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A NOTE ON CERTAIN SET - THEORETICAL FORMULAS

BOLESŁAW SOBOCIŃSKI

In [1], p. 168, Sierpiński notices that the following formula

T *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\aleph_0 \leq \mathfrak{m}$, $\aleph_0 \leq \mathfrak{n}$ *and* $\aleph_0 + \mathfrak{m} = \aleph_0 + \mathfrak{n}$, *then* $\mathfrak{m} = \mathfrak{n}$

is provable without the aid of the axiom of choice in the field of the general set theory.

In this note it will be shown that the following generalization of **T**

$\mathbf{S}_1$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{q}$, *if* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are not finite,* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{q}$ *and* $\mathfrak{m} + \mathfrak{p} = \mathfrak{n} + \mathfrak{q}$, *then* $\mathfrak{p} = \mathfrak{q}$

is equivalent to the formula

$\mathbf{V}_1$ *For any cardinal number* $\mathfrak{m}$ *which is not finite,* $\mathfrak{m} = 2\mathfrak{m}$

and that, on the other hand, the following modification of $\mathbf{S}_1$

$\mathbf{S}_2$ *For cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{q}$, *if* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are not finite,* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{q}$ *and* $\mathfrak{n} + \mathfrak{p} = \mathfrak{m} + \mathfrak{q}$, *then* $\mathfrak{p} = \mathfrak{q}$

and the following formulas

$\mathbf{S}_3$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{q}$, *if* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are not finite,* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{q}$ *and* $\mathfrak{m}\mathfrak{p} = \mathfrak{n}\mathfrak{q}$, *then* $\mathfrak{p} = \mathfrak{q}$

and

$\mathbf{S}_4$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{q}$, *if* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are not finite,* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{q}$ *and* $\mathfrak{n}\mathfrak{p} = \mathfrak{m}\mathfrak{q}$, *then* $\mathfrak{p} = \mathfrak{q}$

and which are, clearly, analogous to $\mathbf{S}_1$ and $\mathbf{S}_2$ are such that each of them is equivalent to the axiom of choice.

Proof:

1. Since, *cf.* [3], p. 115, in the field of the general set theory $\mathbf{V}_1$ is equivalent to

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$\mathbf{V}_2$ *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{n}$ *is not finite and* $\mathfrak{m} < \mathfrak{n}$, *then* $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$,

$\mathbf{S}_1$ is an obvious consequence of $\mathbf{V}_1$. Now, let us assume $\mathbf{S}_1$ and that

(1) $\mathfrak{m}$ *is an arbitrary cardinal which is not finite,*

Clearly, we have the generally valid formulas:

(2) $2\mathfrak{m} + 3\mathfrak{m} = \mathfrak{m} + 4\mathfrak{m}$ and (3) $\mathfrak{m} \leq 2\mathfrak{m}$

If we suppose that the first case of (3), viz. $\mathfrak{m} < 2\mathfrak{m}$, holds, then we have also to accept that

(4) $2\mathfrak{m} < 3\mathfrak{m}$ and (5) $\mathfrak{m} < 4\mathfrak{m}$

are valid. Hence in virtue of $\mathbf{S}_1$ we can conclude from (1), (4), (5) and (2) that

(6) $3\mathfrak{m} = 4\mathfrak{m}$ which gives at once that (7) $4\mathfrak{m} = 5\mathfrak{m}$

Whence, by (6) and (7)

(8) $3\mathfrak{m} = 5\mathfrak{m}$ which implies that (9) $4\mathfrak{m} = 6\mathfrak{m}$

Since the cancellation laws for cardinals are provable without the aid of the axiom of choice and of $\mathbf{V}_1$, *cf.* [2] and [4], we obtain from (9) at once that

(10) $2\mathfrak{m} = 3\mathfrak{m}$

which contradicts (4). Hence our assumption that $\mathfrak{m} < 2\mathfrak{m}$ is not true, and, therefore, the second case of (3), viz.

(11) $\mathfrak{m} = 2\mathfrak{m}$

holds. Thus, it is proved that $\{\mathbf{V}_1\} \rightleftarrows \{\mathbf{S}_1\}$.

2. It is evident that the axiom of choice implies $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$.

2.1. Let us assume $\mathbf{S}_2$, (1) and put $\mathfrak{n} = \aleph_0\mathfrak{m}$. Since, clearly, $\mathfrak{n} = \mathfrak{n} + 1$, we can establish at once, *cf.* e.g. [1], p. 169, that

(12) $\mathfrak{n} + 2^{\mathfrak{n}} = 2^{\mathfrak{n}}$

We have without the aid of the axiom of choice

(13) $\mathfrak{n} < 2^{\mathfrak{n}}$ and (14) $\aleph(2^{\mathfrak{n}}) \leq 2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}})$

where $\aleph(2^{\mathfrak{n}})$ represents the least Hartogs' aleph which is not $\leq 2^{\mathfrak{n}}$, *cf.* e.g. [1], pp. 407-409. And, due to (12) we can establish that

(15) $\aleph(2^{\mathfrak{n}}) + 2^{\mathfrak{n}} = \mathfrak{n} + (2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}}))$

Hence, if we suppose that the first case of (14), viz. $\aleph(2^{\mathfrak{n}}) < 2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}})$, holds, then in virtue of $\mathbf{S}_2$ and due to the fact that $2^{\mathfrak{n}}$ and $2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}})$ are, clearly, not finite cardinals it follows from our assumptions, (13) and (15) that

(16) $2^{\mathfrak{n}} = 2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}})$

which gives an impossible conclusion that $2^{\mathfrak{n}} \geqslant \aleph(2^{\mathfrak{n}})$. Therefore, the second case of (14), viz.

(17) $\aleph(2^{\mathfrak{n}}) = 2^{\mathfrak{n}} + \aleph(2^{\mathfrak{n}})$

holds. Since, by assumption, $\mathfrak{n} = \aleph_0\mathfrak{m}$, (17) implies at once that

(18) $\aleph(2^{\mathfrak{n}}) \geqslant 2^{\mathfrak{n}} > \mathfrak{n} = \aleph_0\mathfrak{m} \geqslant \mathfrak{m}$

which proves that an arbitrary cardinal $\mathfrak{m}$ which is not finite is an aleph. Thus, $\mathbf{S}_2$ implies the axiom of choice.

2.2. Now, let us assume $\mathbf{S}_3$, (1) and put $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$. Hence, clearly,

(19) $\mathfrak{n} = \mathfrak{n}^2$

If $\aleph(\mathfrak{n})$ is the least Hartogs' aleph which is not $\leqslant \mathfrak{n}$, then we have without the aid of the axiom of choice, *cf.* e.g. [1], p. 409, that

(20) $\mathfrak{n} < \mathfrak{n} + \aleph(\mathfrak{n})$ and, therefore, *a fortiori:* (21) $\mathfrak{n} < \mathfrak{n}\aleph(\mathfrak{n})$

On the other hand, by (19)

(22) $\mathfrak{n}(\mathfrak{n} + \aleph(\mathfrak{n})) = \mathfrak{n}^2 + \mathfrak{n}\aleph(\mathfrak{n}) = \mathfrak{n} + \mathfrak{n}\aleph(\mathfrak{n}) = \mathfrak{n}(1 + \aleph(\mathfrak{n})) = \mathfrak{n}\aleph(\mathfrak{n}) = \mathfrak{n}^2\aleph(\mathfrak{n}) = \mathfrak{n}(\mathfrak{n}\aleph(\mathfrak{n}))$

Since, by assumption, $\mathfrak{n} + \aleph(\mathfrak{n})$ and $\mathfrak{n}\aleph(\mathfrak{n})$ are not finite cardinals, $\mathbf{S}_3$ together with (20), (21) and (22) implies that

(23) $\mathfrak{n} + \aleph(\mathfrak{n}) = \mathfrak{n}\aleph(\mathfrak{n})$

Since $\aleph(\mathfrak{n})$ is the least Hartogs' aleph which is not $\leqslant \mathfrak{n}$, it is well-known, *cf.* [5], pp. 148-150, and [1], pp. 419-421, that (23) yields

(24) $\aleph(\mathfrak{n}) \geqslant \mathfrak{n}$

which, since $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$, allows us to conclude that

(25) $\aleph(\mathfrak{n}) \geqslant \mathfrak{n} = \mathfrak{m}^{\aleph_0} \geqslant \mathfrak{m}$

i.e. that an arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph. Thus, $\mathbf{S}_3$ implies the axiom of choice.

2.3. It is obvious that the reasonings entirely analogous to the given above allow us to prove that $\mathbf{S}_4$ implies also the axiom of choice.

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University of Notre Dame
Notre Dame, Indiana