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ON THE CONNECTION OF THE FIRST-ORDER FUNCTIONAL CALCULUS WITH $\aleph_0$ PROPOSITIONAL CALCULUS

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A simply conclusion from papers [2]-[5] is that for each formula E we may construct a $n(E)$-valued propositional calculus such that if E is not a thesis, then E is false in this calculus by a finite interpretation of the quantifiers; by means of a simply extending of the $n(E)$ valued calculus to $\aleph_0$ propositional calculus we may prove in one the converse theorem. This method we have used in [5] and have proved that it is possible to approximate the first-order functional calculus by many valued propositional calculi.

An interest approximation of the first-order functional calculus by $\aleph_0$ propositional calculus follows from [3] and [4]. We obtain it by means of constructing of a correspondence between atomic formulas and sequences of numbers *0* and *1* such that:

1. If the atomic formula is of ≥ 2 arguments, then the correspondents sequence is periodic/we shall give the period/.
2. The difference in this correspondence is in general on atomic formulas of one argument whose we must consider an infinite number.
3. For some formulas, e.g. $\Sigma a_1 \, \Sigma a_2 \, \Pi a_3 \ldots \, \Pi a_k F$ where F is quantifier and individual variable—free, monadic formulas, . . . , the $\aleph_0$ calculus may be replaced by suitable n- or *2*-valued propositional calculus; one follows from a general theorem.

We shall use the notation of all mentioned papers and in particular:

(1) variables: (1°) individual: $x_1, x_2, \ldots$ /or simply x/, (2°) apparent: $a_1, a_2, \ldots$ /or simply a/,
(2) finite numbers of functional variables: $f_1^1, \ldots, f_q^1, f_1^2, \ldots, f_q^2, \ldots, f_1^t, \ldots, f_q^t$ / f_i^m of m-arguments, $m = 1, \ldots, t$ and $i = 1, \ldots q$/
(3) logical constants: (negation), + (alternative), Π(general quantifier),
(4) atomic expression: $R, R_1, R_2, \ldots$; expressions: $E, F, G, E_1, F_1, G_1 \ldots$ [1]

1. Expressions and formulas we define in the usual way; the expression in which an apparent variable a belong to the scope of two quantifiers Πa is not a formula; if a does not occur in E, then ΠaE is not a formula.

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(5) $\{s_m\}$ - the sequence $s_1, \ldots, s_m$; $\{s_{i_m}\}$ - the sequence $s_{i_1}, \ldots, s_{i_m}$,
(6) $w(E)$ - the number of different individual /$p(E)$ - apparent/ variables occurring in the expression E,
(7) $\{i_{w(E)}\}$ or $\{j_{w(E)}\}$ - indices of all different variables occurring in the expression E,
(8) $n(E) = \max \{w(E) \dot{+} p(E), \{i_{w(E)}\}\}$,
(9) $E(u/z)$ - the expression resulting from E by substitution of u for each occurrence of z in E /with knowing conditions/,
(10) $C(E)$ - the set of all significant parts of the formula E: $H \epsilon C(E). \equiv .$[2] $H = E^*$ or there exist F, G, H_1 such that: $(H = F) \vee (E = F') \vee \{(H = F) \vee (H = G)\} \wedge (E = F + G) \vee (\exists i)\{H = H_1(x_i/a)\} \wedge (E = \Pi a H_1)$ Of course, each significant part of the formula E is a formula.
(11) Skt - the set of all formulas of the form $\Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} \ldots \Pi a_k F$[3], where F is quantifierless expression containing no free variables, Πa_j is the sign of the universal quantifier binding a_j and $\Sigma a_j G = (\Pi a_j G')'$, $j = 1, \ldots, k$,
(12) $S(\{i_m\})$ - the set of all atomic formulas R such that all indices of individual variables occurring in R belong to $\{i_m\}$,
(13) $M, M_1, M_2, \ldots$.- functions of all atomic formulas with values 0 and 1; $T, T_1, T_2, \ldots$ - functions on $S(1, \ldots, k)$, for given k, with values 0 and 1 /such functions we shall name "satisfiability functions of the rank k" or simply: functions of the rank k, if k is finite/; $\{\hat{M}\}$-the set which has only one element M,
(14) (K) - for every K; $(\exists K)$ - there exists K such that; $(\{s_m\})$ - for each $\{s_m\}$; $(\exists \{s_m\})$ - there exists $\{s_m\}$ such that;
(15) $\Sigma(F) = 0$, if F is a quantifierless formula,
$\Sigma(F + G) = \max \{\Sigma(F), \Sigma(G)\}$,
$\Sigma(\Pi a F) = \Sigma\{F(x/a)\}$, where x does not occur in F,
$\Sigma(\Sigma a F) = w(F) + 1$, if $\Sigma\{F(x/a)\} = C$[4]
$\Sigma(\Sigma a F) = \Sigma\{F(x/a)\}$, if x does not occur in F and $\Sigma\{F(x/a)\} \neq 0$.

If F is not defined above. then $\Sigma(F) = \max\{\Sigma(G)\}$, for each $G \epsilon C(E)$, where if $G = \Pi a H$, then $\Sigma(G) = w(H) + 1$, $\Sigma(F') = \Sigma(F)$, $\Sigma(F + G) = \max\{\Sigma(F), \Sigma(G)\}$.

For example:

(1°) If $E \epsilon Sks$ and $E = \Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} \ldots \Pi a_k F$, for some F, then $\Sigma(E) = i$.
(2°) If $E = \{\Pi a f_r^m (a, x_2, \ldots, x_m) + f_j^r(x_1, \ldots, x_r)\}'$, then $\Sigma(E) = m$.
(3°) $\Sigma(E) \leq w(E) + p(E) \leq n(E)$.
(4°) $\Sigma(E) = 0 \ . \equiv .\ E$ is an alternative of formulas of the form $\Pi a_1 \ldots \Pi a_m F$ where F is quantifier-free
(16) $w_1, v_1, \ldots$ -numbers 0 and 1.

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2. Dots separated more strongly than parentheses.
3. It is Skolem's normal form for theses.
4. We note that if $\Sigma\{F(x/a)\} = 0$, then $\Sigma\{F(x_i/a)\} = 0$, for each i. In exactly given cases $\Sigma(E)$ may be less than defined above.

The formal proof $E_1, \ldots, E_m$ of the formula E we define in the usual way, see [3], [4], but to the proof of given theorems we also must assume that for each $i = 1, \ldots, m$, E_i is an alternative of significant parts of the formula E and $\Sigma(E_i) = \ldots = \Sigma(E_m) = \Sigma(E)$; the number m we name the length of the formal proof. The thesis is the last element of a formal proof.

Of course:

L.0. If the length of a formal proof of E is m, then the length of some formal proof of $E(x/y)$ also is m.

L.1. For each formula E may be written a formula $F \epsilon Skt$ such that E is a thesis if and only if F is a thesis and $E' + F$ is a thesis/it is possible to replace the assumption $F \epsilon Skt$ by: F is an alternative of formulas belonging to Skt of the form $\Sigma a_1 \ldots \Sigma a_{m-1} \Pi a_m H$ where H is quantifier-free/.

L.1. asserts the existence of Skolem's normal form, [1], for theses.

In the following we shall interpret the signs ' and + as Boolean operations $\neg$/complemention/ and $\dot{+}$ /addition/ respectively; therefore Π is interpreted as an infinite Boolean multiplication. By this interpretation we have extended the function M, see (13), on all formulas and therefore we shall use the symbol $M(E)$ for an arbitrary E.

It is known:

T.1. A formula E is a thesis if and only if for each M we have $M(E) = 1$.

Let $M/s_1, \ldots, s_k/$ be a function on $S(1, \ldots, k)$ such that for an arbitrary $R \epsilon S(1, \ldots, k)$ we have:

$$M/s_1, \ldots, s_k/(R) = M\{R(x_{s_1}/x_1) \ldots (x_{s_k}/x_k)\}^5.$$

Of course:

L.2. If $i_1, \ldots, i_m \leq k$, then:

$$M/s_1, \ldots, s_k//i_1, \ldots, i_m/ = M/s_{i_1}, \ldots, s_{i_m}/.$$

L.3. If T_1, T_2 are functions of the rank k and $r_1, \ldots, r_i\ r_{i+1}, \ldots, r_j,$ $r_{j+1}, \ldots, r_m$ $(m \leq k)$ is a sequence of different natural numbers $\leq k$, then if $T_1/r_1, \ldots, r_i/ = T_2/r_1, \ldots, r_i/$, then there exists a function T° of the rank k such that:

$$T^\circ/r_1, \ldots, r_1, r_{i+1}, \ldots, r_j/ = T_1/r_1, \ldots, r_i, r_{i+1}, \ldots, r_j/,$$
$$T^\circ/r_1, \ldots, r_i, r_{j+1}, \ldots, r_m/ = T_2/r_1, \ldots, r_i, r_{j+1}, \ldots, r_m/.$$

D.1. $M \epsilon Q_k$. $\equiv$. for an arbitrary $s_1, \ldots, s_k, s'_1, \ldots, s'_k$:

If $M/s_1, \ldots, s_k/ = M/s'_1, \ldots, s'_k/$, then $s_1 = s'_1, \ldots, s_k = s'_k$.

$M \epsilon Q_k$ asserts that functions of the form $M/s_1, \ldots, s_k/$ are different; examples may be easily given. It is clear that if $M \epsilon R_1$, then $M \epsilon R_k$, $k = 1, 2, \ldots$

5. If M is defined. We may replace here $1, \ldots, k$ by $i_1, \ldots, i_{w(R)}$.

By an extension of a function M_1 we mean a function M which is equal to M_1 on all formulas for which M_1 is defined.

L.4. Each function M_1 may be extended to $M \epsilon R_1$ / therefore $M \epsilon R_k$, $k = 1, 2, \ldots$ /,

Proof: Let

(0) $(x_1, x_2), (x_1, x_3), (x_2, x_3), \ldots$.

be the sequence of all pairs of different individual variables and $g_1^1, g_2^1, \ldots, g_m^1, \ldots$ an infinite sequence of functional variables of one argument which do not occur in formulas for which M_1 is defined.

Now, we assume that we consider all formulas which are built from $f_1^1, \ldots, f_q^1, \ldots, f_1^t, \ldots, f_q^t$ and also from $g_1^1, g_2^1, \ldots, g_m^1, \ldots$ in the way given above.

Let $M(R) = M_1(R)$, if $M_1(R)$ is defined and:

(1°) if $M_1/1/ = M_1/2/$, then $M\{g_1^1(x_1)\} = 1$ and $M\{g_1^1(x_i)\} = 0$, $i = 2, 3, \ldots$

(2°) if (x_i, x_j) is the m-th pair of the sequence (0), $M_1/i/ = M_1/j/$, then $M\{g_m^1(x_i)\} = 1$ and $M\{g_m^1(x_j)\} = 0$, for $i \neq j$.

Of course $M \epsilon R_1$ and M is an extension of M_1.

Another extension of M_1 to function $M \epsilon R_1$ may be obtained from [2].

In the sequel we shall write $M/\{s_k\}$ instead of $M/s_1, \ldots, s_k/$; $M/\{s_{i_m}\}$ instead of $M/s_{i_1}, \ldots, s_{i_m}/$.

D.2. $T \epsilon M[k] . \equiv . (\exists \{s_k\})(T = M/\{s_k\})$.

$M[k]$ is the set of all functions of the form $M/\{s_k\}$.

We note that if M is defined as in the proof of *L.4.* then $M[k]$ has the following property:

(I) There exists only a finite number $\leq 2^{qtk^t}$ functions belonging to $M[k]$ which differ on atomic formulas of ≥ 2 arguments and:

(1°) for each m and $T \epsilon M[k]$ we have $T\{g_m^1(x_i)\} = 0$, $i = 1, \ldots, k$ or there exists $i \leq k$ such that $T\{g_m^1(x_i)\} = 1$ and $T\{g_m^1(x_j)\} = 0$, for $j \neq i$ and $j \leq k$.

(2°) for each m there exist $T \epsilon M[k]$ and $i \leq k$ such that $T\{g_m^1(x_i)\} = 1$.

By a modification of the proof of *L.4.* the reader may obtain other properties of the considered $M[k]$.

We shall also consider a Boolean algebra whose elements are infinite sequences of numbers *0* and *1* and operations $\urcorner$ (complemention) and $\dot{+}$ (addition); the Boolean algebra determines an $\aleph_0$-valued propositional calculus.

Let k be a natural number and Q a function of atomic formulas $R \epsilon S(1, \ldots, k)$ whose values are infinite sequences of numbers *0* and *1*, such a function Q we shall name *a* sequence function of the rank k, and we shall write briefly $Q(k)$.

The function $Q(k)$ gives a table of infinite sequences of numbers, we name it also Q:

	$R_1, \ldots, R_i, \ldots, R_u$ - *all elements of the set* $S(1,\ldots,k)$.
1	$0 \quad \ldots w_{1i} \quad \ldots \quad w_{1i}$
2	$1 \quad \ldots w_{2i} \quad \ldots \quad w_{2u}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$
j	$0 \quad \ldots w_{ji} \quad \ldots \quad w_{ju}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$

$$Q(R_i) = \begin{pmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{ji} \\ \vdots \end{pmatrix}$$

Each line j of Q determines a function T_jQ of the rank k such that $T_j(R_i) = w_{ji}$, where $i = 1,\ldots,u$ and $j = 1,2,\ldots$

If we consider $T_j(Q)$ on $S(\{j_m\})$, then we shall say that we consider the segment $\{j_m\}$ of the line j of Q.

D.3. $Q^\circ = Q/t_1,\ldots,t_k/ \;.\equiv.\; (j)\{T_j(Q^\circ) = T_j(Q)/t_1,\ldots,t_k/\}$[6].

D.4. $Q/T, \{i_m\} \;.\equiv.\; (\exists j)(\exists \{j_m\})(T_j(Q)/\{j_m\} = T/\{i_m\})$.

$Q/T, \{i_m\}$ asserts that $T/\{i_m\}$ is a segment of some line of Q in the meaning of homomorphism; in this case we shall say briefly: $T/\{i_m\}$ is a segment of some line of Q.

D.5. $Q/Q^\circ \;.\equiv.\; (\exists j)(\exists k)(T)(T^\circ)(\{i_m\})\{(j \geq k \geq m) \wedge Q(j) \wedge Q^\circ(k) \wedge (i_1,\ldots,i_m \leq k) \wedge (T^\circ = T/1,\ldots k/) \rightarrow (Q/T, \{i_m\} \;.\equiv.\; Q^\circ/T^\circ; \{i_m\})\}$[7].

Q/Q° asserts that the relation $Q/T, \{i_m\}$ is invariant for each T, T°, $\{i_m\}$ $(m \leq k)$ such that $T^\circ = T/1,\ldots,k/$.

D.6. $Q \sim M[k] \;.\equiv.\; (T)\{T \epsilon M[k] \;.\equiv.\; (\exists j)(T = T_j(Q))\}$.

D.6. asserts that $M[k]$ is the set of all functions defined by lines of Q. It is easy to show:

L.5. If $Q/T, \{i_m\}$ and $\{i_j\} \subset \{i_m\}, j \leq m,$ then $Q/T, \{i_j\}$.

L.6. If $Q^\circ(k)$, $Q = Q^\circ/1,\ldots,k, k/$, *then* Q/Q°.

L.7. If M is a satisfiability function and $Q \sim M[k]$, then $\{\hat{M}\}/Q$.

L.8. If M is a satisfiability function defined on formulas in which occur only a finite number of functional variables, see (2), and $Q \sim M[k]$, then there exists a function T of the rank $\leq k2^{qtk^t}$ such that $Q \sim T[k]$ and $\{T\}/Q$ (T is a segment of M).[8]

D.7. $Q\{r,k\} \;.\equiv.\; (r \leq k) \wedge Q(k) \wedge (\{i_{m+1}\})(T)\{(m < r) \wedge (i_1,\ldots,i_{m+1}$ are different numbers $\leq k) \wedge Q/T, \{i_m\} \wedge Q/T, i_{m+1} \rightarrow (\exists T_1)(Q/T_1, \{i_{m+1}\} \wedge (\{j_m\})\{(\{j_m\} \subset \{i_{m+1}\}) \wedge Q/T, \{j_m\} \rightarrow (T_1/\{j_m\} = T/\{j_m\})\})\}$.

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6. $T_j(Q^\circ)$ is the function defined by the line j of Q°; the meaning of *D.3.* is simply.
7. If for each line j of Q and each permutation $t_1,\ldots,t_k$ of numbers $\leq k$ $T_j(Q)/\{t_k\}$ also is a line of Q, then $\{j_m\}$ may be replaced by $\{i_m\}$.
8. We understand the word "segment" in the meaning of homomorphism.

$Q\{r, k\}$ asserts that for each $\{i_{m+1}\}$ of different number, $m < r \leq k$ and all T of the rank k, if $T/\{i_m\}$ and T/i_{m+1} are segments of some lines of Q then there exists T_1 of the rank k such that $T_1/\{i_{m+1}\}$ is a segment of some line of Q $T_1/\{i_m\} = T/\{i_m\}$ and for each $\{j_{m-1}\} \subset \{i_m\}$ if $T/\{j_{m-1}\}, i_{m+1}$ is a segment of some line of Q, then $T_1/\{j_{m-1}\}, i_{m+1} = T/\{j_{m-1}\}, i_{m+1}$.

In other words, $Q\{r, k\}$ asserts that for each $\{i_{m+1}\}$ of different numbers k, $m < r \leq k$, if $\{i_m\}$ and i_{m+1} are segments of some lines of Q, then there exists a line n such that $\{i_{m+1}\}$ is a segment of line n of Q and for each $\{j_m\} \subset \{i_{m+1}\}$ if $\{j_m\}$ is a segment of some line s_m of Q for each $\{j_m\}$ we only choose one s_m and if every two of these segments of lines s_m are equal on equal sequences of numbers included in $\{j_m\}$, then $T_n(Q)/\{j_m\} = T_{s_m}(Q)/\{j_m\}$.

L.9. If $Q\{r,k\}$, then:
$(\{i_{m+1}\})(T)\{(m < r) \wedge (i_1, \ldots, i_m$ are different numbers $\leq k) \wedge Q/T, \{i_m\} \wedge Q/T, i_{m+1} \rightarrow (\exists T_1)(Q/T_1, \{i_{m+1}\} \wedge (\{j_s\})\{(j_1, \ldots\ldots, j_s$ are different numbers $\leq k) \wedge Q/T, \{j_s\} \rightarrow (T_1/\{j_s\} = T/\{j_s\})\})\}$.

This lemma follows from *D.7.* by using many times of *L.3*; we note that if in *D.7.* or *L.9.* we have $Q/T, \{i_{m+1}\}$, then $T_1 = T$.

Of course:

L.10. If $M \epsilon R_1$ and $Q \sim M[k]$, then for each $r \leq k$ we have $Q\{r, k\}$.

L.11. If M is a satisfiability function defined only on atomic formulas of one argument and $Q \sim M[k]$, then for each $r \leq k$ we have $Q[\{r, k\}$.

L.12. If $Q \sim M[k]$, then;
1. If T is a function of the rank k, $i, j \leq k$, $Q/T, i$, $Q/T, j$ then there exists T_1 of the rank k such that[9] $Q/T_1, i, j$ and: $T_1/1, \ldots, i\text{-}1, i+1, \ldots, k/ = T/1, \ldots, i\text{-}1, i+1, \ldots, k$, $T_1/1, \ldots, j\text{-}1, j+1, \ldots, k/ - T/1, \ldots, j\text{-}1, j+1, \ldots, k/$.
2. If $k \geq 2$, then $Q\{2, k\}$.

L.13. If $Q^\circ\{r, k\}$ and $Q = Q^\circ/1, \ldots, k, k/$, then $Q\{r, k+1\}$.

D.8. $T, Q/T_1, \{i_m\}; i \,.\equiv.\, (T/\{i_m\} = T_1/\{i_m\}) \wedge Q/T_1, \{i_m\}, i$.

D.9. $H \epsilon A(E) .\equiv. (\exists \{F_j\})(E = F_i + \ldots + F_1 + H + F_{i+1} + \ldots + F_j) \wedge (F)(G)(H \neq F + G)$.

The meaning of *D.8.* and *D.9.* are clear.

Let V be the functional defined for an arbitrary function of the rank k, for each $Q(k)$ and for an arbitrary formula E whose indices of individual variables occurring in it are $\leq k$, in the following way:

(1d) $V\{T, Q, f_j^m(x_{r_1}, \ldots, x_{r_m})\} = 1 .\equiv. T\{f_j^m(x_{r_1}, \ldots, x_{r_m})\} = 1,$
(2d) $V\{T, Q, F'\} = 1 .\equiv. \sim V\{T, Q, F\} = 1 .\equiv. V\{T, Q, F\} = 0,$
(3d) $V\{T, Q, F+G\} = 1 .\equiv. V\{T, Q, F\} = 1 \vee V\{T, Q, G\} = 1,$
(4d) $V\{T, Q, \Pi a E\} = 1 .\equiv. (i)(T_1)\{(i \leq k) \wedge T, Q/T_1, \{i_{w(F)}\}; i \rightarrow V\{T, Q, F(x_i/a)\} = 1\}$.

9. If $Q/T, i, j$, then we assume $T = T_1$. It may be proved the other properties of $Q \sim M[k]$.

D.10. $E\epsilon PQ .\equiv_\circ (T)\{(H)(\{H\epsilon A(E)\} \rightarrow Q/T, \{i_{w(H)}\}) \rightarrow V\{T, Q, E\} = 1\}$.

D.11. $E\epsilon P\{r,k\} .\equiv. (Q)\{Q\{r,k\} \rightarrow (E\epsilon P(Q))\}$.

D.12. $E\epsilon P .\equiv. E\epsilon P\{\Sigma(E), n(E)\}$.

We explain the meaning of ones:

1. $V\{T, Q, E\} = 1$ may be read: T satisfy E relatively to Q.
2. If M is a satisfiability function and $Q \sim M[k]$, then $T_j(Q)$ are segments of M, the number i in (4d) is a name of an arbitrary individual variable and in *D.10 - D.12.* we assume that we only consider segments of M; in *D.12.* we associate to each formula a pair of numbers.
3. Obviously, if E is quantifier-free, then: $E\epsilon P .\equiv. E$ is true.

L.14. Let E° results from E by replacing individual variables with indices $i_1, \ldots, i_{w(E)}$ correspondingly by individual variables $j_1, \ldots, j_{w(E^\circ)}$, $w(E) = w(E^\circ)$[10] and $T/\{i_{w(E)}\} = T^\circ/\{j_{w(E^\circ)}\}$. Then: $V\{T, Q, E\} = 1. \equiv . V\{T^\circ, Q, E^\circ\} = 1$

L.15. Let $k \geq n(E)$, $Q^\circ(k)$, Q/Q° and $T^\circ = T/1, \ldots, k/$; then: $V\{T, Q, E\} = 1 . \equiv V\{T^\circ, Q^\circ, E\} = 1$.

The proofs of *L.14.* and *L.15.* are inductive on the length of the formula E and are analogic to the proofs of *L.12.* and *L.14.* respectively from [2].

L.16. If $E\epsilon P\{r, k\}$ and $k \geq k_0$, then $E\epsilon P\{r, k_0\}$.

L.16. follows from the definitions, *L.6, L.13.* and *L.15*, see [3], [5].

T.2. If $E\epsilon Skt$, $F\epsilon C(E)$, $M\{E\} = 0$, $Q \sim M[k]$, then:

(1) If $M/\{s_{i_{w(F)}}\} = T/\{i_{w(F)}\}$ and $M\{F(\{s_{i_{w(F)}}\})\} = 0$, then $V\{T, Q, F\} = 0$

(2) $E\epsilon P(Q)$, $E\bar{\epsilon}P\{2, k\}$.

(3) If $M\epsilon R_1$, then $E\bar{\epsilon}P$.

Proof: First of all we notice that (2) follows from the assumptions, (1) and *L.12*; however (3) follows from (2) and *L.10*.

The proof of (1) is inductive on the number of quantifiers occurring in F and is analogous to *T.2.* of [3].

T.3. If E is a thesis, then $E\epsilon P\{\Sigma(E), k\}$, for each $k \geq n(E)$.

The proof of *T.3.* is inductive on the length of the formalized proof of the formula E; we use here *L.0, L.2, L.3, L.5, L.9, L.14, L.16.* and definitions; the whole proof is analogous to the proof of *T.3.* from [3].

T.4. A formula E is a thesis if and only if $E\epsilon P$.

T.4. follows from *T.1-3, L.1, L.7,* and *L.15.*, see [3]. It is easy to see:

1. *T.4.* remains true if we shall only consider $Q(k)$ with property (I), $p \ldots$, where $M[k]$ is replaced by the set of all $T_j(Q)$ and $k = n(E)$; therefore Q has properties given on p. 73.

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10. Then E results from E° by an inverse substitution.

2. If $E \epsilon Skt$ and $E = \Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} \ldots \Pi a_k F$, then E is a thesis if and only if $E \epsilon P \{k, i\}$.
3. The classes $P\{1, k\}$ and $P\{2, k\}$ are decidable, $k = 1, 2, \ldots$ (follows from *L.12*, *T.3.* and *T.4.*).

The monadic first-order functional calculus is decidable (follows from *L.11*, *T.3.* and *T.4.*)}.

From *L.8.* and *L.15.* it also follows that in *T4.* we may assume that Q has only one line whose rank is $\leq k2^{qtk^t}$, where $k = n(E)$.

The above consideration describes a method of decidabling of arbitrary formulas; the examples we shall give in [6].

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