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AXIOMATIZATION OF PROPOSITIONAL CALCULUS WITH SHEFFER FUNCTORS[1]

THOMAS W. SCHARLE

The two binary functors of the (two-valued) propositional calculus known as Sheffer functors have the property that all other functors are definable by each of them.[2] Hence, one is able to base a functionally complete propositional calculus on either of these functors, and it is of interest to axiomatize such systems, which is the main purpose of this paper. We will employ the parenthesis-free notation of Łukasiewicz, in which the Sheffer functors are given by

$Dpq = NKpq$, i.e., not both p and q
$Spq = KNpNq$, i.e., neither p nor q

It is well known that this problem has been investigated before. From the first, it was seen that it would be easier to work with D rather than with S, especially because 1) the shortest tautologies expressible with the Sheffer functors are $DDppp$ and $SSSpppSSppp$ (indicating that tautologies in S are generally longer than ones in D), and 2) the rules of detachment can be made simpler for D: with D we are allowed such rules as

D1 $\dfrac{D\alpha D\beta\gamma \quad \alpha}{\gamma}$ **D2** $\dfrac{D\alpha D\beta\gamma \quad \alpha}{\beta}$ **D3** $\dfrac{D\alpha D\beta\beta \quad \alpha}{\beta}$

while with S, the simplest rules are of the sort

S1 $\dfrac{SSS\alpha\beta\gamma\delta \quad \alpha}{\gamma}$ **S2** $\dfrac{SSS\alpha\beta\gamma SS\alpha\beta\gamma \quad \alpha}{\gamma}$ **S3** $\dfrac{SSS\alpha\alpha\beta\gamma \quad \alpha}{\beta}$

For such reasons, all investigations have been for D axioms, using the rule **D1**.[3]

Following Nicod [6], the conventional rule of detachment for D is **D1**. Clearly, this is a stronger rule than, say, **D3**, for we are allowed more freedom in the first line. However, the only investigations carried out have

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been with the rule **D1**, and we continue the practice in this paper. For *S*, we take the corresponding rule **S1**, which again is a quite strong rule.

As mentioned above, the aim of this paper is to give axioms for the Sheffer functors. To establish the point that this is still necessary for *D*, we devote the first section to the history of *D* axioms, and it will be shown that no proof of completeness has been given for any of the famous axioms.

To fill the gap, in the second section we give a completeness proof. Finally, in the third section we investigate the properties of *S* which show the way to giving an axiom set for *S*, and a demonstration of the completeness of the propositional calculus based on this axiom set.

J. Nicod intended to simplify the propositional calculus of *Principia Mathematica* by giving a system with only one undefined functor and a single axiom, viz.:

DDpDqrDDtDttDDsqDDpsDps

and the rules of substitution and detachment. Other axioms, closely related, have been given by Łukasiewicz [5]:

DDpDqrDDsDssDDsqDDpsDps
DDpDqrDDpDrpDDsqDDpsDps

and Wajsberg [12]:

DDpDqrDDDsrDDpsDpsDpDpq

from which they derived the axiom of Nicod.

From his basis, Nicod sketched deductions of what purported to be the axioms of *Principia Mathematica:*

***1.2**	*CAppp*
***1.3**	*CqApq*
***1.4**	*CApqAqp*
***1.5**	*CApAqrAqApr*
***1.6**	*CCqrCApqApr*

At points in Nicod's presentation, it is not clear what the steps are to be, and a more detailed exposition was to be desired, as was given by J. Jørgensen in [3], pp. 150-158. Unfortunately, there was an error made in the derivation of ***2.08**, one of the early steps, the theorem *DpDpp*. This error invalidated the subsequent steps[4]. Corrections were given by Łukasiewicz [5], and shortly thereafter (and independently) by W. V. Quine [9].

It was then noticed by B. A. Bernstein in [1] that in the demonstrations of the axioms of *Principia Mathematica,* it was assumed that they were in fact the formulas ***1.2** - ***1.6**. This could not be the case, for the primitive functors of this system are not implication and alternation but alternation and negation, with implication being a defined functor. This would ordinarily be of no difficulty, except that it was also noted by Bernstein that the definitions of alternation, negation, and implication by *D* given by Nicod were incompatible with the definitions given in *Principia Mathematica*. To make this point clearer, let us arrange these definitions in a table:

DEFINIENDUM	DEFINIENS AS GIVEN BY		
	NICOD	*PRINCIPIA MATHEMATICA*	BERNSTEIN
Np	*Dpp*	(undefined)	*Dpp*
Apq	*DDppDqq*	(undefined)	*DDppDqq*
Cpq	*DpDqq*	*ANpq*	*DDDppDppDqq*
Kpq	*DDpqDpq*	*NANpNq*	*DDDDppDppDDqq-DqqDDDppDppDDqqDqq*

Bernstein went on to show that the equivalences between Nicod's and his definitions obtained in Nicod's system, hence this point was cleared up. This seems to be the end of any contributions to the problem.

But even now, there is another oversight, for to show the completeness of an axiom set in one set of primitives, it is not sufficient to derive another axiom set with a different set of primitives. To be specific, all that can be claimed for any of the axioms for D is that one can prove that there is a model for a complete propositional calculus contained in the consequences of the axiom.[5]

It might be possible to finish a proof of completeness simply by filling in a few steps, but it seems worthwhile to give a unified presentation, with all deductions explicitly presented. Here we intend to do that, with the first axiom of Łukasiewicz. Since this axiom is only a substitution in Nicod's axiom, the argument given below is equally valid for Nicod's axiom, hence also for any other system containing this axiom (and the rules). First of all, we must obtain certain consequences of this axiom.[6] In so doing, we use a certain short-hand notation by way of indicating what substitutions and detachments are performed: $i.\ p/\alpha,\ q/\beta,\ r/\gamma, \ldots$ refers to the formula resulting from formula number i when for p we substitute α, for q, β, for r, γ, etc. Given our rule of detachment, if a formula of the form $D\ i\ D\ j - k$ is a consequence of the axiom, and also formula number i, then the formula numbered k is a consequence (here, the j serves no purpose as far as the detachment is concerned).

1. *DDpDqrDDsDssDDsqDDpsDps*
   1, p/DpDqr, q/DsDss, r/DDsqDDpsDps, s/t = D1 D6 - 2
2. *DDtDsDssDDDpDqrtDDpDqrt*
   1, p/DtDsDss, q/DDpDqrt, r/DDpDqrt, s/w = D2 D6, t/w - 3
3. *DDwDDpDqrtDDDtDsDsswDDtDsDssw*
   3, w/DpDqr, p/s, q/s, r/s, t/DDsqDDpsDps = D1 D4 - 4
4. *DDDDsqDDpsDpsDtDttDpDqr*
   2, t/DDDstDDtsDtsDtDtt, s/t = D4, q/t, p/t, r/t D5-5
5. *DDpDqrDDDstDDtsDtsDtDtt*
   5, p/DpDqr, q/DDstDDtsDts, r/DtDtt = D5 D7 - 6
6. *DtDtt*
   1, p/t, q/t, r/t = D6 D6, t/s - 7.

7. *DDstDDtsDts*
   7, *s/t*, *t/Dtt* = *D6 D8* - 8
8. *DDttt*
   7, *s/Dst*, *t/DDtsDts* = *D7 D9* - 9
9. *DDDtsDtsDst*
   1, *p/DDtsDts*, *q/s*, *r/t*, *s/p* = *D9 D6*, *t/p* - 10
10. *DDpsDDDDtsDtspDDDtsDtsp*
   10, *p/Dpp*, *s/p*, *t/s* = *D8*, *t/p D11* - 11
11. *DDDspDspDpp*
   7, *s/DDspDsp*, *t/Dpp* = *D11 D12* - 12
12. *DDppDDspDsp*
   1, *p/Dpp*, *q/Dsp*, *r/Dsp*, *s/r* = *D12 D6*, *t/r* - 13
13. *DDrDspDDDpprDDppr*
   13, *r/DpDqr*, *s/DtDtt*, *p/DDsqDDpsDps* = *D1 D14* - 14
14. *DDDDsqDDpsDpsDDsqDDpsDpsDpDqr*
   7, *s/DDsqDDpsDps*, *t/DpDqr* = *D14 D15* - 15
15. *DDpDqrDDDsqDDpsDpsDDsqDDpsDps*
   15, *p/Dst*, *q/Dts*, *r/Dts*, *s/p* = *D7 D16* - 16
16. *DDpDtsDDDstpDDstp*
   7, *s/DpDts*, *t/DDDstpDDstp* = *D16 D17* - 17
17. *DDDDstpDDstpDpDts*
   16, *p/DDDstpDDstp*, *t/p*, *s/Dts* = *D17 D18* - 18
18. *DDDtspDDDstpDDstp*
   7, *s/DDtsp*, *t/DDDstpDDstp* = *D18 D19* - 19
19. *DDDDstpDDstpDDtsp*
   15, *p/DpDqr*, *q/DDsqDDpsDps*, *r/DDsqDDpsDps*, *s/t* =
   D15 D20 - 20
20. *DDtDDsqDDpsDpsDDDpDqrtDDpDqrt*
   20, *t/DDDqsDDpsDpsDDqsDDpsDps* = *D19*, *p/DDpsDps*, *s/q*, *t/s*
   D21 - 21
21. *DDpDqrDDDqsDDpsDpsDDqsDDpsDps*
   1, *p/DDtsDts*, *q/s*, *r/t*, *s/p* = *D9 D6*, *t/p* - 22
22. *DDpsDDDDtsDtspDDDtsDtsp*
   22, *s/Dpp* = *D6*, *t/p D23* - 23
23. *DDDtDppDtDppp*
   7, *s/DDtDppDtDpp*, *t/p* = *D23 D24* - 24
24. *DpDDtDppDtDpp*
   1, *p/DDtsp*, *q/DDstp*, *r/DDstp*, *s/q* = *D18 D6*, *t/q* - 25
25. *DDqDDstpDDDDtspqDDDtspq*
   25, *q/DpDts*, *s/Dst*, *t/p*, *p/DDstp* = *D16 D26* - 26
26. *DDDpDstDDstpDpDts*
   7, *s/DDpDstDDstp*, *t/DpDts* = *D26 D27* - 27
27. *DDpDtsDDpDstDDstp*
   21, *p/DpDts*, *q/DpDst*, *r/DDstp*, *s/q* = *D27 D28* - 28
28. *DDDpDstqDDDpDtsqDDpDtsq*
   28, *p/Dts*, *q/DDDstDpqDDstDpq*, *s/p*, *t/q* = *D18*, *p/Dpq D29* - 29
29. *DDDtsDqpDDDstDpqDDstDpq*
   1, *p/DDtsDqp*, *q/DDstDpq*, *r/DDstDpq*, *s/r* = *D29 D6*, *t/r* - 30

30. *DDrDDstDpqDDDDtsDqprDDDtsDqpr*
    30, p/t, q/Dpp, r/p, t/Dpp = D24 D31 - 31
31. *DDDDppsDDppsp*
    7, s/DDDppsDDpps, t/p = D31 D32 - 32
32. *DpDDDppsDDpps*

We will work with certain substitutions in these laws:

8 = 8, t/Dpp = DDDppDppDpp*
21 = 21, r/q, s/Drr = DDpDqqDDDqDrrDDpDrrDpDrrDDqDrrDDpDrrDpDrr*
24 = 24, t/Dpp = DpDDDppDppDDppDpp*
32 = 32, s/Dqq = DpDDDppDqqDDppDqq*

Note that we can use the definitions of *C* and *N*, rewriting:

8#. CCNppp
21#. CCpqCCqrCpr
32#. CpCNpq

These last three laws are the well-known axiom set of Łukasiewicz [4] for a complete propositional calculus. Moreover, it is obvious that the rules of substitution and detachment for *C* and *N* hold in our system. So we know that if any formula is a tautology expressed in *C* and *N*, it follows from these three, and *a fortiori* from the single axiom. Also, we know that *CpNq* (i.e., *DpDDqqDqq*) has the same truth-values as *Dpq*, so that to every tautology expressed in *D*, there is a corresponding tautology, expressed in *C* and *N*, which is a consequence of the axiom. This can be expressed more formally as:

Lemma: If α is a tautology (with D as the only functor), and β results from α by replacing every part of α of the form $D\gamma\delta$ by $D\gamma DD\delta\delta D\delta\delta$, then β is a consequence of axiom 1.

If we had the equivalence of *Dpq* and *DpDDqqDqq* and a rule for the substitutability of equivalents, completeness would follow. So we define an equivalence relation and show the required propositions.

Definition: $\alpha \sim \beta$ if and only if $D\alpha D\beta\beta$ and $D\beta D\alpha\alpha$ are consequences of the axiom.

From the definition we obtain immediately:

$\alpha \sim \alpha$	(by *6*)
If $\alpha \sim \beta$ *and* $\beta \sim \gamma$ *then* $\alpha \sim \gamma$	(by *21**)
If $\alpha \sim \beta$ *then* $\beta \sim \alpha$	(by definition)
$\alpha \sim DD\alpha\alpha D\alpha\alpha$	(by *8** and *24**)

Lemma: If $\alpha \sim \beta$, and γ results from δ by replacing some occurrence of α by β, then $\gamma \sim \delta$.

Proof: (By induction)

If $\delta = \alpha$, then $\gamma = \beta$, and $\gamma \sim \delta$.

Suppose $\gamma_0 \sim \delta_0$, we show that $D\gamma_0\varepsilon \sim D\delta_0\varepsilon$ and $D\varepsilon\gamma_0 \sim D\varepsilon\delta_0$:

a) $D\gamma_0 D\delta_0\delta_0$
b) $D\delta_0 D\gamma_0\gamma_0$
c) $DD\gamma_0\varepsilon DD\delta_0\varepsilon D\delta_0\varepsilon$ (by *21** and *a*)
d) $DD\delta_0\varepsilon DD\gamma_0\varepsilon D\gamma_0\varepsilon$ (by *21** and *b*)
e) $DDD\varepsilon\delta_0 D\varepsilon\delta_0 D\gamma_0\varepsilon$ (by *30* and *c*)
f) $DD\varepsilon\gamma_0 DD\varepsilon\delta_0 D\varepsilon\delta_0$ (by *16* and *e*)
g) $DDD\varepsilon\gamma_0 D\varepsilon\gamma_0 D\delta_0\varepsilon$ (by *30* and *d*)
h) $DD\varepsilon\delta_0 DD\varepsilon\gamma_0 D\varepsilon\gamma_0$ (by *16* and *g*)
i) $D\gamma_0\varepsilon \sim D\delta_0\varepsilon$ (by *c* and *d*)
j) $D\varepsilon\gamma_0 \sim D\varepsilon\delta_0$ (by *f* and *h*)

We can then show in an obvious way:

Lemma: If α is a consequence of the axiom, and β results from α by replacing all occurrences of the form $DD\gamma\gamma D\gamma\gamma$ by γ, then β is a consequence of the axiom.

Theorem: If α is a tautology, α follows from the axiom.

Having shown the completeness of an axiom set with D alone, we can construct an axiom set for the functor S alone. In this section, we will prove the theorem:

If α and β constitute a complete axiom set for D, with the rule for detachment

$$\frac{\begin{array}{c} D\xi D\eta\zeta \\ \xi \end{array}}{\zeta}$$

then

S1. $SSSSpSqSrsSpSppSSSStqStqStsSpSpp\alpha^*$
S2. $SSSSpqSpqSqp\beta^*$

constitute a complete axiom set for S, with the rule for detachment

$$\frac{\begin{array}{c} SSS\xi^*\eta^*\zeta^*\theta^* \\ \xi^* \end{array}}{\zeta^*}$$

where α^* and β^* are α and β, respectively, with every occurrence of "D" in them replaced by "S".

This theorem requires for its proof a few lemmas. In the discussion to follow, we will adopt these conventions:
1) Placing an asterisk after a formula indicates the replacing of every "D" in it by an "S".
2) The use of the rules of substitution and detachment are indicated in a manner similar to that for D above, viz.:

$$m.\ p/\alpha^*, \ldots = SSS\ n\ (\ ) - k - (\ )$$

means: in thesis *Sm*, for "p" substitute α^*, etc., obtaining a formula, from which detach thesis *Sn*, yielding formula *Sk* as a thesis.

We prove the completeness of $\{S1, S2\}$ by first drawing attention to the fact that every proposition in S, say $S\xi^*\eta^*$, is tautologous if and only if ξ and η are tautologies (in D). Then we can show by some straightforward deductions that if ξ and η follow from α and β, then $S\xi^*\eta^*$ follows from $S1$ and $S2$.

Lemma 1. *$S\xi^*\eta^*$ is a tautology if and only if ξ and η are tautologies.*

Proof: Since S and D are mutually dual, ξ is true precisely when ξ^* is false. But $S\xi^*\eta^*$ is true when ξ^* and η^* are both false, and conversely.

Lemma 2. *If γ is deduced from α and β, then there is such a δ^* that $S\delta^*\gamma^*$ is deducible from $S1$ and $S2$.*

Proof: By induction on the number of detachments from α and β to γ.
i) If $\gamma = \alpha$ or $\gamma = \beta$, then $S1$ or $S2$ satisfies the lemma.
ii) Suppose we deduce γ from $D\xi D\eta\gamma$ and ξ, where we have (by induction hypothesis) results from $S1$ and $S2$:

$S3$. $S\zeta^*S\xi^*S\eta^*\gamma^*$
$S4$. $S\theta^*\xi^*$

Then we may proceed as follows:

1. $p/\zeta^*,\ q/\xi^*,\ r/\eta^*,\ s/\gamma^* = SSS3\ (\ )\ -\ 5\ -\ (\ )$

$S5$. $SSSSt\xi^*St\xi^*St\gamma^*S\zeta^*S\zeta^*\zeta^*$
5. $t/\theta^* = SSS4\ (\ )\ -\ 6\ -\ (\ )$

$S6$. $S\theta^*\gamma^*$

The induction is completed.

Lemma 3. *If ε and ζ are tautologies, then $S\varepsilon^*\zeta^*$ is provable.*

Proof: Since ε and ζ are true, there are formulas γ and δ which are true, and such that

$S9$. $S\gamma^*\varepsilon^*$
$S10$. $S\delta^*\zeta^*$

are provable. Likewise,

$$\lambda = DDD\gamma\varepsilon D\gamma\varepsilon DDDD\delta\zeta D\delta\zeta D\varepsilon\zeta DpDpp$$

is true, whence for some η,

$S7$. $S\eta^*\lambda^*$

and detach:

2, $p/\eta^*,\ q/\lambda^* = SSS\ 7\ (\ )\ -8-\ (\ )$

$S8$. $S\lambda^*\eta^*$
8 $= SSS\ 9\ (\ )\ -\ 11\ -\ (\ )$

$S11$. $SSSS\delta^*\zeta^*S\delta^*\zeta^*S\varepsilon^*\zeta^*SpSpp$
11 = SSS 10 ( ) - 12 - ( )

$S12$. $S\varepsilon^*\zeta^*$

Theorem. If α and β are complete axioms for D, then S1 and S2 are complete axioms for S.

Proof: We may see from lemma 1 and inspection that *S1* and *S2* are true; and by lemmas 1 and 3, every true proposition follows from *S1* and *S2*.

By applying this theorem to the results for *D*, we arrive at the following two-axiom set:

SSSSpSqSrsSpSppSSSStqStqStsSpSppSpSpp
SSSSpqSpqSqpSSpSqrSSsSssSSsqSSpsSps

The question of the independence of the axioms is left open.

The theorem may be generalized for different rules of detachment, with essentially the same proof. Without providing all of the steps required, which for the most part are identical to those given above, we can point out that

DpDpp
DDpDqqDDDsqDDpsDpsDDsqDDpsDps

are an axiom set for *D* with the rule of detachment **D3**, hence, by this generalization

SSSSpSqSrsSpSqSrsSSSStqStqStsSStqStqStsSpSpp
SSSSpqSpqSqpSSpSqqSSSsqSSpsSpsSSsqSSpsSps

are an axiom set for *S* with the rule of detachment **S3**.

NOTES

1. The author would like to thank Prof. B. Sobociński, especially for pointing out the paper [5] of Łukasiewicz and for suggesting the problem of axiomatization of *S*.

2. C. S. Peirce discovered these two functors *ca.* 1880, but his discovery was not published until 1933 – see [7] par. 12-20 and 264-265, wherein it is shown that all binary functors are definable with either functor. The first published indication of this property was by E. Stamm [11], and later, more explicitly, by H. M. Sheffer [10]. E. Żyliński [13] then showed that this holds for no other binary functor. He also pointed out that no proof had been published that *all* functors (ternary, quaternary, etc.) were definable (Peirce claimed this to be true in [7] par. 265). For such a proof, we can only refer to a proof of functional completeness as given in, say, E. L. Post [8] or A. Church [2].

3. See J. Nicod [6], p. 40. We should qualify these remarks to the extent that some logicians, Sheffer for one, used *S* in the basis for a Boolean algebra, but since *S* and *D* are duals, a Boolean algebra based on one is isomorphic to a Boolean algebra based on the other, i.e., the two functors are indistinguishable.

4. The error was first detected by S. Leśniewski in the work of Nicod – see [5], *ad init.* Quine, in the work mentioned below, contended that the error was not Nicod's, and that the deductions given by Quine were the ones indicated by Nicod. For the record, we should mention that two other errors occur in the presentation of Jørgensen: 1) an improper substitution at step (98) in the deduction of ***1.5** (p. 155), and 2) unwarranted use of the rule: from *α* and *β* to infer *Kαβ*, in the deductions of the equivalences ***1.01**, ***4.13**, ***3.01**, which are respectively the definition of implication

by negation and alternation, the law of double negation, and the definition of conjunction by negation and alternation.

5. Suppose that we translate every tautology of C and N into a tautology of D by the definitions $Cpq = DpDqq$ and $Np = Dpp$. Then it can easily be checked that the resulting formulas are all verified by the matrix

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 \end{array}$$

where 1 is the designated value *truth*, while the tautology $DDpqDDqpDqp$ is not verified. Hence it is independent of any C-N axiom system.

6. The steps here shown are largely adapted from [5] and from [3].

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Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana