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# BOOLEAN ALGEBROIDS

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1. The concept of a ringoid used here was given in [1]. In the same spirit we define in section 2, a Boolean algebroid and prove that:

*Any Boolean algebroid is a disjoint union of Boolean algebras.*

In section 3, we give lattice-like axioms for a Boolean algebroid and prove the same results. That the two systems are equivalent is trivial.

2. We recall the

Definition 1. *A collection* $\mathbf{R}$ *of elements is called a ringoid if operations of addition and multiplication are defined for certain pairs of elements of* $\mathbf{R}$ *and the following axioms are satisfied for any* $a$, $b$, $c$ *in* $\mathbf{R}$*:*

a) i) *Given* $a \in \mathbf{R}$, *there exists* $0_a \in \mathbf{R}$ *such* $a + 0_a = a$ *and* $0_a + x = x$ *whenever for* $x \in \mathbf{R}$, $0_a + x$ *is defined.*

ii) *Given* $a \in \mathbf{R}$, *there exists* $b \in \mathbf{R}$ *such that* $a + b = b + a = 0_a$.

b) *The following hold if either side is defined i.e. if one side is defined the other is also defined and the two are equal:*

i) $a + (b+c) = (a+b) + c$;

ii) $a + b = b + a$;

iii) $a(bc) = (ab)c$;

iv) $a(b+c) = ab + ac$;

v) $(b+c)a = ba + ca$.

c) The conditions $\alpha$), $\beta$) and $\gamma$) given below are satisfied. Define for $a \in \mathbf{R}$,

$\mathbf{L}(a) = \{x \in \mathbf{R} : xa \text{ is defined}\}$ *and* $\mathbf{R}(a) = \{x \in \mathbf{R} : ax \text{ is defined}\}$.

$\alpha$) *For every* $a \in \mathbf{R}$, $\mathbf{L}(a) \neq \phi$ *and* $\mathbf{R}(a) \neq \phi$.

$\beta$) *For every* $a \in \mathbf{R}$, *there is an element* $b$ *different from* $a$ *of* $\mathbf{R}$ *such that* $a + b$ *is defined.*

$\gamma$) *If* $\mathbf{L}(a) \cap \mathbf{L}(b) \neq \phi$ *and* $\mathbf{R}(a) \cap \mathbf{R}(b) \neq \phi$ *then* $a + b$ *is defined.*

It was proved in [1] that

$\delta$) *If for* $a, b \in \mathbf{R}$, $a + b$ *is defined then* $\mathbf{L}(a) = \mathbf{L}(b)$ *and* $\mathbf{R}(a) = \mathbf{R}(b)$.

It is easy to see that a ringoid can be written as a union of disjoint additive abelian groups. We prove more, namely;

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*Theorem* 1. *Let* **R** *be a ringoid as given by the axioms in definition 1. Then* **R** *can be written as a disjoint union of additive abelian groups,*

$\mathbf{R} = \bigcup_{0 \equiv u \in \Gamma^*} \mathbf{A}_u$, *where* $\Gamma^*$ *is a semi-group with zero. The multiplication in* **R** *is given by*

$$\mathbf{A}_u \cdot \mathbf{A}_v \rightarrow \mathbf{A}_w,$$

*whenever* $uv = w$ *in* $\Gamma^*$, *and is bilinear and associative (whenever triple products are defined).*

Proof: If $a + b$ is defined then $0_a = 0_b = 0_{a+b}$. Conversely, if $0_a = 0_b$, we can write $a = a + 0_a = a + 0_b = a + (b-b)$, and therefore $a + b$ is defined. Hence $a + b$ is defined if and only if $0_a = 0_b$. This divides **R** into a union of disjoint abelian groups, say, $\mathbf{R} = \bigcup_{u \in \Gamma} \mathbf{A}_u$. By virtue of $\delta$), we can define a partial multiplication on the index set $\Gamma$, by setting $ij = k$ if and only if for $x \in \mathbf{A}_i$ and $y \in \mathbf{A}_j$, $xy$ is defined and is in $\mathbf{A}_k$. Then $\Gamma$ forms a partial semi-group; i.e. multiplication is sometimes defined, and $(ij)k = i(jk)$ if either side exists. Finally, adjoin $0$ to $\Gamma$, to obtain $\Gamma^*$ and put $u0 = 0u = 0$ for all $u \in \Gamma$ and also $uv = 0$ for $u, v \in \Gamma$, if $uv$ is undefined in $\Gamma$. Then $\Gamma^*$ is a semi-group with zero. This completes the proof.

*Definition 2. A ringoid* **R** *is called a Boolean algebroid if* $x \cdot x$ *is defined and* $x \cdot x = x$ *for all* $x \in \mathbf{R}$.

*Theorem 2. Let* **A** *be a Boolean algebroid then* **A** *can be written as a union of disjoint Boolean algebras.*

Proof: It suffices to prove:

a) *For* $x,y \in \mathbf{A}$ *if* $x + y$ *is defined then* $xy$ *and* $yx$ *are defined.*

and

b) *For* $x,y \in \mathbf{A}$ *if* $xy$ *is defined the* $x + y$ *is defined.*

Since $(x+y) = (x+y)(x+y) = x^2 + xy + yx + y^2$, a) is clear.
Now to prove b), suppose $xy$ is defined. Since $x^2$ and $y^2$ are defined $x \in \mathbf{L}(y) \cap \mathbf{L}(x)$ and $y \in \mathbf{R}(y) \cap \mathbf{R}(x)$. Thus by $\gamma$) $x + y$ is defined. This completes the proof.

*Corollary* 1. $xy = yx$ *whenever either side is defined, for* $x,y \in \mathbf{A}$.

*Corollary* 2. $2x = 0$ *for all* $x \in \mathbf{A}$.

3. Consider the following axioms for a system $(\mathbf{L},\cup,\cap)$. These characterize a Boolean algebroid:

*Preface L1, L2, and L5 by, "If one side of the equation exists, then the other side also exists, and"*

Let $a,b,c \in \mathbf{L}$.

*L1* $a \cup b = b \cup a$, $a \cap b = b \cap a$

*L2* $(a \cup b) \cup c = a \cup (b \cup c)$, $(a \cap b) \cap c = a \cap (b \cap c)$

*L3* $a \cup a = a$, $a \cap a = a$

*L4* *If the left hand side of the equation exists,* $(a \cup b) \cap a = a$, $(a \cap b) \cup a = a$

*L5* $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

*D1* $\mathbf{Z} = \{x \in \mathbf{L} \mid$ *for any* $y \in \mathbf{L}$, *if* $x \cup y$ *or* $x \cap y$ *exists, then* $y \cup x = y = x \cup y$ *and* $y \cap x = x = x \cap y\}$.

*L6'* *For any* $a \in \mathbf{L}$ *there is an* $x \in \mathbf{Z}$, *so that* $a \cup x$ *exists.*

*D2* $\mathbf{N} = \{x \in \mathbf{L} \mid$ *for any* $y \in \mathbf{L}$, *if* $x \cup y$ *or* $x \cap y$ *exists, then* $y \cup x = x = x \cup y$ *and* $y \cap x = y = x \cap y\}$.

*L7'* *For any* $a \in \mathbf{L}$, *there is an* $x \in \mathbf{N}$, *so that* $a \cup x$ *exists.*

*L8'* *For any* $a \in \mathbf{L}$, *there are* $x \in \mathbf{L}$, $z \in \mathbf{Z}$ *and* $n \in \mathbf{N}$ *so that* $a \cup x = n = x \cup a$ *and* $a \cap x = z = x \cap a$.

First we shall prove the uniqueness of zero, one, and complement.

*T1* *If* $x,y \in \mathbf{Z}$, $a \in \mathbf{L}$, $a \cup x$ *and* $a \cup y$ *exist, then* $x = y$.

Proof. $a \cup y = [L6'] \ (a \cup x) \cup y = [L2] \ a \cup (x \cup y)$. *Therefore* $x \cup y$ *exists.* $x \cup y = [L6'] \ y = y \cup x = [L6'] \ x$.

*L6* *For any* $a \in \mathbf{L}$, *there is a unique* $0_a \in \mathbf{Z}$ *so that* $a \cup 0_a = a = 0_a \cup a$ *and* $a \cap 0_a = 0_a = 0_a \cap a$. [*L6'*, *D1*, *T1*]

*T2* *If* $x,y \in \mathbf{N}$, $a \in \mathbf{L}$, $a \cap x$ *and* $a \cap y$ *exist, then* $x = y$. [*L2*, *L7'*]

*L7* *For any* $a \in \mathbf{L}$, *there is a unique* $1_a \in \mathbf{N}$ *so that* $a \cap 1_a = a = 1_a \cap a$ *and* $a \cup 1_a = 1_a = 1_a \cup a$. [*L7'*, *D2*, *T2*]

*T3* *If* $a \cup b$ *exists, then* $0_a = 0_b$.

Proof. $a \cup b = [L6] \ a \cup (0_b \cup b) = [L2] \ (a \cup 0_b) \cup b$. Therefore $a \cup 0_b$ exists. By *L6*, $a \cup 0_a$ exists. Then by *T1*, $0_a = 0_b$.

*T4* *If* $a \cap b$ *exists, then* $1_a = 1_b$. [*L2*, *L7*, *T2*]

*T5* *If* $a \cap b$ *exists, then* $0_a = 0_{a \cap b}$.

*Proof. Since* $a \cap b$ *exists,* $0_{a \cap b} = [L6] \ 0_{a \cap b} \cap (a \cap b) = [L2] \ (0_{a \cap b} \cap a) \cap b = [L6] \ (a \cap 0_{a \cap b}) \cap b$. *Therefore* $a \cap 0_{a \cap b}$ *exists. By L6 and T1,* $0_a = 0_{a \cap b}$.

*T6* *If* $a \cup b$ *exists, then* $1_a = 1_{a \cup b}$. [*L2*, *T2*, *L7*]

*T7* *If* $z \in \mathbf{Z}$, *then* $0_z = z$.

Proof. By *D1*, $z \in \mathbf{L}$, so $z \cup 0_z$ exists by *L6*. By the last sentence of the proof of *T1*, $z = 0_z$.

*T8* *If* $n \varepsilon \mathbf{N}$, *then* $1_n = n$. [*D2*, *L7*, *T2*]

*T9* *If* $z \varepsilon \mathbf{Z}$ *and* $a \cap b = z$, *then* $z = 0_a$.

Proof. $0_a = [T5]\ 0_{a \cap b} = 0_z = [T7]\ z$.

*T10* *If* $n \varepsilon \mathbf{N}$ *and* $a \cup b = n$, *then* $n = 1_a$. [*T6*, *T8*]

*T11* *For any* $a \varepsilon \mathbf{L}$, *there is* $x \varepsilon \mathbf{L}$ *so that* $a \cap x = 0_a = x \cap a$ *and* $a \cup x = 1_a = x \cup a$. [*L8'*, *T9*, *T10*]

*T12* *If* $a \cup x = 1_a$, $a \cap x = 0_a$, $a \cup y = 1_a$, *and* $a \cap y = 0_a$, *then* $x = y$.

*Proof.* $x = [L7]\ x \cap 1_x = [T4]\ x \cap 1_a = x \cap (a \cup y) = [L5]\ (x \cap a) \cup (x \cap y) = [L6]\ 0_a \cup (x \cap y) = [T3]\ 0_x \cup (x \cap y) = [T5]\ x \cap y$. Interchanging $x$ and $y$ we also have $y = y \cap x$. So by *L1*, $x = y$.

*L8* *For any* $a \varepsilon \mathbf{L}$ *there is a unique* $a' \varepsilon \mathbf{L}$ *so that* $a \cap a' = 0_a = a' \cap a$ *and* $a \cup a' = 1_a = a' \cup a$. [*T11*, *T12*]

*T13* *If* $0_a = 0_b$, *then* $a \cap b$ *exists.*

*Proof.* $0_a = [L6]\ a \cap 0_a = a \cap 0_b = [L6]\ a \cap (b \cap 0_b) = [L2]\ (a \cap b) \cap 0_b$. Therefore $a \cap b$ exists.

*T14* *If* $1_a = 1_b$, *then* $a \cup b$ *exists.* [*L7*, *L2*]

*T15* $a \cup b$ *exists if and only if* $a \cap b$ *exists.*

Proof. If $a \cup b$ exists, then by *T3*, $0_a = 0_b$; consequently $a \cap b$ exists by *T13*. For the 'if' part, use *T4* and *T14*.

*T16* *If* $a \cup x$ *and* $a \cup y$ *exist, then* $x \cup y$ *exists.*

Proof. By *T3*, $0_a = 0_x$ and $0_a = 0_y$. Therefore by *T13* $x \cap y$ exists and consequently by *T15*, $x \cup y$ exists.

*T17* *If* $a \cap x$ *and* $a \cap y$ *exist, then* $x \cap y$ *exists.* [*T4*, *T14*, *T15*]

*Theorem 3. Any Boolean algebroid is the disjoint union of Boolean algebras.*

Proof. By *T15* and *T16* we have that a Boolean algebroid is the disjoint union of systems having the operations $\cup$ and $\cap$. By *L1* through *L8*, these systems are Boolean algebras.

Remark. The proof has been arranged so that *L1* for $\cup$, *L3*, and *L4* are not used; and *L1* for $\cap$, and *L5* are used only in the proof of the uniqueness of the complement.

## REFERENCE

[1] S. K. Sehgal: Jacobson Theory of Ringoids, *Notre Dame Journal of Formal Logic*, Vol. IV (1963), pp. 206-215.

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