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A REMARK ON CONTINUOUS SELECTORS

HENRYK FAST

1. For a set S and a class $\mathfrak{F}$ of its subsets a function $f: \mathfrak{F} \to S$ with the property that $f(X) \epsilon X$ for $X \epsilon \mathfrak{F}$ is called a *selector* on $\mathfrak{F}$. If S and $\mathfrak{F}$ are topological spaces we can talk about continuous selectors.

We shall restrict ourselves in our considerations to metric spaces only. Let S be a metric space with the metric d; $\mathfrak{A}$ and $\mathfrak{X}$, $\mathfrak{A} \subset \mathfrak{X}$ be the classes of all arcs and of all continua of S. Metrize $\mathfrak{X}$ using the Hausdorff metric $d_H(X, Y) = \inf\{r: C_r(X) \supset Y, C_r(Y) \supset X\}$, $X, Y \subset \mathfrak{X}$.[1] Consider also a stronger metric $d_H^*(X, Y) = \max [d_H(X, Y), d_H(\partial X, \partial Y)]$ on $\mathfrak{X}$.

About the space S, make the following assumptions: S in compact, connected, **L.C.** (locally connected) everywhere and has the property (*): A domain G in S is not decomposable by an (compact) arc in G having one point only on its boundary. Notice, that a closed ball for example in the spaces E_n and S_n (Euclidean and spherical n-dimensional) satisfies these conditions.[2]

In the following statement the density is meant in d_H metric, the (*)-continuity and the (*)-density in d_H^* metric; $\mathfrak{X}^*$ denotes the subclass of $\mathfrak{X}$ consisting of all continua with empty interior.

Proposition: *Any selector on* $\mathfrak{A}$ (*and the more on* $\mathfrak{X}$) *is* (*)*-discontinuous on a set which is dense in* $\mathfrak{X}$ *and* (*)*-dense in* $\mathfrak{X}^*$.

This proposition answers the question on the possibility of existence in a local sense of a continuous selector on $\mathfrak{A}$, giving an negative answer to the question asked by Prof. Morton Brown and communicated here by Prof. K. Kuratowski in his recent lecture. The question concerned the existence of a continuous selector on a class of arcs (in E_n, for instance) with Hausdorff metric.

1. We denote $C_r(E) = \{x: \text{dist}\,(x,E) < r\}$, $E \subset S$.
2. The property (*) may be verified here for instance by the application of the "sweep away" theorem [c.f. [1], th.4 and 4a, p. 350] (as an arc is continuously retractible in itself to a point).

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2. *Proof*: The proof will be conducted in two steps (a) and (b):
(a) $\mathfrak{A}$ is dense in $\mathfrak{X}$. Let $\delta > 0$ and let C the component of $C_\delta(X)$ containing X. C being relatively compact has a δ-net $N = \{x_0, x_1, \ldots x_m\}$: $C_\delta(N) \supset C$. Due to our supposition about S, C being connected and **L.C.** is arcwise connected. There exists in C an arc A passing through all the points $x_0, x_1, \ldots x_m$ in this order. This may be established inductively: supposing that such an arc A_k exists for the points $x_0, x_1 \ldots x_k$, $k = 0, 1, \ldots, m - 1$ we can extend it to an A_{k+1} due to the property (*) of S, (applied to $G = C - A_k$; for $k = 0$ to $C - \{x_0\}$, which is a domain). From the inclusions $X \subset C \subset C_\delta(N) \subset C_\delta(A)$ and $A \subset C \subset C_\delta(X)$ follows $d_H(X, A) \leq \delta$ i.e. (a).

(b) Set of points of (*)-discontinuity of f is dense in $\mathfrak{A}$. Take an arbitrary $X \subset \mathfrak{A}$. We have $x = f(X) \epsilon X$. Let $y \epsilon X$, $y \neq x$ and let an ϵ is chosen such that $0 < \epsilon < d(x, y)$.

Let $0 < \delta < \epsilon$ such that

$$d(f(Y), x) < \epsilon \quad \text{as} \quad Y \epsilon \mathfrak{A},\ d_H(Y, X) < \delta. \tag{2.1}$$

By (a) there exists an arc A such that $d_H(X, A) < \delta$ passing through a net $N = \{x_0, x_1, \ldots x_{2p}\}$ where the additional requirements $y = x_p$ and $C_\delta(N_i) \supset C$, $i = 1, 2$, with $N_1 = \{x_0, x_1, \ldots x_p\}$, $N_2 = \{x_{p+1}, \ldots x_{2p}\}$ clearly may be added.

Let A be represented by the homeomorphism $g: [0, 1] \to S$ of an interval into S and let $g(\frac{1}{2}) = y$. We have the following relations:

$$g([t, 1]) \subset A = g([0, 1]) \subset C_\delta(X) \text{ and } X \subset C_\delta(N_2) \subset C_\delta(g[t, 1])$$

for $0 \leq t \leq \frac{1}{2}$, which yields

$$d_H(g([t, 1]), X) < \delta, \quad 0 \leq t \leq \tfrac{1}{2} \tag{2.2}$$

whence, putting $z(t) = f(g([t, 1])$ we have by the continuity condition (2.1)

$$d(z(t), x) < \epsilon, \quad 0 \leq t \leq \tfrac{1}{2}. \tag{2.3}$$

From the definition of selector, $z(t) \epsilon g([t, 1])$. By (2.3) and the choice of ϵ, $z(t) \neq g(\frac{1}{2}) = y$, hence, $z(t) \epsilon g([0, \frac{1}{2}))$ or $z(t) \epsilon g((\frac{1}{2}, 1])$. $t \to g([t, 1])$ is a continuous mapping from $[0, 1]$ into the space $\mathfrak{A}$. Were f (*)-continuous in the δ-neighbourhood of X, so would be by (2.2) $z(t)$ for $0 \leq t \leq 1$ and, assuming for instance that $z(0) \epsilon g([0, \frac{1}{2}])$, we would have $z(t) \epsilon g([t, \frac{1}{2}])$ for $0 \leq t \leq 1$ and this is impossible, since this implies $\lim_{t \to 1/2} z(t) = y$, which contradicts (2.3).

Since (*)-density of $\mathfrak{A}$ in $\mathfrak{X}^*$ follows obviously from (a), this ends the proof.

REFERENCE

[1] C. Kuratowski, *Topologie* II, Monografie Matematyczne 21, Warszawa 1952.

University of Notre Dame
Notre Dame, Indiana