1]) have not been construed in logic as just another way of expressing universal propositions. Rather—and this is what this writer finds more objectionable—they have played a sort of double role semantically, as usually indicated, though often rather obscurely, in the explanatory text accompanying their formal introduction.

The point is most simply explained at first by reference to logical systems, or their applications, in which no use is made of place markers (as distinct from variables subject to quantification) for predicates and/or statements. In such contexts, well-formed formulas with free variables may be said to be construed as statements expressing universal propositions when the rule of substitution or that of universal generalization is being applied to them. They are also clearly so construed, in those contexts, in their role as initial premises or final conclusion of an argument. Yet, in their capacity as lines in a derivation by *modus ponens*, the same expressions as a rule are construed not as statements, but rather as statement forms, with their variables, wherever they are free therein, standing for constants, i.e. they are construed as lines in an argument form. So even in one and the same occurrence, a well-formed formula with free variables may play this double semantical role. The process in its entirety, from premises expressing universal propositions (whether by means of free variables or by means of universal quantifiers) to a conclusion expressing a universal proposition (again, either through the use of free variables or by means of universal quantifiers) via such an argument form often is characterized (with specific reference to cases in which one individual variable is free in the formulas) as "reasoning about an unspecified object satisfying certain conditions to conclude something about every object satisfying those conditions," or in some such way. Thus we have argument forms used as parts of, rather than as patterns for, arguments, an algorythm which—this writer feels—logic ought to justify rather than sanction primitively.[2]

Of course, when place markers for predicates or statements are used in place of predicates or statements, the whole procedure as described above is represented by an argument form. Yet essentially the situation is the same in so far as such argument forms, if free variables are also used therein, are construed as patterns for arguments which have argument forms as parts.

The issue just raised concerns a habitual interpretation of well-formed formulas in traditional systems of logic rather than anything in the formal systems themselves. It might seem therefore that, except for possible objections to the duplication of ways of expressing universal propositions, it can be easily disposed of. Nothing prevents one, in fact, from always construing well-formed formulas with free variables as synonymous with their respective closures (understanding by this term the results of prefixing to them, in any order, universal quantifiers binding each of their free variables), as Carnap, for instance, does explicitly, stressing his divergence from the first edition of *Principia* on this point ([3], p. 22), and Church at least by implication (see quotation above). But it should be noted that if we thus construe our well-formed formulas with free variables, the rule of *modus ponens* assumes a different, less transparent role than it is usually conceived to have, as when it is originally introduced in so-called propositional logic. For, if well-formed formulas with free variables are construed as synonymous with their closures, to derive a well-formed formula ψ from a well-formed formula ϕ with free variables and the well-formed formula $\phi \supset \psi$ (to use a familiar metalinguistic device to refer to unspecified expressions of a specific form) amounts to deriving ψ, or a closure of it if any variable is free in it, from a closure of ϕ and a closure of $\phi \supset \psi$, a procedure that cannot be truth-functionally analyzed as *modus ponens* usually is.

To see the import of what has been just noted, consider two open sentences (i.e. expressions differing from statements only in exhibiting individual variables in place of individual constants[3]) ϕ and χ with one free variable, respectively α and β, in each. If the open sentences ϕ and $\phi \supset (\exists \beta)\chi$ are construed as statement forms in which α, wherever it is free therein, stands for a constant, the derivation of $(\exists \beta)\chi$ from them constitutes a valid argument form on purely truth-functional grounds: without exception, the statement $(\exists \beta)\chi$ follows from any two statements respectively of the form of the open sentences ϕ and $\phi \supset (\exists \beta)\chi$; only, should our universe of discourse be empty, then either there would not be any two such statements to deduce it from, or, if constants without denotation are countenanced in our language, one of the two statements would be false. On the other hand, if the open sentences ϕ and $\phi \supset (\exists \beta)\chi$ are construed as synonyms of their closures, i.e. as statements as they stand, then, if the universe is empty, they are true (think of them as synonyms of $\sim(\exists \alpha)\sim\phi$ and $\sim(\exists \alpha)\sim(\phi \supset (\exists \beta)\chi)$ respectively), while $(\exists \beta)\chi$ is false.

We are thus brought to the consideration of another objection that has been made to traditional logical systems, notably by Russell himself ([16], p. 203, footnote). In traditional formulations of logic, by the device of applying *modus ponens* to theorems with free individual variables, expressions such as '$(\exists x)(Fx \vee \sim Fx)$' and '$(x)Fx \supset (\exists x)Fx$' (or, in other systems, statements of the form of such expressions with an actual predicate in place of 'F'), which are not valid in the empty domain,[4] are proved as theorems. But if logic is to be independent of empirical knowledge, it ought to say nothing about the existence of objects. To be sure, every axiomatic theory of any interest outside pure logic postulates a non-empty domain, and logic ought to investigate that part of the structure of those theories which they have in common (just as it investigates the common substructure of all axiomatic theories using the predicate of identity, with the specific axioms pertaining to it). At the same time, however, logic ought sharply to separate from all others, as theorems of its own, those well-formed formulas occurring as lines in derivations that are valid in every domain.[5]

What is most disturbing in traditional formulations of logic about the controversial theorems referred to above is the surreptitious manner in which they make their way into the system. The point is forcefully made by Rosenbloom when he writes:

> In some formulations where the notion of "variable" is used rather freely, T6 [in his system, the theorem that all statements of the form of '$(x)Fx \supset (\exists x)Fx$' are true] is proved without this assumption [the assumption that there are individuals], but the deduction, while formally correct, smacks of sleight of hand. One may doubt that a formal system in which such a deduction is valid is a correct representation of our admittedly vague ideas of valid inference.[6]

Habitually, the source of the controversial theorems is seen in such axioms as '$(x)Fx \supset Fy$' and/or '$Fx \supset (\exists y)Fy$' (or, in formulations with an infinity of axioms, in their substitution instances taken as axioms, or in expressions of their form with an actual predicate in place of 'F'). Yet, if those axioms are construed as synonymous with their closures, then they are valid also in the empty domain. Semantically, the controversial theorems may be seen as obtained by construing those axioms, or theorems derived from them by substitution, and other theorems with free individual variables of tautological form needed for the derivation, as statement forms, with the individual variables, wherever they are free therein, standing for constants, and, at the same time, assuming that there are statements of their form with individual constants that have denotation. Yet, in most systems, those same axioms are also needed for the proof of theorems that are valid in every domain (e.g. their own closures), in a manner whereby they ought rather to be construed as synonymous with their closures. If we consistently construe all well-formed formulas with free variables as synonyms of their closures, as we are urged to do by Carnap and Church, then all axioms are valid in every domain and the source of the controversial theorems becomes even more subtly concealed in the rule of *modus ponens,* which then, as we have seen, no longer preserves the truth of statements, nor hence the validity of statements or statement forms in the empty domain.

§2. *The Primitive Basis of the System* **L**. We will now introduce a new system of pure logic, the System **L**, which meets the demands implicit in the criticism of traditional formulations of logic that we have voiced. Some other systems already meet those demands, as we will see in §3.

We will use an infinity of axioms for **L**, syntactically determined by *metaaxioms*. However, instead of borrowing predicates from ordinary language or special theories, we give logic an object language of its own, thus making the definition of 'well-formed formula', perfectly precise, by using place markers for predicates.[7] As such, we adopt the usual letters '*F*', '*G*', '*H*' with superscripts (that can be most of the time omitted in practice as a way of abbreviation) indicating the intended number of argument places, and with or without numerical subscripts; we will refer to these symbols as *predicate symbols*—by way of abbreviation, as *pss*, and to one of them as a *ps*.

For the rest, we adopt a familiar vocabulary for L, with '$\sim$' and '$\supset$' as primitive truth-functional connectives, and hence there is no need to spell it out. But it is better to give our formation rules in full, since they are less permissive than has become customary, being rather like those in the original Hilbert-Ackermann formulation. In stating them, we will use the familiar device of referring to unspecified object-language expressions of a specific form by the appropriate sequence of Greek letters and object-language symbols. Further, we agree to say of a variable α that it is *free* in a linear sequence of object language symbols, or *formula*, if it occurs therein, but the formula (α) does not occur therein, and to say that it is bound in it if the formula (α) occurs therein. Our formation rules, recursively defining 'well-formed formula' (we will use the usual abbreviations 'wf', 'wff', 'wffs') are as follows:

1. *If λ is an n-place ps, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are variables (not necessarily all distinct), then $\lambda\alpha_1\alpha_2 \ldots \alpha_n$ is wf.*
2. *If ϕ is wf and α is a variable free in it, then $(\alpha)\phi$ is wf.*
3. *If ϕ is wf, then $\sim\phi$ is wf.*
4. *If ϕ, ψ are wf, and if no variable is free in one of them and bound in the other, then $(\phi \supset \psi)$ is wf.*

No formula is wf unless it is so by virtue of rules 1-4 above.

We will not bother to define such familiar terms as 'quantifier', 'scope', and the like. Note that by L's formation rules, a quantifier cannot occur more than once with overlapping scopes, or vacuously in a wff.

We will use customary abbreviations of wffs, as by omitting parentheses, or by employing familiar defined symbols such as defined connectives and '$\exists$'. For perspicuity, we will use brackets and braces instead of parentheses, as needed.

Definitions. If some variable is free in a formula ϕ, then ϕ is said to be *open*. A wff that is not open is referred to as a *statement form*, or as an *sf* (more than one of them as *sfs*). A *closure* of a formula is the result of prefixing to it, in any order, universal quantifiers binding each of its free variables, and no other quantifiers. (Thus any closure of a wff is an sf, and any sf is its own only closure.) The *standard closure*, or *stcl* of a formula ϕ is that closure of ϕ in which all variables occurring at the left of ϕ occur there from left to right in alphabetical order (which order we may assume given in any arbitrary way whereby there is no last variable). A sequence of quantifiers $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$ in which the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct and occur from left to right in alphabetical order, is referred to as a *standard prefix*. An sf $(\alpha)\phi$ wherein ϕ is wf (i.e. is the scope of (α)) is referred to as a *universal sf*.

Only sfs will occur as lines in derivations in L. In pure logic, as the subject is here understood, open wffs are significant only as fragments of sfs (though for certain particular purposes, as the definition of 'validity' considered in §6, they may be regarded as subject to interpretations in non-empty domains).

Convention. For any syntactical expression used to refer to a formula ψ, as 'ψ' or '$\phi \supset \psi$', the result of prefixing to it such an expression as '$\frac{\alpha}{\beta}$', '$\frac{\alpha_1}{\beta_1} \frac{\alpha_2}{\beta_2} \cdots \frac{\alpha_n}{\beta_n}$', or the like in which other symbols used to refer to variables occur in place of the ones therein, will be used to refer to the formula that is like ψ except for exhibiting each of the variables referred to in that expression on top of an occurrence of '—' wherever ψ instead exhibits the variable referred to in that expression underneath that occurrence of '—'.

The axioms of our system L are determined by the following *meta-axioms*:

MA1. If $(\sim\phi \supset \phi) \supset \phi$ *is wf, then its stcl is an axiom.*
MA2. If $\phi \supset (\sim\phi \supset \psi)$ *is wf, then its stcl is an axiom.*
MA3. If $(\phi \supset \psi) \supset [(\psi \supset \chi) \supset (\phi \supset \chi)]$ *is wf then its stcl is an axiom.*
MA4. If $(\alpha)(\phi \supset \psi) \supset (\phi \supset (\beta)\frac{\beta}{\alpha}\psi)$ *is wf, then its stcl is an axiom.*
MA5. If $(\alpha)(\phi \supset \psi) \supset ((\alpha)\phi \supset \frac{\beta}{\alpha}\psi)$ *is a wff in which* α *occurs in* ψ *and* β *is not bound in* ψ, *then its stcl is an axiom.*

Observe that a formula such as is referred to in *MA4* is not wf, and hence is not an axiom, if, e.g., α is free in ϕ or if β is bound in ψ. It is clear that there would have been no advantage for us in adopting more liberal formation rules. Especially if we had allowed quantifiers to occur vacuously in wffs, the resulting duplication of ways of expressing a same thing would not have been compensated, as it is in other systems, by a greater simplicity of formulation at the syntactical level; on the contrary, the syntactical determination of exactly the wanted axioms would have been a great deal more elaborate.

Instances of axioms by *MA4* are:

$$(x)((y)Fy \supset Gx) \supset ((y)Fy \supset (x)Gx)$$
$$(y)[(x)((\exists z)Fzy \supset Gxy) \supset ((\exists z)Fzy \supset (u)Guy)]$$

Instances of axioms by *MA5* are:

$$(y)[(x)(Fx \supset Gx) \supset ((x)Fx \supset Gy)]$$
$$(y)(z)[(x)(Fx \supset Gzx) \supset ((x)Fx \supset Gzy)]$$
$$(y)[(x)(Fxy \supset Fxy) \supset ((x)Fxy \supset Fyy)]$$

Our only rule of immediate derivation is *modus ponens* (whereby, clearly, from sfs only sfs can be derived). Besides *proofs* (the term being defined as usual) the last lines of which are *theorems of* L, we explicitly also recognize *derivations from assumption forms*, in which sfs that are neither axioms, nor derivable from preceding sfs by *modus ponens* are admitted as lines under the name of *assumption forms*.

If $\phi, \psi_1, \psi_2, \ldots, \psi_n$ are sfs, we will say of ϕ that it is *derivable from* $\psi_1, \psi_2, \ldots, \psi_n$ if it is the last line of a derivation in which all assumption forms, if any, are among the sfs $\psi_1, \psi_2, \ldots, \psi_n$. (Hence, if every ψ_i is a theorem, ϕ is a theorem.)

Semantically, derivations from assumption forms are to be understood as argument forms, i.e. patterns for arguments, in which the assumption forms stand for in general non-logical premises, or *axioms in a theory*.[8]

§3. *Comparison of the System* **L** *with Some Other Systems.* In his *Mathematical Logic* [12], Quine dispenses with the use of free variables in the formulas that function as lines in derivations, while retaining the rule of *modus ponens* in its original form (rather than making it applicable to the closures of a conditional and its antecedent). It is therefore somewhat disappointing to discover that by his principles of quantification (Chapter 3), statements with existential import are still provable as theorems, especially since the source of such theorems is still not brought into the system in broad daylight. Of course, in that particular work, those principles are introduced only to be used in class theory, where there is no question of the domain being empty, and the theorems under discussion are then true (though still not valid in every domain as determined by letting the interpretation of the symbol 'ε' as a 2-place predicate vary). But it may be argued that this should have been one more reason for not letting the principles of quantification as such, which could find a more general application elsewhere, suffice for proving statements with existential import; for axioms asserting the existence of specific classes were to be introduced later in the work anyway.[9]

At any rate, the system syntactically schematized in Chapter 2 of [12], which has been of some inspiration for our own system **L**, interests us here only as it is there informally construed, i.e. as a system for quantificational logic in which primitive predicates other than 'ε' may be employed, rather than as it is later restricted in its vocabulary for a special purpose. Statements not valid in the empty domain are provable in that system because some of them are unperspicuously let into it as axioms in the company of others that are valid in every domain.

To fix our ideas and better to see the relationship between a system determined by Quine's principles for quantification in [12] and our own system L, let us assume the former to have the same vocabulary as ours (thereby its closed formulas being statement forms rather than statements).

The axioms introduced by Quine's Principle ***101** fall into four classes, being differentiated by the presence or absence in their various parts of free occurrences of the variable designated as α in the statement of the principle. If we delete the vacuous occurrences of quantifiers in those axioms they become respectively (using 'closure' and 'formula' in Quine's sense):

1. The closures of formulas $(\phi \supset \psi) \supset (\phi \supset \psi)$.
2. The closures of formulas $(\alpha)(\phi \supset \psi) \supset (\phi \supset (\alpha)\psi)$ wherein α is free in ψ but not in ϕ.

3. The closures of formulas $(\alpha)(\phi \supset \psi) \supset ((\alpha)\phi \supset (\alpha)\psi)$ wherein α is free both in ϕ and ψ.
4. The closures of formulas $(\alpha)(\phi \supset \psi) \supset ((\alpha)\phi \supset \psi)$ wherein α is free in ϕ but not in ψ.

Some of the formulas in group 4 are not valid in the empty domain; e.g. '$(x)(Fx \supset (\exists y)Fy) \supset ((x)Fx \supset (\exists y)Fy)$' is not.

The formulas in group 4 above are provable in Quine's theory of quantification with the help of axioms by ***103**, some of which, e.g. '$(x)(\exists y)Fy \supset (\exists y)Fy$',[10] are not valid in the empty domain, at least if we construe all universal statements as true in the empty domain. It is of course possible, though hardly elegant in a calculus intended to be open to interpretations in any domain, to construe any formula with vacuous quantification as semantically equivalent to the result of deleting its vacuous quantifier occurrences. With the latter construction, all axioms by ***103** are valid in every domain, but then among those axioms by ***101** which by such construction are synonymous with the formulas in group 4 above some are not valid in the empty domain.

A calculus in which only formulas valid in every domain are provable as theorems (an *inclusive* calculus, as Quine refers to such in [14], referring to the customary 1st order calculi as *exclusive*) and in which variables do not occur free in formulas appearing as lines in derivations, is obtained from Quine's theory of quantification in [12] by modifying its ***103** so as to read (omitting the quasi-quotations):

> If α is free in ϕ, and if ϕ' is like ϕ except for containing free occurrences of α' wherever ϕ contains free occurrences of α, then $\vdash (\alpha)\phi \supset \phi'$.

The resulting system was proposed by Quine in [14] as a simplification of an inclusive system proposed earlier by Hailperin [7], which was based on the theory of quantification in the first edition of *Mathematical Logic*.

Both in the theory of quantification of [12] and in the inclusive system of [14], the closures (in Quine's sense of the word) of formulas $(\alpha)(\beta)\phi \supset (\beta)(\alpha)\phi$ are proved as theorems by a stratagem discovered by Berry [2], which makes use of vacuous quantification. The procedure is somewhat artificial, and in both systems it can be dispensed with if we slightly modify them. To this end, in both systems, one should understand by 'closure' what was meant by that term in the first edition of *Mathematical Logic* (i.e. what we mean by 'standard closure' in our System **L**), and modify ***101** (Principle (2) of [14]) to read:

***101'**. $\vdash (\alpha)(\phi \supset \psi) \supset \left((\alpha)\phi \supset (\beta)\frac{\beta}{\alpha}\psi\right)$

where $\frac{\beta}{\alpha}\psi$ is understood as like ψ except for exhibiting free occurrences of β wherever ψ exhibits free occurrences of α. In both the systems under consideration thus modified, one can prove '$\vdash (\alpha)(\beta)\phi \supset (\beta)(\alpha)\phi$', or another metatheorem to the same effect, substantially as (and indeed more simply than) we will in the System **L**. (Cf. *MT3* and *MT7* in §4.)

In the two systems just obtained by modifying those in [12] and in [14], vacuous quantification is no longer needed for use in Berry's procedure, and at least in the inclusive one nothing is gained by retaining that duplicate way of expressing what can be expressed without it: the number of principles need not be increased if we further modify the inclusive system to proscribe vacuous quantification. In fact, inspired by our previous analysis of the axioms by ***101** as falling into four groups when their vacuous quantifier occurrences are dropped, we obtain another inclusive system from that of [12] if we (a) adopt the same formation rules that we formulated for the System **L**, accordingly modifying our understanding of the principles that we retain, as referring to the standard closures of wffs of the form specified, (b) drop ***102**, which now is inconsistent with the definition of 'theorem', and (c) replace ***101** with the following two principles:

***101a.** If $(\alpha)(\phi \supset \psi) \supset (\phi \supset (\beta)\frac{\beta}{\alpha}\psi)$ is wf, then its stcl is a theorem.

***101b.** If $(\alpha)(\phi \supset \psi) \supset ((\alpha)\phi \supset (\alpha)\psi)$ is wf, then its stcl is a theorem.

Observe that in this new system ***103** is automatically restricted as in [14] by the new formation rules. Except for the different treatment of the standard closures of tautologies, the resulting system still has one more principle or metaaxiom than the System **L**, but its much greater simplicity of development may make it didactically preferable to the latter. Both as it stands, and as further modified by replacing ***100** with the first three metaaxioms of **L**—whereby all axioms, as distinct from metaaxioms, are made independent—this system is hereby proposed as yet another inclusive system, with distinct advantages of its own. The system **L** is obtained therefrom by replacing ***101b** with the more powerful *MA5*, and dropping ***103**.

Mention should be made here of other inclusive systems that have been proposed, which allow variables to occur free in lines in derivations within a syntax whereby from such formulas no closed formulas not valid in the empty domain can be derived. In order to keep our subject within bounds, we will not here discuss the merits of thus countenancing free variables in an inclusive system.

The system by Hailperin [7] referred to above was itself a simplification and improvement of an earlier system by Mostowski [11]. In the latter, formulas with free variables were admitted as lines in derivation and construed so as to be valid in the empty domain. The proof of formulas not valid in the empty domain was prevented by a suitable restriction on the use of *modus ponens*. An awkward feature of Mostowski's system was the provision whereby vacuous quantifier occurrences could be always eliminated, on the semantical ground that the latter should be regarded as always superfluous. Thereby the quantification of a false statement was construed as false for the empty domain. This necessitated the sacrifice of the principle of extensionality.[11]

In [8], Hintikka used an inclusive system countenancing free variables in the formulas appearing as lines in derivation, construing wffs with free variables as valid in the empty domain. The system has no axioms, but six rules of equivalence transformation, formulated in terms of mutual replaceability of wf parts.

In [17], Schneider proposes an inclusive system with identity in which all tautologies and all formulas patterned in accordance with one or the other of two axiom schemata are axioms. Further, the system has six rules of derivation. Formulas with free variables are admitted as lines in derivation and construed as valid in the empty domain.

In the inclusive systems considered so far which countenance free variables in the lines of derivations, wffs with free variables are theorems, and are regarded as valid in every domain if and only if their closures are theorems (and valid in every domain), though the authors differ somewhat in the semantical analysis of such wffs. (In the system of [8], a wff ϕ is a theorem, or demonstrable, if it is transformable by the primitive rules into $\phi \vee \sim \phi$.) By contrast Hintikka in [9] and Lambert in a paper [10] which is forthcoming in the *Notre Dame Journal of Formal Logic* at the time when this is written propose systems of first order logic with identity, countenancing free variables in lines of derivation, but in which not all formulas with free variables whose closures are theorems are themselves theorems. Thus in both systems '$Fx \vee \sim Fx$' and '$x = x$' are theorems, but '$Fx \supset (\exists y)Fy$' and '$(\exists y)y = x$' are not. The rationale for this distinction among formulas whose closures are theorems is that the free variables in them are construed as place markers for individual constants, or names, which need not have denotation. (Hintikka actually uses different symbols for such place markers and for bound variables.) The merits of such an approach, and more specifically the completeness of these systems, can be only determined in the context of a semantics for non-referential names fully stipulating which statements containing non-referential names are to be regarded as true and which as false. Though they discuss the use of non-referential names in ordinary language at some length, both authors fail to furnish their respective calculi with such a semantics. If 'Pegasus = Pegasus' is to be regarded as true by being construed as synonymous with some such statement as 'if there is such a thing as Pegasus, then it is identical with Pegasus', one may just as well regard 'Pegasus eats daisies $\supset (\exists x)x$ eats daisies' as true by analogously construing it as synonymous with 'if there is such a thing as Pegasus, then if it eats daisies, there is something that eats daisies'. And the same remark applies to statements of the form of their paradigm non-theorem '$(\exists y)y = x$'. But more of this in some later paper.

Hintikka's system in [9], which is clearly derived from that of [8], has no axioms, and only three rules of equivalence transformation for the part that does not include identity. But this economy of the primitive apparatus is to some extent only apparent. To begin with, Hintikka's first rule lumps together all pairs of tautologically equivalent wffs with the same free varables as consisting of two mutually replaceable wffs. For a fair comparison, as regards economy of the primitive apparatus, of a system containing such a redundantly strong rule with the System L, one should first modify the latter by replacing its first three metaaxioms with a single one asserting the standard closures of all tautologies to be axioms. Even after this change, the System L appears to have one more principle than that part of

Hintikka's which does not include identity, for it has three metaaxioms and one rule of derivation versus Hintikka's three rules. But Hintikka's rules, unlike *modus ponens*, serve only for equivalence transformations, and are hence, as such, not adequate for all derivations from premises. Implication, as distinct from equivalence, in a syntactical sense, is introduced by Hintikka by definition as follows: 'ϕ implies ψ' means 'ϕ is equivalent to $\phi \cdot \psi$'. But thereby, in effect, an additional rule of derivation is adjoined to those of equivalence transformation, authorizing the dropping of the left hand conjunct of a conjunction.

§4. *Some Fundamental Metatheorems of L.* We will refer to sfs that are axioms by virtue of *MA1*, *MA2*, *MA3*, *MA4*, or *MA5* respectively as *MA1 axioms, MA2 axioms*, etc. Analogously, with reference to metatheorems asserting sfs of a specific form to be theorems, we will speak of *MT1 theorems*, etc.

By definition, any sf is a truth-functional compound of one or more universal sfs. Thus viewed, sfs may be tautologies, or some of them may tautologically imply another. The prefixless among our *MA1*, *MA2* and *MA3* axioms are known to constitute with *modus ponens* a complete primitive basis for the truth-functional calculus. Hence we record as our first metatheorem:

MT1.

a. *Every tautological sf is a theorem.*
b. *If an sf is tautologically implied by sfs* $\chi_1, \chi_2, \ldots, \chi_n$, *then it is derivable from them.*

Definition. If ϕ *is an sf derivable from* $\chi_1, \chi_2, \ldots, \chi_n$, *by MT1b, we will say that it is t-derivable from them.*

We next record:

MT2. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi \supset \psi) \supset ((\beta_1)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\psi)$ *is an sf in which the formulas* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$, $(\beta_1)(\beta_2)\ldots(\beta_m)$ *are standard prefixes, and in which no* γ_i *is* α_j *for a* $j < i$, *then it is a theorem.*

Note, that, for the hypothesis in *MT2* to be satisfied, every variable free in ϕ must be free in ψ, i.e. each β_i must be α_j for some j. Note further that the hypothesis of *MT2* may be satisfied if, as a special case, γ_i is α_i for every i. The formulas said to be theorems by *MT2* are *MA4* axioms if $n = 1$ and $m = 0$, and a look at *MA4* and *MA5* will make it clear that they are theorems if $n = 1$ and $m = 1$. Before proving *MT2* for the general case let us note the following corollaries that follow from it immediately:

Cor1-MT2. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi \supset \psi)$ *is an sf such that* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\psi$ *is the stcl of* ψ *and* $(\beta_1)(\beta_2)\ldots(\beta_m)\phi$ *is the stcl of* ϕ, *then any sf*

$$(\beta_1)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\psi$$

in which no γ_i *is* α_i *for a* $j < i$ *is derivable from it.*

Cor2-MT2. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi \supset \psi)$ *is an sf such that* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\,\psi$ *is the stcl of* ψ, *then any sf* $(\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\,\psi$ *in which no* γ_i *is* α_j *for a* $j < i$ *is derivable from it and the stcl of* ϕ.

Proof of MT2. For $n = 0$, *MT2* theorems are *MT1* theorems. Hence to prove *MT2* by induction it suffices to establish that for any $k \geq 0$, if *MT2* is true for $n = k$, then it is true for $n = k + 1$. Let us therefore assume that *MT2*, and hence its corollaries hold for $n = k \geq 0$, and let us consider an sf as specified in *MT2* and in which:

Case 1. $\alpha_n = \alpha_{k+1}$ occurs in ϕ, i.e. is β_m. Then, if ζ is a variable alphabetically subsequent to any in ϕ or ψ:

(i) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)\left\{(\zeta)\left[(\alpha_{k+1})(\phi \supset \psi) \supset \left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right)\right] \supset \left[(\alpha_{k+1})(\phi \supset \psi) \supset (\zeta)\left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right)\right]\right\}$$

is an *MA4* axiom;

(ii) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\zeta)\left[(\alpha_{k+1})(\phi \supset \psi) \supset \left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right)\right]$$

is an *MA5* axiom;

(iii) $$(\alpha_2)(\alpha_2)\ldots(\alpha_k)\left[(\zeta)\left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right) \supset \left((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\,\psi\right)\right]$$

is an *MA4* axiom;

(iv) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)\left[(\alpha_{k+1})(\phi \supset \psi) \supset (\zeta)\left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right)\right]$$

is a theorem by (i), (ii), and *Cor2-MT2* for $n = k$;

(v) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset (\alpha_1)(\alpha_2)\ldots(\alpha_k)(\zeta)\left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\,\psi\right)$$

is a theorem by (iv) and *Cor1-MT2* for $n = k$.

(vi) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\zeta)\left((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\right) \supset (\alpha_1)(\alpha_2)\ldots(\alpha_k)\left((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\,\psi\right)$$

is a theorem by (iii) and *Cor1-MT2* for $n = k$.

(vii) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)\left((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\,\psi\right) \supset \left((\beta_1)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_k)(\gamma_{k+1})\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_k}{\alpha_k}\frac{\gamma_{k+1}}{\alpha_{k+1}}\,\psi\right)$$

is a theorem by *MT2* for $n = k$;

(viii) $$(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset \left((\beta_2)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_k)(\gamma_{k+1})\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_k}{\alpha_k}\frac{\gamma_{k+1}}{\alpha_{k+1}}\,\psi\right)$$

is t-derivable from the theorems referred to in (v), (vi), (vii).

Case 2. $\alpha_n = \alpha_{k+1}$ does not occur in ϕ. Then:

(i) $(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big[(\alpha_{k+1})(\phi \supset \psi) \supset \Big(\phi \supset (\gamma_{k+1}) \frac{\gamma_{k+1}}{\alpha_{k+1}} \psi\Big)\Big]$

is an *MA4* axiom;

(ii) $(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset (\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big(\phi \supset (\gamma_{k+1}) \frac{\gamma_{k+1}}{\alpha_{k+1}} \psi\Big)$

is a theorem by (i) and *Cor1-MT2* for $n = k$;

(iii) $(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big(\phi \supset (\gamma_{k+1}) \frac{\gamma_{k+1}}{\alpha_{k+1}} \psi\Big) \supset \Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_k)$

$(\gamma_{k+1}) \frac{\gamma_1}{\alpha_1} \frac{\gamma_2}{\alpha_2} \ldots \frac{\gamma_k}{\alpha_k} \frac{\gamma_{k+1}}{\alpha_{k+1}} \psi\Big)$

is a theorem by *MT2* for $n = k$;

(iv) $(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset \Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi \supset (\gamma_1)(\gamma_2)\ldots(\gamma_k)(\gamma_{k+1})$

$\frac{\gamma_1}{\alpha_1} \frac{\gamma_2}{\alpha_2} \ldots \frac{\gamma_k}{\alpha_k} \frac{\gamma_{k+1}}{\alpha_{k+1}} \psi\Big)$

is t-derivable from the sfs referred to in (ii) and (iii).

The result of changing the order of occurrence of the quantifiers in the prefix of an axiom is a theorem, i.e.:

MT3. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi$ *is an axiom, then every sf* $(\beta_1)(\beta_2)\ldots(\beta_n)\phi$ *is a theorem.*

Outline of proof. If the two formulas referred to in the metatheorem are sfs, clearly, every β_i is α_j for some j, and conversely. Let $(\zeta_1)(\zeta_2)\ldots(\zeta_n)$ be a standard prefix whose variables are alphabetically subsequent to all those of ϕ. Then, by virtue of *Cor2-MT2*, $(\zeta_1)(\zeta_2)\ldots(\zeta_n)\frac{\zeta_1}{\beta_1}\frac{\zeta_2}{\beta_2}\ldots\frac{\zeta_n}{\beta_n}(\phi \supset \phi)$ clearly is derivable from appropriate *MA1, MA2,* and *MA3* axioms, and according to *Cor1-MT2* $(\zeta_1)(\zeta_2)\ldots(\zeta_n)\frac{\zeta_1}{\beta_1}\frac{\zeta_2}{\beta_2}\ldots\frac{\zeta_n}{\beta_n}\phi \supset (\beta_1)(\beta_2)\ldots(\beta_n)\phi$ is derivable from it. If the hypothesis of the metatheorem is satisfied, the antecedent of the latter conditional obviously is an axiom, and its consequent then is a theorem derivable from it and its antecedent by *modus ponens*.

Note that the axioms are *MT3* theorems wherein $\alpha_i = \beta_i$ for every i. *MT3* theorems differing from *MA1*, *MA2*, *MA3*, *MA4*, or *MA5* axioms at most by the order of occurrence of the quantifiers in their prefixes will hereafter be referred to as *MT3-1*, *MT3-2*, *MT3-3*, *MT3-4*, or *MT3-5* theorems respectively.

MT4. If

$$(\delta_1)(\delta_2)\ldots(\delta_\ell)\Big[(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)\supset\Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi\supset(\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\psi\Big)\Big]$$

is an sf in which all variables in $(\beta_1)(\beta_2)\ldots(\beta_m)$ *occur in* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$ *in the same order, and in which no* γ_i *is* α_j *for a* $j<i$, *then it is a theorem.*[12]

As in the case of *MT2*, the following corollaries may be conveniently used without circularity in the proof of *MT4* itself:

Cor1-MT4. If $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)$ *and*

$$(\delta_1)(\delta_2)\ldots(\delta_\ell)\Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi\supset(\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\psi\Big)$$

are sfs such that all variables in $(\beta_1)(\beta_2)\ldots(\beta_m)$ *occur in* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$ *in the same order, and such that no* γ_i *is* α_j *for a* $j<i$, *then the second is derivable from the first.*

Cor2-MT4. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)$ *and* $(\beta_1)(\beta_2)\ldots(\beta_m)\phi$ *are sfs wherein all variables in* $(\beta_1)(\beta_2)\ldots(\beta_m)$ *occur in* $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$ *in the same order, then any sf* $(\gamma_1)(\gamma_2)\ldots(\gamma_n)\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_n}{\alpha_n}\psi$ *in which no* γ_i *is* α_j *for a* $j<i$ *is derivable from them.*

To see how *Cor1-MT4* follows from *MT4*, observe that for the case $\ell = 0$ it follows immediately from it, and that *Cor1-MT4* for $\ell = 0$ can be applied to the *MT4* theorems themselves. *Cor2-MT4* of course follows immediately from *Cor1-MT4* for $\ell = 0$.

Outline of the proof of MT4. The proof of *MT4* for $\ell = 0$ is the same as that of *MT2*, except that wherever in the latter reference is made to an axiom, in the proof of *MT4* for $\ell = 0$ the reference instead is to the corresponding *MT3* theorem, and wherever in the proof of *MT2* appeal is hypothetically made to the authority of *MT2* or its corollaries for $n = k$, in the proof of *MT4* for $\ell = 0$ the appeal instead is correspondingly to *MT4* for $\ell = 0$ and $n = k$, *Cor1-MT4* for $\ell = 0$ and $n = k$, or to *Cor2-MT4* for $n = k$.[13] With the proof of *MT4* for $\ell = 0$, also *Cor1-MT4* for $\ell = 0$ and *Cor2-MT4* are established. For $\ell > 0$, $n = m = 0$, any *MT4* theorem clearly is derivable from the appropriate *MT3-1*, *MT3-2*, and *MT3-3* theorems in virtue of *Cor2-MT4*. Assuming therefore *MT4* and *Cor1-MT4* to be established for $\ell = 0$ and for $n = 0 \leq \ell$, and *Cor2-MT4* hence also to be established, to complete the proof of *MT4* by induction it remains to show, on that basis, that, for any $k \geq 0$, if *MT4* holds for $n = k$, then it holds for $n = k + 1$. To this end, let us assume *MT4* to hold for $n = k$ and consider any sf such as is specified in *MT4* and in which:

Case 1. $\alpha_n = \alpha_{k+1}$ occurs in ϕ, i.e. is β_m. Then, if ζ is a variable foreign to ϕ and ψ:

(i) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big\{(\zeta)\Big[(\alpha_{k+1})(\phi \supset \psi) \supset \Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big)\Big] \supset$

$\Big[(\alpha_{k+1})(\phi \supset \psi) \supset (\zeta)\Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big)\Big]\Big\}$

is an *MT3-4* theorem;

(ii) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\zeta)\Big[(\alpha_{k+1})(\phi \supset \psi) \supset \Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big)\Big]$

is an *MT3-5* theorem;

(iii) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big[(\zeta)\Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big) \supset \Big((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\psi\Big)\Big]$

is an *MT3-4* theorem;

(iv) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big[(\alpha_{k+1})(\phi \supset \psi) \supset (\zeta)\Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big)\Big]$

is a theorem by (i), (ii), and *Cor2-MT4;*

(v) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset (\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)$

$(\zeta)\Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big)$

is a theorem by (iv) and *Cor1-MT4* for $\ell = 0$;

(vi) $(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\zeta)\Big((\beta_m)\phi \supset \frac{\zeta}{\alpha_{k+1}}\psi\Big) \supset$

$(\delta_1)(\delta_2)\ldots(\delta_\ell)(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\psi\Big)$

is a theorem by (iii) and *Cor1-MT4* for $\ell = 0$;

(vii) $(\delta_1)(\delta_2)\ldots(\delta_\ell)\Big[(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big((\beta_m)\phi \supset (\gamma_{k+1})\frac{\gamma_{k+1}}{\alpha_{k+1}}\psi\Big) \supset \Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi$

$\supset (\gamma_1)(\gamma_2)\ldots(\gamma_{k+1})\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_k}{\alpha_k}\frac{\gamma_{k+1}}{\alpha_{k+1}}\psi\Big)\Big]$

is a theorem by *MT4* for $n = k$;

(viii) $(\delta_1)(\delta_2)\ldots(\delta_\ell)\Big[(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\alpha_{k+1})(\phi \supset \psi) \supset \Big((\beta_1)(\beta_2)\ldots(\beta_m)\psi \supset$

$(\gamma_1)(\gamma_2)\ldots(\gamma_k)(\gamma_{k+1})\frac{\gamma_1}{\alpha_1}\frac{\gamma_2}{\alpha_2}\ldots\frac{\gamma_{k+1}}{\alpha_{k+1}}\psi\Big)\Big]$

is t-derivable from the theorems referred to in (v), (vi), and the one that, according to *Cor1-MT4* for $\ell = 0$, can be derived from that referred to in (vii).

Case 2. $\alpha_n = \alpha_{k+1}$ does not occur in ϕ. The proof that on the hypothesis that *MT4* holds for $n = k$ the sf is provable in this case parallels the corresponding part in the proof of *MT2* and that in the proof of *MT4* for $\ell = 0$ in the same way as it does for the case just considered.

We have already had occasion to call attention to the fact that the stcl of any wff $\psi \supset \psi$ is a theorem. More in general we have:

MT5. If ϕ is a tautological wff, then every closure of it is a theorem.

Outline of proof. From what is known in truth-functional theory, it can be easily shown that in a system **P** with **L**'s vocabulary and formation rules, admitting open wffs as lines in derivations, using **L**'s *MA1*, *MA2*, and *MA3* axioms minus their prefixes as axioms, and *modus ponens* as sole rule of derivation, there exists a proof of every tautological wff in which *modus ponens* is used only on wffs $\chi_1 \supset \chi_2$ and χ_1 such that every variable free in χ_1 is free in χ_2. Such a proof in **P** is transformed into a proof in **L** abbreviated by authority of *MT3* and *Cor2-MT4*, if we replace the last wff therein with any of its closures and each of the other wffs therein by an appropriate closure of it.

It may be noted that *MT1*, *MT3-1*, *MT3-2*, and *MT3-3* theorems are *MT5* theorems.

MT6. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi \supset (\beta_1)(\beta_2)\ldots(\beta_n)\frac{\beta_1}{\alpha_1}\frac{\beta_2}{\alpha_2}\ldots\frac{\beta_n}{\alpha_n}\phi$ is an sf, then it is a theorem.

Proof. Let $\delta_1, \delta_2, \ldots, \delta_n$ be n distinct variables foreign to ϕ and distinct from all the β_i. Then $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi \supset \phi)$ and $(\delta_1)(\delta_2)\ldots(\delta_n)\frac{\delta_1}{\alpha_1}\frac{\delta_2}{\alpha_2}\ldots\frac{\delta_n}{\alpha_n}(\phi \supset \phi)$ are *MT5* theorems, and the two sfs $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi \supset (\delta_1)(\delta_2)\ldots(\delta_n)\frac{\delta_1}{\alpha_1}\frac{\delta_2}{\alpha_2}\ldots\frac{\delta_n}{\alpha_n}\phi$ and $(\delta_1)(\delta_2)\ldots(\delta_n)\frac{\delta_1}{\alpha_1}\frac{\delta_2}{\alpha_2}\ldots\frac{\delta_n}{\alpha_n}\phi \supset (\beta_1)(\beta_2)\ldots(\beta_n)\frac{\beta_1}{\alpha_1}\frac{\beta_2}{\alpha_2}\ldots\frac{\beta_n}{\alpha_n}\phi$ are respectively derivable from them according to *MT1-4*. $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi \supset (\beta_1)(\beta_2)\ldots(\beta_n)\frac{\beta_1}{\alpha_1}\frac{\beta_2}{\alpha_2}\ldots\frac{\beta_n}{\alpha_n}\phi$ is *t*-derivable from the latter two conditionals.

MT7. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi \supset (\beta_1)(\beta_2)\ldots(\beta_n)\phi$ is an sf, then it is a theorem.

Outline of proof. We will prove *MT7* for the case in which the prefix of the consequent of the conditional specified is obtained by replacing a segment $(\alpha_j)(\alpha_{j+1})\ldots(\alpha_k)$ $(1 \leq j < k \leq n)$ of the prefix of its antecedent with $(\alpha_k)(\alpha_j)(\alpha_{j+1})\ldots(\alpha_{k-1})$ (in other words, by moving one of the quantifiers any number of places to the left, and leave all others in their original order). It is clear how, in the general case, the antecedent and the consequent of the conditional to be proved can be linked through a chain of conditionals each of which is a theorem by *MT7* for the special case just described. Let then, in a conditional such as is specified in the metatheorem, $(\beta_1)(\beta_2)\ldots(\beta_n)\phi$ be

$(\alpha_1)(\alpha_2)\ldots(\alpha_{j-1})(\alpha_k)(\alpha_j)(\alpha_{j+1})\ldots(\alpha_{k-1})(\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi$, where $1 \leq j < k \leq n$. Then if γ is foreign to ϕ:

(i) $(\alpha_1)(\alpha_2)\ldots(\alpha_{j-1})(\gamma)(\alpha_j)(\alpha_{j+1})\ldots(\alpha_{k-1})\Big[(\alpha_k)\Big((\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi \supset (\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi\Big) \supset \Big((\alpha_k)(\alpha_{k+1})\ldots(\alpha_n)\phi \supset (\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\frac{\gamma}{\alpha_k}\phi\Big)\Big]$

is an *MT3-5* theorem;

(ii) $(\alpha_1)(\alpha_2)\ldots(\alpha_k)\Big((\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi \supset (\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi\Big)$

is an *MT5* theorem;

(iii) $(\alpha_1)(\alpha_2)\ldots(\alpha_{j-1})(\gamma)(\alpha_j)(\alpha_{j+1})\ldots(\alpha_{k-1})\Big((\alpha_k)(\alpha_{k+1})\ldots(\alpha_n)\phi \supset (\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\frac{\gamma}{\alpha_k}\phi\Big)$

is a theorem by (i), (ii), and *Cor2-MT4.*

(iv) $(\alpha_1)(\alpha_2)\ldots(\alpha_n)\phi \supset (\alpha_1)(\alpha_2)\ldots(\alpha_{j-1})(\alpha_k)(\alpha_j)(\alpha_{j+1})\ldots(\alpha_{k-1})(\alpha_{k+1})(\alpha_{k+2})\ldots(\alpha_n)\phi$

is a theorem by (iii) and *Cor1-MT4.*

As an illustration, here is the proof of an *MT7* theorem for $n = 2$, as abbreviated by appeal to earlier metatheorems:

$$(z)(x)[(y)(Fxy \supset Fxy) \supset ((y)Fxy \supset Fxz)] \tag{1}$$

$$(x)(y)(Fxy \supset Fxy) \tag{2}$$

$$(z)(x)((y)Fxy \supset Fxz) \tag{3}$$

$$(x)(y)Fxy \supset (y)(x)Fxy \tag{4}$$

(1) is an *MT3-5* theorem. (2) is an *MT5* theorem. (3) is derivable from (1) and (2) by *Cor2-MT4*. (4) is derivable from (3) according to *Cor1-MT4*.

From *Cor2-MT4, MT6,* and *MT7* clearly follows:

MT8. If $\phi \supset \psi$ and ϕ are wffs in which every variable free in ϕ is free in ψ, then any closure of ψ is derivable from any two closures of $\phi \supset \psi$ and ϕ respectively. (*Cf*. *111, [12], p. 90)

MT9. If $(\alpha)\phi \supset \frac{\beta}{\alpha}\phi$ is a wff in which β is not bound in ϕ, then every closure of it is a theorem. (*Cf*. *103, [12], p. 88.)

Proof. If the hypothesis of the metatheorem is satisfied, any closure of $(\alpha)(\phi \supset \phi) \supset \Big((\alpha)\phi \supset \frac{\beta}{\alpha}\phi\Big)$ is an *MT3-5* theorem, and any closure of $(\alpha)(\phi \supset \phi)$ is an *MT5* theorem. Any closure of $\Big((\alpha)\phi \supset \frac{\beta}{\alpha}\phi\Big)$ is derivable from two sfs such as have been just mentioned by *MT8*.

MT10. If $(\alpha)\phi \supset (\exists\alpha)\phi$ is an open wff, then every closure of it is a theorem.

Proof. The wff in question being open, let β be free in it. Then:

(i) the closures of $\left((\alpha)\phi \supset \frac{\beta}{\alpha}\phi\right) \supset \left[\left(\frac{\beta}{\alpha}\phi \supset (\exists\alpha)\phi\right) \supset \left((\alpha)\phi \supset (\exists\alpha)\phi\right)\right]$ are MT5 theorems;

(ii) the closures of $(\alpha)\phi \supset \frac{\beta}{\alpha}\phi$ are *MT9* theorems;

(iii) the closures of $(\alpha)\sim\phi \supset \frac{\beta}{\alpha}\sim\phi$ are *MT9* theorems;

(iv) the closures of $\left((\alpha)\sim\phi \supset \frac{\beta}{\alpha}\sim\phi\right) \supset \left(\frac{\beta}{\alpha}\phi \supset (\exists\alpha)\phi\right)$ are *MT5* theorems;

(v) the closures of $\frac{\beta}{\alpha}\phi \supset (\exists\alpha)\phi$ are theorems by (iii), (iv) and *MT*8.

(vi) in virtue of *MT8*, the closures of $(\alpha)\phi \supset (\exists\alpha)\phi$ are each derivable from three sfs as specified in (i), (ii) and (v), respectively, and hence are theorems.

Observe that the last step made as an abbreviation by authority of *MT8* in the object-language proof indicated above would not be authorized by that metatheorem if β were not free in ϕ. Sfs $(\alpha)\phi \supset (\exists\alpha)\phi$ are not theorems in L.

The object-language proofs indicated in the proofs of the next three metatheorems, which metatheorems we will have occasion to use later, may serve as illustrations of the analogy between the proofs of familiar theorems in L and in traditional systems.

MT11.

a. If $(\alpha)(\chi_1 \cdot \chi_2 \cdot \ldots \cdot \chi_n) \supset \left((\alpha_1)\frac{\alpha_1}{\alpha}\chi_1 \cdot \ldots \cdot (\alpha_n)\frac{\alpha_n}{\alpha}\chi_n\right)$ *is an sf, then it is a theorem.*

b. If $\left((\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2 \cdot \ldots \cdot (\alpha_n)\chi_n\right) \supset (\alpha)\left(\frac{\alpha}{\alpha_1}\chi_1 \cdot \frac{\alpha}{\alpha_2}\chi_2 \cdot \ldots \cdot \frac{\alpha}{\alpha_n}\chi_n\right)$ *is an sf, then it is a theorem.*

We will prove *MT11a* and *MT11b* for $n = 2$. Their generalization presents no difficulty.

Proof MT11a for $n = 2$. If $(\alpha)(\chi_1 \cdot \chi_2) \supset \left((\alpha_1)\frac{\alpha_1}{\alpha}\chi_1 \cdot (\alpha_2)\frac{\alpha_2}{\alpha}\chi_2\right)$ is an sf (observe that then α must occur both in χ_1 and χ_2), then:

(i) $(\alpha)[(\chi_1 \cdot \chi_2) \supset \chi_1]$ is an *MT5* theorem;

(ii) $(\alpha)[(\chi_1 \cdot \chi_2) \supset \chi_2]$ is an *MT5* theorem;

(iii) $(\alpha)(\chi_1 \cdot \chi_2) \supset (\alpha_1)\frac{\alpha_1}{\alpha}\chi_1$ is a theorem by (i) and *Cor2-MT4;*

(iv) $(\alpha)(\chi_1 \cdot \chi_2) \supset (\alpha_2)\frac{\alpha_2}{\alpha}\chi_2$ is a theorem by (ii) and *Cor2-MT4;*

(v) $(\alpha)(\chi_1 \cdot \chi_2) \supset \left((\alpha_1)\frac{\alpha_1}{\alpha}\,\chi_1 \cdot (\alpha_2)\frac{\alpha_2}{\alpha}\,\chi_2\right)$ is t-derivable from the theorems referred to in (iii) and (iv).

Proof of MT11b for n = 2. If $\left((\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2\right) \supset (\alpha)\left(\frac{\alpha}{\alpha_1}\chi_1 \cdot \frac{\alpha}{\alpha_2}\chi_2\right)$ is an sf and β is foreign to it:

(i) $(\beta)\left((\alpha_1)\chi_1 \supset \frac{\beta}{\alpha_1}\,\chi_1\right)$ is an *MT9* theorem;

(ii) $(\beta)\left((\alpha_2)\chi_2 \supset \frac{\beta}{\alpha_2}\chi_2\right)$ is an *MT9* theorem;

(iii) $(\beta)\left[\left((\alpha_1)\chi_1 \supset \frac{\beta}{\alpha_1}\chi_1\right) \supset \left\{\left((\alpha_2)\chi_2 \supset \frac{\beta}{\alpha_2}\chi_2\right) \supset \left[\left((\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2\right) \supset \left(\frac{\beta}{\alpha_1}\chi_1 \cdot \frac{\beta}{\alpha_2}\,\chi_2\right)\right]\right\}\right]$

is an *MT5* theorem;

(iv) $(\beta)\left[\left((\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2\right) \supset \left(\frac{\beta}{\alpha_1}\chi_1 \cdot \frac{\beta}{\alpha_2}\chi_2\right)\right] \supset \left[\left((\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2\right) \supset (\alpha)\left(\frac{\alpha}{\alpha_1}\chi_1 \cdot \frac{\alpha}{\alpha_2}\chi_2\right)\right]$

is an *MT3-4* theorem;

(v) The consequent of the last conditional, which is the sf to be proved, is derivable from the theorems referred to in (i), (ii), (iii) and (iv) by virtue of *MT8*.

Not all sfs $(\alpha)(\phi \cdot \psi) \supset ((\alpha)\phi \cdot \psi)$ are theorems in L; if they were, we could derive '$(x)\,(Fx \vee \sim Fx) \cdot (\exists y)\,(Gy \vee \sim Gy)$', which is not valid in the empty domain, from '$(x)\,[(Fx \vee \sim Fx) \cdot (\exists y)(Gy \vee \sim Gy)]$', which is valid in every domain and a theorem. However, the closures of open wffs $(\alpha)(\phi \cdot \psi) \supset ((\alpha)\phi \cdot \psi)$ are provable in L (as are those of all open wffs that are theorems in traditional systems). Moreover the converse conditionals $((\alpha)\phi \cdot \psi) \supset (\alpha)(\phi \cdot \psi)$ without free variables, as well as the closures of open wffs of that form are theorems in L, i.e., we have:

MT12. If $((\alpha)\phi \cdot \psi) \supset (\alpha)(\phi \cdot \psi)$ *is wf, then every closure of it is a theorem.*

Proof. If $((\alpha)\phi \cdot \psi) \supset (\alpha)(\phi \cdot \psi)$ is wf and β is foreign to it, then:

(i) the closures of $(\beta)\left((\alpha)\phi \supset \frac{\beta}{\alpha}\phi\right)$ are *MT9* theorems;

(ii) the closures of $(\beta)\left\{\left((\alpha)\phi \supset \frac{\beta}{\alpha}\phi\right) \supset \left[\left((\alpha)\phi \cdot \psi\right) \supset \left(\frac{\beta}{\alpha}\phi \cdot \psi\right)\right]\right\}$ are MT5 theorems;

(iii) the closures of $(\beta)\left[\left((\alpha)\phi \cdot \psi\right) \supset \left(\frac{\beta}{\alpha}\phi \cdot \psi\right)\right]$ are theorems by (i), (ii) and *MT8*;

(iv) the closures of $((\alpha)\phi \cdot \psi) \supset (\alpha)(\phi \cdot \psi)$ are theorems by (iii) and *Cor1-MT4*.

MT13. If $(\phi \supset (\alpha)\psi) \supset (\alpha)(\phi \supset \psi)$ *is wf, then every closure of it is a theorem.*

Proof. If the formula specified is wf, and β is foreign to it:

(i) the closures of $(\beta)\left\{\left((\alpha)\psi \supset \frac{\beta}{\alpha}\psi\right) \supset \left[(\phi \supset (\alpha)\psi) \supset \left(\phi \supset \frac{\beta}{\alpha}\psi\right)\right]\right\}$ are *MT5* theorems;

(ii) the closures of $(\beta)\left((\alpha)\psi \supset \frac{\beta}{\alpha}\psi\right)$ are *MT9* theorems;

(iii) the closures of $(\beta)\left[(\phi \supset (\alpha)\psi) \supset \left(\phi \supset \frac{\beta}{\alpha}\psi\right)\right] \supset \left[(\phi \supset (\alpha)\psi) \supset (\alpha)(\phi \supset \psi)\right]$ are *MT3-4* theorems;

(iv) in virtue of *MT8*, every closure of $(\phi \supset (\alpha)\psi) \supset (\alpha)(\phi \supset \psi)$ is derivable from three sfs as specified in (i), (ii) and (iii) respectively.

Finally, in this section, we record the metatheorem of deduction, whose proof is the same as in traditional systems.

MT14. If ψ *is an sf derivable from sfs* $\phi_1, \phi_2, \ldots, \phi_n$, *then* $\phi_n \supset \psi$ *is an sf derivable from* $\phi_1, \phi_2, \ldots, \phi_{n-1}$, *and* $\phi_1 \supset (\phi_2 \supset (\ldots \supset (\phi_n \supset \psi)) \ldots)$ *is a theorem.*

§5. *Derivations From Existence Assumption Forms.* Probably the most commonly assumed non-logical premises are those to the effect that there are objects. Such assumptions are made as a matter of course in argumentation outside specific axiomatic theories, and explicitly in the latter, at least when they are not fully formalized. In geometry, e.g., it is assumed that there are points (if 'to be a point' is taken as one of the primitive predicates), as when it is axiomatically asserted that there are at least three points. In other cases, a non-empty domain *D* is postulated, as e.g. in group theory or in Boolean algebra. When the latter is done, 'to be in *D*' is really taken as a primitive predicate, often explicitly used in every statement, or at least in the axioms, when the theory is not formalized; but in formalized theories, by way of abbreviation, often the non-empty domain *D* is taken as the *universe of discourse*, i.e. is assumed to constitute the whole universe, and then its non-emptiness has traditionally be implicitly assumed in the logic used rather than explicitly postulated.[14]

The system L has no theorems which become statements with existential import if the pss therein are given an interpretation as predicates, and in its applications any premise to the effect that the universe is not empty, or that there are objects of a specific kind must be explicitly stated. We will therefore consider derivations from assumption forms standing for such premises. The most general of such premises, to the effect that there are objects of some kind or other in the universe, can be expressed by statements of the form of sfs $(\exists\alpha)(\chi \vee \sim \chi)$, as the sf '$(\exists x)(Fx \vee \sim Fx)$'. We will examine derivations from such assumption forms next.

MT15. If $(\exists\alpha)(\chi \vee \sim\chi)$ *is an sf, then the closures of any wff* $(\beta)\phi \supset (\exists\beta)\phi$ *are derivable from it.*

Proof. If $(\beta)\phi \supset (\exists\beta)\phi$ is an open wff, then its closures are *MT10* theorems. If $(\exists\alpha)(\chi \vee \sim\chi)$ and $(\beta)\phi \supset (\exists\beta)\phi$ are sfs, let γ be foreign to both. Then:

(i) $(\gamma)\left[\frac{\gamma}{\beta}(\phi\cdot\sim\phi) \supset \frac{\gamma}{\alpha}\sim(\chi \vee \sim\chi)\right]$ is an *MT5* theorem;

(ii) $(\gamma)\frac{\gamma}{\beta}(\phi\cdot\sim\phi) \supset (\alpha)\sim(\chi \vee \sim\chi)$ is a theorem by (i) and *Cor1-MT4;*

(iii) $((\beta)\phi\cdot(\beta)\sim\phi) \supset (\gamma)\frac{\gamma}{\beta}(\phi\cdot\sim\phi)$ is an *MT11* theorem;

(iv) $((\beta)\phi\cdot(\beta)\sim\phi) \supset (\alpha)\sim(\chi \vee \sim\chi)$ is *t*-derivable from the theorems referred to in (ii) and (iii);

(v) $(\exists\alpha)(\chi \vee \sim\chi) \supset ((\beta)\phi \supset (\exists\beta)\phi)$ is *t*-derivable from the theorem referred to in (iv);

(vi) the antecedent of the conditional referred to in (v) is our assumption form, and its consequent is the sf to be derived from it; hence the latter is derivable from the former.

By virtue of *MT15,* obviously, if $(\beta)\phi$ is any theorem, then $(\exists\beta)\phi$ is derivable from any sf $(\exists\alpha)(\chi \vee \sim\chi)$. Thus, e.g., since every sf $(\beta)(\chi \vee \sim\chi)$ is an *MT5* theorem, any sf $(\exists\beta)(\chi \vee \sim\chi)$, as '$(\exists x)(Gx \vee \sim Gx)$', is derivable from any other, as '$(\exists x)(Fx \vee \sim Fx)$'.

MT16. If $(\beta)(\phi \supset \psi) \supset ((\beta)\phi \supset \psi)$ *is wf, then every closure of it is derivable from any sf* $(\exists\alpha)(\chi \vee \sim\chi)$.[15]

Proof. If the hypothesis is satisfied:

(i) the closures of $((\beta)\phi\cdot\sim\psi) \supset (\beta)(\phi\cdot\sim\psi)$ are *MT12* theorems;

(ii) the closures of $(\beta)(\phi\cdot\sim\psi) \supset (\exists\beta)(\phi\cdot\sim\psi)$ are derivable from any sf $(\exists\alpha)(\chi \vee \sim\chi)$ according to *MT15;*

(iii) the closures $((\beta)\phi\cdot\sim\psi) \supset (\exists\beta)(\phi\cdot\sim\psi)$ are each derivable from two closures of conditionals as specified in (i) and (ii) respectively and the appropriate *MT3-3* theorem in virtue of *MT8*, and hence are derivable from any sf $(\exists\alpha)(\chi \vee \sim\chi)$;

(iv) the closures of $(\beta)\sim\sim(\phi \supset \sim\sim\psi) \supset ((\beta)\phi \supset \psi)$ are each derivable from the closure of a conditional as specified in (iii) and the appropriate *MT5* theorem in virtue of *MT8*, and hence from any sf $(\exists\alpha)(\chi \vee \sim\chi)$;

(v) the closures of $(\beta)(\phi \supset \psi) \supset (\beta)\sim\sim(\phi \supset \sim\sim\psi)$ are each derivable from the appropriate *MT5* theorem by *Cor2-MT4;*

(vi) the closures of $(\beta)(\phi \supset \psi) \supset ((\beta)\phi \supset \psi)$ are each derivable from two closures of conditionals as specified in (iv) and (v) respectively and the appropriate *MT3-3* theorem in virtue of MT8 and hence are each derivable from any sf $(\exists\alpha)(\chi \vee \sim\chi)$.

MT17. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)\supset\Big((\beta_1)(\beta_2)\ldots(\beta_m)\phi\supset(\delta_1)(\delta_2)\ldots(\delta_\ell)\frac{\delta_1}{\gamma_1}\frac{\delta_2}{\gamma_2}\ldots\frac{\delta_\ell}{\gamma_\ell}\psi\Big)$ *is an sf, then it is derivable from any sf* $(\exists\alpha)(\chi\vee\sim\chi)$.

Cor1-MT17. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)$, $(\beta_1)(\beta_2)\ldots(\beta_m)\phi\supset(\delta_1)(\delta_2)\ldots(\delta_\ell)\frac{\delta_1}{\gamma_1}\frac{\delta_2}{\gamma_2}\ldots\frac{\delta_\ell}{\gamma_\ell}\psi$ *are sfs, then the second is derivable from the first and any sf* $(\exists\alpha)(\chi\vee\sim\chi)$.

Cor2-MT17. If $(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\phi\supset\psi)$, $(\beta_1)(\beta_2)\ldots(\beta_m)\phi$, and $(\delta_1)(\delta_2)\ldots(\delta_\ell)\frac{\delta_1}{\gamma_1}\frac{\delta_2}{\gamma_2}\ldots\frac{\delta_\ell}{\gamma_\ell}\psi$ *are sfs, then the last one is derivable from the first two and any sf* $(\exists\alpha)(\chi\vee\sim\chi)$. (*Cf*. *111 in [12], p. 90.)

Outline of proof. In the sf specified, clearly the β_i and the γ_i together make up the α_i. Without loss of generality we may assume that the β_i occur in $(\beta_1)(\beta_2)\ldots(\beta_m)$ in the same order as they do in $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$, that further the order of occurrence of the δ_i in $(\delta_1)(\delta_2)\ldots(\delta_\ell)$ is such that the γ_i occur in $\frac{\gamma_1}{\delta_1}\frac{\gamma_2}{\delta_2}\ldots\frac{\gamma_\ell}{\delta_\ell}(\delta_1)(\delta_2)\ldots(\delta_\ell)$ in the same order as they do in $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$, and finally that no δ_i is α_j for a $j<i$ (since if the sf in question does not satisfy these conditions, clearly it is t-derivable from one that does and the appropriate *MT6* and/or *MT7* theorems). The proof then proceeds as that of *MT4* for $\ell = 0$, except that a third case must be added to the two considered there (following the pattern of the proof of *MT2*) in the second part of the proof by induction. For now $\alpha_n = \alpha_{k+1}$ may occur in ϕ (i.e. be β_m) and not in ψ. In this case: $(\alpha_1)(\alpha_2)\ldots(\alpha_k)[(\alpha_{k+1})(\phi\supset\psi)\supset((\beta_m)\phi\supset\psi)]$ is derivable from our assumption form $(\exists\alpha)(\chi\vee\sim\chi)$ according to MT16; in virtue of Cor1-MT17 for the case $n=k$, the desired sf clearly is derivable from the assumption form and from the sf just said to be derivable from the same.

It is now fairly clear, and in the next section we will establish rigorously, that, used in conjunction with a general assumption having only the import that the universe of discourse is not empty, L has the force of traditional systems of 1st order logic. But L has that force also with more specific assumptions, such as that there are objects of such-and-such a specific kind or that there are at least so-and-so many objects of such-and-such a kind, or that there is a unique object of such-and-such a kind, etc. We have in fact:

MT18. If $(\exists\beta)(\chi\vee\sim\chi)$ *is an sf, then it is derivable from any sf* $(\exists\alpha)\phi$.

Proof. If $(\exists\beta)(\chi\vee\sim\chi)$ is an sf, and if γ is foreign to ϕ and χ:

(i) $(\gamma)\Big[\frac{\gamma}{\beta}\sim(\chi\vee\sim\chi)\supset\frac{\gamma}{\alpha}\sim\phi\Big]$ is an *MT5* theorem;

(ii) $(\gamma)\frac{\gamma}{\beta}\sim(\chi\vee\sim\chi)\supset(\alpha)\sim\phi$ is a theorem by (i) and *Cor1-MT4;*

(iii) $(\beta)\sim(\chi\vee\sim\chi)\supset(\alpha)\sim\phi$ is t-derivable from the last theorem and the appropriate *MT6* theorem;

(iv) $(\exists\alpha)\phi \supset (\exists\beta)(\chi \vee \sim\chi)$ is t-derivable from the theorem referred to in (iii), and hence is a theorem;

(v) by (iv) clearly $(\exists\beta)(\chi \vee \sim\chi)$ is derivable from $(\exists\alpha)\phi$.

§6. *The Completeness* of **L**. For the sfs of **L**, we adopt the traditional concept of validity in a non-empty domain: an interpretation of a k-place ps λ in a domain D is a class of ordered pairs $\langle\langle\lambda,\theta\rangle,t\rangle$, where t is the truth-value truth, and θ, in the ordered pairs $\langle\lambda,\theta\rangle$, is an ordered k-tuple of objects in D (thereby there being only one interpretation, the null-class, of any ps in the empty domain); under an interpretation in a non-empty domain D of each of its pss, a wff is in accordance with familiar rules associated with exactly one of the two truth value t and f[16] (becomes a true or a false statement) for every assignment of a value in D to each of its free variables; an sf is valid in a non-empty domain D if and only if it becomes a true statement under every interpretation of its pss in D.

Definition. An sf valid in every non-empty domain is said to be existentially valid.

MT19. If ψ is an existentially valid sf, then it is derivable from any sf $(\exists\alpha)\phi$.

Proof. Consider a system **S**, with the same vocabulary and formation rules as **L**, but admitting open wffs as lines in derivations, its rules of immediate derivation being *modus ponens* and universal generalization, and its axioms being:

(1) all wffs $(\sim\phi \supset \phi) \supset \phi$;

(2) all wffs $\phi \supset (\sim\phi \supset \psi)$;

(3) all wffs $(\phi \supset \psi) \supset [(\psi \supset \chi) \supset (\phi \supset \chi)]$;

(4) all wffs $(\alpha)\phi \supset \frac{\beta}{\alpha}\phi$ in which β is not bound in ϕ;

(5) all wffs $(\alpha)(\phi \supset \psi) \supset (\phi \supset (\alpha)\psi)$.

Such a system is known to be complete in the traditional sense: every wff whose closures are existentially valid is a theorem in it. To prove our *MT19*, it suffices therefore to show that to every proof in **S** of an sf ψ, there corresponds in **L** a derivation of ψ from any assumption form $(\exists\alpha)\phi$. Let $\chi_1, \chi_2, \ldots, \chi_n$, in this order, be the wffs which, written in column form, constitute a proof of an sf ψ in **S**, and consider a sequence of sfs $(\exists\beta)(\chi \vee \sim\chi)$, χ_1', χ_2', ..., χ_n' where, for each i, χ_i' is a closure of χ_i and hence $\chi_n' = \chi_n = \psi$. I say that in the latter sequence, every χ_i' is either a theorem of **L** or derivable in **L** from preceding sfs, i.e. that each χ_i', and hence in particular $\chi_n' = \psi$, is derivable in **L** from $(\exists\beta)(\chi \vee \sim\chi)$, and hence, in virtue of *MT18*, from any sf $(\exists\alpha)\phi$. In fact, by hypothesis, for every i, χ_i is one of the following:

(a) an axiom of **S** in group (1), (2), or (3);

(b) an axiom of **S** in group (4);

(c) an axiom of **S** in group (5);

(d) a wff derivable from some χ_j, χ_k, where $j, k < i$ by *modus ponens*;

(e) a wff derivable from some χ_j where $j < i$ by universal generalization.

In case (a), χ_i' is an *MT3-1*, *MT3-2*, or *MT3-3* theorem; in case (b), χ_i' is an *MT9* theorem; in case (c), χ_i' is an *MT3-4* theorem; in case (d), χ_i' is derivable in **L** from χ_j', χ_k' and $(\exists\beta)(\chi \vee \sim\chi)$ by *Cor2-MT17*; in case (e), χ_i' is derivable from χ_j' and the appropriate MT7 theorem by *modus ponens*.

Thus it is clear that if we adjoin any existentially valid sf $(\exists\alpha)\phi$, e.g., '$(\exists x)(Fx \vee \sim Fx)$', to the axioms of **L** as an additional axiom, the resulting system is equivalent to traditional systems of 1st-order logic, in the sense that its theorems are exactly the closures of the theorems of those systems (except for differences in vocabulary or conditions of well-formedness). Such a system may be referred to as a system of *existence logic*.

Let us now give more precision to the concept of validity in an empty domain, and hence of validity in any given domain (empty or not). In accordance with the logician's understanding of the universal quantifier, 'valid in the empty domain' may be sharply defined as follows, with reference to the familiar truth-functional interpretation of '$\sim$' and '$\supset$'.

Definition. An sf is said to be valid in the empty domain if and only if under the truth-functional interpretation of '$\sim$' and '$\supset$', it acquires the truth-value t when all the universal sfs that are its elementary truth-functional components are assigned the truth-value t.

It should be noted, however, that it is quite possible, and desirable, to define 'valid in a domain D' for sfs in a general way which is consistent with the above definition of 'valid in the empty domain.' In fact, the rules referred to in the first paragraph of this section, may be formulated so that by them every universal sf becomes a true statement under the interpretation of its pss in the empty domain. To see this, it suffices to note that when, e.g., it is said that $(\alpha)\lambda\alpha$, where λ is a *1*-place ps, is true under a given interpretation of λ in a domain D if and only if $\lambda\alpha$ is under that interpretation associated with t for every assignment of a value in D to α, $(\alpha)\lambda\alpha$ becomes a true statement under the interpretation of λ in the empty domain, for in that domain there are no assignments of values to α (no objects it may be assigned to name).

Definition. An sf is said to be valid if and only if it is valid in every domain (i.e. it is both existentially valid and valid in the empty domain).

As a conclusion to this article, we will next show that the system **L** is complete, meaning thereby that the following metatheorem holds in it:

MT20. Every valid sf is a theorem.

To prove *MT20*, we will prove:

A. If every valid universal sf is a theorem, then every valid sf is a theorem.

B. Every valid universal sf is a theorem.

Proof of A. As can be determined by inspection, every axiom is valid, and *modus ponens* preserves validity. Hence if two sfs are derivable from each other, then both or neither are valid. As is well known from standard truth-functional theory, for every sf there is a tautologically equivalent *normal alternation,* i.e. a tautologically equivalent sf which may be abbreviated as an alternation of one or more conjunctions of one or more universal sfs and negations of such. Thus, since tautologically equivalent sfs are derivable from each other according to *MT1,* if every valid normal alternation is a theorem, every valid sf is a theorem. To be valid in the empty domain, and hence to be valid, a normal alternation, as written in the abbreviated form whence it gets its name, clearly must have at least one alternant in which no conjunct is a negation. Let ϕ be any normal alternation satisfying the latter condition, $(\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2 \cdot \ldots \cdot (\alpha_n)\chi_n$ one of the alternants in one of its characteristic abbreviations, and ψ the alternation of the other alternants in that abbreviation. Then, if β is foreign to ψ, and to each of the χ_i:

(i) $\sim\psi \supset (\alpha_1)\chi_1 \cdot (\alpha_2)\chi_2 \cdot \ldots \cdot (\alpha_n)\chi_n$ and ϕ are each *t*-derivable from the other;

(ii) $\sim\psi \supset (\beta)\left(\frac{\beta}{\alpha_1}\chi_1 \cdot \frac{\beta}{\alpha_2}\chi_2 \cdot \ldots \cdot \frac{\beta}{\alpha_n}\chi_n\right)$ and the conditional referred to in (i) are each t-derivable from the other and the appropriate MT11 theorems;

(iii) $(\beta)\left[\sim\psi \supset \left(\frac{\beta}{\alpha_1}\chi_1 \cdot \frac{\beta}{\alpha_2}\chi_2 \cdot \ldots \cdot \frac{\beta}{\alpha_n}\chi_n\right)\right]$ and the conditional referred to in (ii) are each derivable from the other in virtue of *MT4* and *MT13.*

Thus, for every valid normal alternation there is a valid universal sf from which it is derivable. Hence, if all valid universal sfs are provable, so are all valid normal alternations and hence all valid sfs.

Proof of B. Since every valid sf is existentially valid, to establish *B* it suffices to show that all existentially valid universal sfs (all of which are valid, since all universal sfs are valid in the empty domain) are theorems. Let $(\alpha)\phi$ be any existentially valid universal sf, and $(\exists\beta)(\chi \vee \sim\chi)$ an sf such that α is foreign to χ and β to ϕ. Then:

(i) $(\exists\beta)(\chi \vee \sim\chi) \supset (\alpha)\phi$ is a theorem by *MT19* and *MT14*;

(ii) $(\alpha)[(\exists\beta)(\chi \vee \sim\chi) \supset \phi]$ is derivable from the theorem referred to in (i) and the appropriate *MT13* theorem;

(iii) $(\alpha)[\sim\phi \supset (\beta)\sim(\chi \vee \sim\chi)]$ is derivable from the theorem referred to in (ii) and the appropriate *MT5* theorem in virtue of *MT8*;

(iv) $(\alpha)(\beta)[\sim\phi \supset \sim(\chi \vee \sim\chi)]$ is derivable from the theorem referred to in (iii) and the appropriate *MT13* theorem in virtue of *MT8*;

(v) $(\alpha)(\beta)[(\chi \vee \sim\chi) \supset \phi]$ is derivable from the theorem referred to in (iv) and the appropriate *MT5* theorem in virtue of *MT8.*

(vi) $(\alpha)\left\{(\beta)\left[(\chi \vee \sim\chi) \supset \phi\right] \supset \frac{\alpha}{\beta}\left[(\chi \vee \sim\chi) \supset \phi\right]\right\}$ is an *MT9* theorem;

(vii) $(\alpha)(\beta)[(\chi \vee \sim\chi) \supset \phi] \supset (\alpha)\frac{\alpha}{\beta}[(\chi \vee \sim\chi) \supset \phi]$ is a theorem by (vi) and *Cor2-MT4;*

(viii) $(\alpha)\left[\frac{\alpha}{\beta}(\chi \vee \sim\chi) \supset \phi\right]$ is a theorem by (v) and (vii);

(ix) $(\alpha)\frac{\alpha}{\beta}(\chi \vee \sim\chi)$ is an *MT5* theorem;

(x) $(\alpha)\phi$ is a theorem by (viii), (ix) and *MT8*.

NOTES

1. I use 'generalized upon' as defined in [4], p. 172.

2. This technique, of course, has been taken over by logic from mathematics, to the development of which it has been essential, and where it has been used at least since classical antiquity (notably in geometry). But it is not as simple, or "natural" as it may seem to those today who have been conditioned to it since childhood; it took mankind a long time to "see" it, or at least to gain self-assurance in its use (*vide* Egyptian and Babylonian mathematics).

   In mathematics, so-called variables are used in several ways. The procedure described above corresponds to that employed in the proofs of identities in algebra, as when '$(x+y)(x-y)=x^2-y^2$' is proved by deriving it from '$(x+y)(x-y)=x^2+xy-xy+y^2$' and other algebraic propositions. What is meant by '$(x+y)(x-y)=x^2-y^2$' as a statement of course is that $(x)(y)[(x+y)(x-y)=x^2-y^2]$. But historically, the name 'variable' was introduced in mathematics in connection with equations, rather than identities, in two or more variables, because of the association these equations had with laws governing physical changes. In solving or otherwise transforming equations, we may be doing one of two different things. If the equation is directly applied to a particular physical problem, then no variables at all are involved, only so-called unknowns. Thus if we know that a certain stick A of 6 inches measures as much as another stick B of 4 inches plus 3 times a stick C of unknown length, to find the length of the latter in inches we may write down the equation '$3x+4=6$' and solve it for x obtaining '$x=\frac{2}{3}$'. Throughout this operation, 'x' has a definite denotation, it means 'the number expressing the length of stick C in inches'. The equation as originally written and the final expression giving the solution are statements, and, in solving the equation, the latter is deduced from the former. But when the solution of the same equation is carried out as a problem in "pure" mathematics '$3x+4=6$' and '$x=\frac{2}{3}$' are not statements, for what can they possibly mean? What is really being proved is '$(x)(3x+4=6 \equiv x=\frac{2}{3})$'. Or when the equation '$4x^2-9x+2=0$' is being solved, what is being proved in pure mathematics is the statement '$(x)[4x^2-9x+2=0 \equiv (x=2 \vee x=\frac{1}{4})]$'. And when '$3x+9y=6$', is being solved for x, what is being proved in pure mathematics is '$(x)(y)(3x+9y=6 \equiv x=2-3y)$'. In all these cases, the transformation of the equation amounts to an argument form, in which the variables stand for the names or definite descriptions of numbers (the fact that in the first example the variable turns out to have only one possible value is incidental); then, by what logicians refer to as the theorem of deduction, a biconditional between the first and the last line in the derivation is established as being derivable from general propositions of arithmetic as applied to unspecific numbers; finally on the basis of the existence of the latter argument form, it is regarded as legitimate to generalize upon the variables so as to get a statement. Altogether, this is anything but a logically simple procedure, which logic ought to, and can justify in terms of forms of arguments in which every line is a statement deduced from preceding ones. The logical analysis of the process given above need not of course precisely correspond to the actual train of thought of the person carrying it out.

3. An open sentence of course is not a sentence as the term is often used in logic in place of 'statement'. As to the reasons for preferring 'open sentence' to the older term 'matrix', which goes back to *Principia Mathematica,* see [13], p. 90.

4. For the purpose of the above informal discussion, with no definite system to refer to, we may say that, roughly speaking, an expression that is construed either as a statement or as a statement form is valid in a given domain if all statements of its form are true on the assumption that the universe coincides with that domain. The concept will be sharpened with reference to the expressions of a particular system in §6.

5. Cf [3], p. 140. The modifications to one of the systems in [16] which Carnap there suggests in order to meet the above objections do not appear to yield a complete system, i.e. one in which all well-formed formulas valid in every empty or non-empty domain are theorems, as far as can be judged from the sketchy description of it that he gives. Moreover, the author envisages no other method of proving statements with existential import in a theory using such new logical system than by the use of constants (or names as the author calls them, since he uses 'constant' in a different sense). But it must be possible in a theory to postulate that there are objects in the universe of discourse, and draw consequences therefrom, without being able to name any.

6. [15], p. 79-80. Contrary to what might appear from the above quotation, Rosenbloom's system in the context of which those remarks occur cannot be used or adapted for the purpose of our reconstruction as we have outlined it in our introductory remarks. Besides the fact that it is an axiomatic system *about* propositions, individuals and properties rather than a logistic system, the assumption that there are individuals (which in a reformulation of the system in logistic terms would correspond to the introduction of constants, implicitly assumed to have denotation, in the vocabulary) cannot be dropped therefrom without making it impossible to prove the truth also of some statements or "propositions" valid in every domain (including the empty one).

7. In this paper, we will not discuss the controversial question of whether it is ever necessary or legitimate to construe the symbols occupying the place of predicates in statement forms as variables subject to quantification.

8. Though it finds its applications in particular theories, the concept of derivation from assumption forms definitely belongs to pure logic, where 'derivation from assumption form, must be defined syntactically. In the application of the concept to particular theories, the possible derivations from a number of fixed assumptions (the axioms of the theory) are studied systematically. Also, in a particular theory, our pss may be interpreted as definite predicates, (or at least one or more such interpretations of them may be kept in mind as the important), and/or be replaced by special symbols (such as '<'). But the dividing line between pure and applied logic is not a sharp one, as when logic investigates such very general theories as the so-called logic of identity, or, even more clearly so (since identity may be regarded as a logical concept), when it investigates what we will later call *'the logic of existence.'*

Though truth-functional and quantificational logic in *Principia Mathematica* form what is there referred to as the theory of deduction, the tendency among logicians whose lines of thought have remained close to those in that work has often been to stress the use of variables, subject to substitution and quantification, rather than that of place markers, and consequently to neglect the consideration of derivations from assumption forms in pure logic. This has occasionally created the impression that the axiomatic-logistic approach in logic, as opposed to the so-called natural deduction approach, does not furnish a theory of deduction at all. (See, e.g., [1], p. 74). Actually, the import of both methods is the same. Both methods serve to establish a theory of validity for both statements and arguments. An axiomatic-logistic system can indeed be easily reworded so as to appear to stress deduction the way natural deduction systems do: what are usually introduced as axioms may instead be introduced only in the context of a rule of derivation saying that such-and-such well-formed formulas may at any point be introduced as lines in derivations from premises. The natural deduction approach serves certain didactical purposes since it rapidly codifies procedures well established in mathematics, and is useful for certain particular investigations related to intuitionism, but as a general method in pure logic, the axiomatic-logistic approach seems to this writer to be the more basic one.

9. In [14] Quine presents his reasons for disregarding the empty domain in 1st order calculi.

10. As the author later specifies the atomic formulas for his system, in place of 'F' such an axiom would exhibit such a predicate as, say '$(z)z\varepsilon$'. However, in the system of [12] as a whole, such axioms are non-independent. As will appear from our §5, the formulas in group 4 above, or the axioms by ***103** themselves if vacuous quantification is admitted, which are not valid in the empty domain, are derivable from any existential assumption in a system whose theorems are valid in every domain. Of course, already as determined in Chapter 2, Quine's system contains non-independent axioms on account of ***100**.

11. Cf. the comments on [11] in [7] and [14].

12. The *MT4* theorems in which $n=1, m=0$ are of course the *MT3-4* theorems. The well-formed by our rules among the formulas that we classified under 3 in our discussion of Quine's ***101** are *MT4* theorems in which $n=m=1$.

13. If we did not mind having non-independent axioms, we could have adopted all *MT3* theorems as axioms and saved ourselves the separate proof of *MT2* and *MT3*. Cf. [6].

14. Sometimes existence assumptions are introduced implicitly by the use of individual constants which, by the logic used, are implicitly assumed to have denotation. This is not the place to discuss the logical status of individual constants or proper names in every day discourse or natural science. But it can be said that at least in abstract axiomatic theories, or in such that deal only with abstract entities, as sets or numbers, since any reference to naming by pointing is there out of the question, the only objects ever named are those that can be identified by the fact that they uniquely satisfy certain conditions, as is asserted axiomatically or proved of them, and hence the constants can be dispensed with in the primitive vocabulary (though they may be conveniently used in abbreviations as definite descriptions). Thus in some formulations of Boolean algebra, for instance, '*0*' is primitively introduced as a constant to name an individual in the domain which is then axiomatically asserted to satisfy such-and-such conditions, and later proved to be the only individual in the domain satisfying those conditions; clearly, it is more proper axiomatically to assert that there exists an object in the domain satisfying such-and-such conditions (one axiom may have to be used in place of several used before); then, after having proved the uniqueness of that object, a special symbol, as '0', may be used as a definite description to abbreviate statements which contain a clause asserting its existence and uniqueness.

15. Thereby, the well-formed by our rules among the formulas that we classified under 4 in our discussion of *101 in §3 are derivable in L from any sf$(\exists\alpha)(\chi \vee \sim \chi)$.

16. See, e.g. the rules b_1-f in [4], p. 175 and their reinterpretation by the author *ibid*, p. 227-28. For the purposes of our calculus those rules have of course to be reworded, as by translating 'functional variable' with 'ps,' etc.

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