142
Notre Dame Journal of Formal Logic
Volume VI, Number 2, April 1965

A MODAL EXTENSION OF INTUITIONIST LOGIC

R. A. BULL

1. In [3] (pp. 38, 39) Prior gives a modal extension of **IC** by adding to it the rules

R1. $C\alpha\beta \Longrightarrow CL\alpha\beta$
R2. $C\alpha\beta \Longrightarrow C\alpha M\beta$
R3. $C\alpha\beta \Longrightarrow C\alpha L\beta$, *if* α *is fully modalised,*[1]
R4. $C\alpha\beta \Longrightarrow CM\alpha\beta$, *if* β *is fully modalised.*

This system, which he calls **MIPQ**, is analogous to S5, in the sense that adding *ANpp* to it yields S5, and is intuitionistically plausible, in the sense that collapsing the modal operators yields **IC**. The purpose of this paper is to give a characterization of the normal models for **MIPQ** (in section 2) and show that it has the finite model property (in section 3). From this last result it follows immediately that **MIPQ** is decidable, since its normal models are strong models for the rules.

Before I proceed with this work I wish to refer briefly to a related system. The question as to whether **MIPQ**—or any other modal extension of **IC**—formalises concepts which an intuitionist philosopher would regard as modal is quite distinct from the formal ones answered in this paper. As Prior points out, one could regard the propositions of **MIPQ** as predicates in one individual variable, x say, and regard L and M as Πx and Σx. Perhaps this would give a suitable intuitionist interpretation of modality, but I prefer a rather stronger system in which $L\alpha$ and $M\alpha$ can be interpreted as 'α is the case in all possible worlds' and 'α is the case in some possible world'. This system has for its models those obtained by taking any model for **IC**, $\mathfrak{M}$ say, and any $n \geq 1$ and

(1) Taking as truth-values sequences of n elements of $\mathfrak{M}$.

(2) Designating $\langle 1, 1, \ldots, 1\rangle$, where 1 is the designated element of $\mathfrak{M}$.

(3) Determining non-modal operators by applying the operators of $\mathfrak{M}$ to corresponding terms of sequences.

1. Ie if every occurrence of a variable in α is an occurrence in the argument of a modal operator.

Received April 6, 1964

(4) Taking $L\langle x_1, x_1, \ldots, x_n\rangle$ to be $Kx_1 \ldots Kx_{n-1}x_n$, and taking $M\langle x_1, x_2, \ldots, x_n\rangle$ to be $Ax_1 \ldots Ax_{n-1}x_n$.

That this system is stronger than **MIPQ** may be confirmed by noting that $CLALpqALpLq$ holds in it but not in **MIPQ.** I conjecture that **MIPQ** plus $CLALpqALpLq$ is in fact sufficient for this system, but I cannot prove it. A more elegant presentation of the axiom system can be obtained using Schutte's notion of positive and negative parts. Let $f(\alpha)(g(\alpha))$ be any word with an occurrence of α as a positive (negative) part, this occurrence of α, and possibly its parts, being the only parts of the word not in the argument of a modal operator. Let $f(L\alpha)(g(M\alpha))$ be the word obtained from $f(\alpha)(g(\alpha))$ by replacing this occurrence of α by $L\alpha$ $(M\alpha)$. Then **R1, R2, R3, R4** and $CLALpqALpLq$ are equivalent to the rules

$$f(\alpha) \Longleftrightarrow f(L\alpha)$$
$$g(\alpha) \Longleftrightarrow g(M\alpha).$$

One can also set up a system, presumably the same one, formalizing this concept of possible worlds, by combining Kripke's semantic analyses of **IC** and S5; but again I have been unable to obtain any completeness results.

2. By a normal model for **MIPQ** I mean an 8-tuble $\langle H, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ which verifies **MIPQ** under the usual interpretation,[2] and in which the relation $\leq$ on H defined by

$$x \leq y \textit{ if and only if } y \dot{-} x = 1$$

is a partial ordering. By a canonical model I mean a 9-tuple $\langle H, K, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ where

(i) $\langle H, +, ., \dot{-}, 0\rangle$ is a Heyting algebra with unit 1.

(ii) $\langle K, +, ., \dot{-}, 0\rangle$ is a sub-Heyting algebra of $\langle H, +, ., \dot{-}, 0\rangle$.

(iii) Under the usual ordering on $\langle H, +, ., \dot{-}, 0\rangle$ there is a greatest element of K below every element of H and a least element of K above every element of H.

(iv) $\mathfrak{r}$ and $\mathfrak{t}$ are defined on H by taking $\mathfrak{r}x$ as the greatest element of K below x and $\mathfrak{t}x$ as the least element of K above x.

It is easy to check that if $\langle H, K, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ is a canonical model then $\langle H, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ is a normal model for **MIPQ.** I shall prove below that if $\langle H, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ is a normal model for **MIPQ** and

$$K = \{x \mid \textit{all } x = \mathfrak{r}y \textit{ for some } y \textit{ in } H\}$$

then $\langle H, K, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$ is a canonical model. Thus the normal models for **MIPQ** coincide with the canonical models. It is worth noting that a similar result can be obtained for S5, replacing 'Heyting algebra' by 'Boolean algebra' in (i) and (ii).

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2. H is the set of truth-values, with 1 the designated value and 0 its negation; Axy, Kxy, Cxy, Nx, Lx, Mx are represented by $x+y$, $x.y$, $y \dot{-} x$, $0 \dot{-} x$, $\mathfrak{r}x$, $\mathfrak{t}x$.

First note that by definition of being a normal model for **MIPQ**,

$$f(x_1, x_2, \ldots, x_n) \leq g(x_1, x_2, \ldots, x_n) \textit{ for all } x_1, x_2, \ldots, x_n \textit{ in } H$$

holds in $\langle H, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t} \rangle$ if

$$CF(p_1, p_2, \ldots, p_n)G(p_1, p_2, \ldots, p_n)$$

is a thesis of **MIPQ**, where f and g are the functions of the model corresponding to the logical functions F and G; and further

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \textit{ for all } x_1, x_2, \ldots, x_n \textit{ in } H$$

holds in $\langle H, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t} \rangle$ if

$$CF(p_1, p_2, \ldots, p_n)G(p_1, p_2, \ldots, p_n)$$

and

$$CG(p_1, p_2, \ldots, p_n)F(p_1, p_2, \ldots, p_n)$$

are theses of **MIPQ**. Now to satisfy the conditions for canonical models:

(i) $\langle H, +, ., \dot{-}, 0 \rangle$ is known to be a Heyting algebra with unit 1.[3]

(ii) K is closed under $+, ., \dot{-}$, for

$$\mathfrak{r}x + \mathfrak{r}y = \mathfrak{r}(\mathfrak{r}x + \mathfrak{r}y),\ \mathfrak{r}x . \mathfrak{r}y = \mathfrak{r}(\mathfrak{r}x . \mathfrak{r}y),\ \mathfrak{r}x \dot{-} \mathfrak{r}y = \mathfrak{r}(\mathfrak{r}x \dot{-} \mathfrak{r}y)$$

since $CALpLqLALpLq$, $CLALpLqALpLq$, $CKLpLqLKLpLq$, $CLKLpLqKLpLq$, $CCLpLqLCLpLq$, $CLCLpLqCLpLq$ can be derived in **MIPQ** with **R1** and **R3**. 0 is in K, for

$$0 = \mathfrak{r}0$$

since $COLO$ and $CLOO$ can be derived in **MIPQ**. Thus $\langle K, +, ., \dot{-}, 0 \rangle$ is a sub-Heyting algebra of $\langle H, +, ., \dot{-}, 0 \rangle$.

(iii) and (iv) Note that for each x in H, $\mathfrak{t}x$ is in K, for $\mathfrak{t}x = \mathfrak{r}\mathfrak{t}x$ since $CMpLMp$ and $CLMpMp$ can be derived in **MIPQ** with **R1** and **R3**. For each x in H,

$$\mathfrak{r}x \leq x \leq \mathfrak{t}x$$

since $CLpp$ and $CpMp$ can be derived in **MIPQ** with **R1** and **R2**. Given any element $\mathfrak{r}y$ of K,

$$\mathfrak{r}(x \dot{-} \mathfrak{r}y) \leq \mathfrak{r}x \dot{-} \mathfrak{r}y$$

since $CLCLpqCLpLq$ is a thesis of **MIPQ**

$(CCLpqCLpq$	$\Longrightarrow CLCLpqCLpq$	**R1**
	$\Longrightarrow CKLCLpqLpq$	
	$\Longrightarrow CKLCLpqLpLq$	**R3**
	$\Longrightarrow CLCLpqCLpLq)$,	

so

3. Cf [1], Theorem 4.1.

$\mathfrak{r}y \leq x$ implies $x \dot{-} \mathfrak{r}y = 1$
implies $\mathfrak{r}(x \dot{-} \mathfrak{r}y) = 1$ since the rule $\alpha \Longrightarrow L\alpha$ can be derived from **R3**
implies $\mathfrak{r}x \dot{-} \mathfrak{r}y = 1$
implies $\mathfrak{r}y \leq \mathfrak{r}x$.

Again,

$$\mathfrak{r}(\mathfrak{r}y \dot{-} x) \leq \mathfrak{r}y \dot{-} \mathfrak{t}x$$

since $CLCpLqCMpLq$ is a thesis of **MIPQ**

$$(CCpLqCpLq \Longrightarrow CLCpLqCpLq \quad \textbf{R1}$$
$$\Longrightarrow CpCLCpLqLq$$
$$\Longrightarrow CMpCLCpLqLq \quad \textbf{R4}$$
$$\Longrightarrow CLCpLqCMpLq),$$

so

$x \leq \mathfrak{r}y$ implies $\mathfrak{r}y \dot{-} x = 1$
implies $\mathfrak{r}(\mathfrak{r}y \dot{-} x) = 1$
implies $\mathfrak{r}y \dot{-} \mathfrak{t}x = 1$
implies $\mathfrak{t}x \leq \mathfrak{r}y$.

Thus for each x in H, $\mathfrak{r}x$ is the greatest element of K below x and $\mathfrak{t}x$ is the least element of K above x.

3. It is easy to show that **MIPQ** has a characteristic normal model, to wit the Lindenbaum model of the equivalence classes of words in it. Therefore to prove that it is characterised by finite models it is sufficient to show that any word rejected by a canonical model is rejected by a finite canonical model. Let us suppose then that a word is rejected by a canonical model $\langle H, K, \{1\}, +, ., \dot{-}, 0, \mathfrak{r}, \mathfrak{t}\rangle$, its parts taking values $a_1, a_2, \ldots, a_m$ in a rejecting allocation. I now define a finite model $\langle H', K', \{1\}, +, ., \dot{-}', 0, \mathfrak{r}', \mathfrak{t}'\rangle$ by

(a) Taking $\langle H', +, ., 0\rangle$ as the sub-lattice of $\langle H, +, ., 0\rangle$ generated by $0, a_1, a_2, \ldots, a_m$. (H' is finite and closed under + and ., so *sup* and *inf* are defined on it.)

(b) Taking $K' = K \cap H'$. (K' is finite since H' is, and closed under + and since K and H' are, so *sup* and *inf* are defined on it.)

(c) Defining $\dot{-}'$ on H' by

$$x \dot{-}' y = sup\{z \mid \text{all } z \text{ in } H' \text{ such that } z \leq x \dot{-} y\}.$$

(d) Defining $\mathfrak{r}'$ and $\mathfrak{t}'$ on H' by

$$\mathfrak{r}'x = sup\{y \mid all\ y\ in\ K'\ such\ that\ y \leq x\},$$
$$\mathfrak{t}'x = inf\{y \mid all\ y\ in\ K'\ such\ that\ y \geq x\}.$$

The model $\langle H', K', \{1\}, +, ., \dot{-}', 0, \mathfrak{r}', \mathfrak{t}'\rangle$ satisfies the conditions for being a canonical model, for:

(i) $\langle H', +, ., \dot{-}', 0\rangle$ is known to be a Heyting algebra with unit 1.[4]

(ii) K' is closed under + and ., and contains 0, since both K and H' have these properties. To prove that $\langle K', +, ., \dot{-}', 0\rangle$ is a sub-Heyting algebra of $\langle H', +, ., \dot{-}', 0\rangle$ it remains to show that K' is closed under $\dot{-}'$. Let x and y be elements of K', and let $\{z_1, z_2, \ldots, z_n\}$ be all the elements of H' such that $z_i \leq x \dot{-} y$, $1 \leq i \leq n$. Then

$$\begin{aligned} x \dot{-}' y &= sup\{z_1, z_2, \ldots, z_n\} \\ &\leq sup\{\mathfrak{t}'z_1, \mathfrak{t}'z_2, \ldots, \mathfrak{t}'z_n\} \end{aligned}$$

since $z_i \leq \mathfrak{t}'z_1$, for each $1 \leq i \leq n$; also

$$\{\mathfrak{t}'z_1, \mathfrak{t}'z_2, \ldots, \mathfrak{t}'z_n\} \subseteq \{z_1, z_2, \ldots, z_n\}$$

since $\mathfrak{t}'z_i$ is in K' and $\mathfrak{t}'z_i \leq \mathfrak{t}'(x \dot{-} y) = x \dot{-} y$, for each $1 \leq i \leq n$, so

$$\begin{aligned} x \dot{-}' y &= sup\{z_1, z_2, \ldots, z_n\} \\ &\geq sup\{\mathfrak{t}'z_1, \mathfrak{t}'z_2, \ldots, \mathfrak{t}'z_n\}; \end{aligned}$$

therefore

$$x \dot{-}' y = sup\{\mathfrak{t}'z_1, \mathfrak{t}'z_2, \ldots, \mathfrak{t}'z_n\}.$$

Now each element $\mathfrak{t}'z_i$, $1 \leq i \leq n$, is the *inf* of a set of elements of K', so $x \dot{-}' y$ is a sum of products of elements of K', and is therefore itself a member of K'.

(iii) and (iv) It is clear from (d) that for each element x in H', $\mathfrak{r}'x$ is the greatest element of K' below x and $\mathfrak{t}'x$ is the least element of K' above x.

It will be noted that when the operators of the original canonical model are defined from H' to H' they take the same values as the corresponding operators of the new canonical model. It follows that if all the parts of a function $f(x_1, x_2, \ldots, x_n)$ on the original canonical model are in H' then the corresponding function $f'(x_1, x_2, \ldots, x_n)$ on the new canonical model takes the same value. Therefore the given word is rejected by $\langle H', K', \{1\}, +, ., \dot{-}', 0, \mathfrak{r}', \mathfrak{t}'\rangle$. Thus we have constructed a finite canonical model rejecting the given word.

REFERENCES

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[3] A. N. Prior, *Time and Modality*. Oxford, 1957.

Wadham College
Oxford University, Oxford, England

4. Cf [2], Theorem 1.11.