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ON THE PROPOSITIONAL SYSTEM A OF VUČKOVIĆ AND ITS EXTENSION. I

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In [4], pp. 56-62, Goodstein defines in recursive arithmetic a model for the classical propositional calculus. And, in [3] he constructs in a similar way models for a certain class of the finite many-valued propositional calculi. No recursive model for Heyting's intuitionistic propositional calculus is known.

In [12] Vučković has constructed a formal system called the recursive arithmetic of words which differs from the ordinary recursive arithmetic, e.g., presented in [4], in this respect that instead of the usual successor function S in Vučković's system we have two or, even, arbitrary finite number of different successor functions: $S_0, S_1, \ldots, S_n$, for any $n \geq 0$. And, in [13] he defines in the field of his recursive system a model for a certain system of the propositional calculus which he calls system **A**. In the manner of Goodstein's [4] he associates with each propositional functor C, N, K and A a recursive function belonging to his arithmetic of words. Thus, having a propositional formula, say Γ, we obtain a recursive function γ replacing in Γ all occurrences of functors C, N, K and A by those functions associated with them. And, formula Γ can be considered as true, if the equation $\gamma = 0$ is provable in the recursive arithmetic of words. In [13] an axiom-system with suitable rules of procedure for system **A** is established. But, in connection with this, my paper, Prof. Vučković observed that it was stated there erroneously that all axioms and rules given in [13] are verified by his recursive model. In fact, this model verifies the rules of procedure and all axioms except the last, given below, axiom *A16*. In his paper, forthcoming in this Journal, Vučković will explain this error and present such modification of his recursive model that it will verify all axioms which he accepted. Contrary to the situation which we have in Goodstein's recursive system for the classical propositional calculus, in Vučković's system there are propositional formulas whose corresponding equations are verified by his model published in [13] and by his, yet unpublished, modified model, and which are not consequences of the axiom-system given in [13]. In constructing system **A** the author wished to obtain a recursive model for a system as similar to Heyting's system as possible. But, due to the peculiar

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properties of recursive arithmetic of words, system **A**, and even its part which is axiomatized in [13], neither contains nor is contained in Heyting's system. Thus, e.g. the intuitionistic thesis *CCpNpNp* is not true in **A**, and in the latter system the unintuitionistic thesis *ApCpq* is valid. On the other hand, e.g. the thesis *CNNpp* is not true in both systems.

In this paper I shall investigate the propositional calculus defined by the axiom-system and the rules of procedure given in [13] disregarding the problems connected with its model in the recursive arithmetic of words. First of all a simplification of the axiom-system given in [13] will be established. And later, I shall present a quite natural extension of propositional system **A** of Vučković. It will be shown that this extension which I call system $\mathcal{A}$ contains all theses verified by both, above mentioned, recursive models. Namely, I shall establish below such logical matrices for functors *C*, *N*, *K* and *A* occuring in system **A**, that they will be the characteristic matrices of system $\mathcal{A}$. A complete axiom-system of $\mathcal{A}$ will be given which together with the matrices will show that from the logical point of view system $\mathcal{A}$ is a weakly complete, partial three-valued propositional calculus with one designated value. Moreover, it will be proved that the addition of any well-formed propositional formula which is not valid in $\mathcal{A}$ to its complete axiom-system generates at least the bi-valued propositional calculus, i.e. that system $\mathcal{A}$ possesses the 3-rd degree of completeness[1].

As one could remark already, instead of the symbols used in [4] and [13] I am using the well-known Łukasiewicz's symbolism in this paper. The following abbreviations will be employed here: **A** means the propositional system of Vučković; $\mathcal{A}$—its extension discussed in this paper; **H**—the propositional system of Heyting; **P**—an arbitrary complete axiom system of positive logic; **C**—an arbitrary complete axiom system of the bi-valued implicational propositional calculus; {*C*}—formula {or - system, - axiom, etc}—a pure implicational propositional formula, similarly: {*C*;*N*}-, {*C*;*N*;*K*}-, a.s.o. Symbol $\vdash \alpha$ means always: formula α is a consequence of the axiom-system under consideration. The axiom-system of **A** given in [13] will be called **B1**.

1. Axiom-system **B1**. In [13], p. 71, the following axioms

A1 *ApCpq*
A2 *CpKpp*
A3 *CKpqKqp*
A4 *CCpqCKprKqr*
A5 *CKCpqCqrCpr*
A6 *CqCpq*
A7 *CKpCpqq*
A8 *CpApq*
A9 *CApqAqp*
A10 *CKCprCqrCApqr*
A11 *CNpCpq*
A12 *CKCNpqCNpNqNNp*
A13 *CKCNNpqCNNpNqNp*

A14 *CNKpqANpNq*
A15 *CNNApqANNpNNq*
A16 *CNNKpqKNNpNNq*

and the following rules of procedure:

RI Rule of substitution ordinarily used in the propositional calculus and adjusted to the primitive functors *C*, *N*, *K* and *A*.
RII Rule of detachment: *If* $\vdash \alpha$ *and* $\vdash C\alpha\beta$, *then* $\vdash \beta$.
RIII Rule of adjunction: *If* $\vdash \alpha$ *and* $\vdash \beta$, *then* $\vdash K\alpha\beta$.

are adopted as an axiom-system which I call **B1** of **A.** In [13], p. 77 and p. 83, Vučković mentions that he could not prove from **B1** the following theses

W1 *NNApNp*

and

W2 *NNCNNpp*

although they are verified by his recursive model. We shall see later that these theses are independent of **B1**.

It is obvious that the axioms *A2-A13*, *A16* and the theses *W1* and *W2* are intuitionistic theses provable in $\mathbf{H}^2$. On the other hand, axioms *A1*, *A14* and *A15* are clearly unintuitionistic formulas.

2. Simplification of **B1**. It will be shown here that in **B1** axiom *A12* is redundant and that **B1** is equivalent to the other, more simple and convenient axiom-system.

2.1 It is evident that **B1** is formulated in the manner of [5] in which the first axiom-system of **H** is published. And, it is well known that this axiom-system is equivalent to several other systems, cf., e.g., [9], [6], and [7], in which only two rules of procedure are accepted, viz. **RI** and **RII.** Since a proof that the first axiom-system of **H** established by Heyting yields all theses needed for a construction of the axiom-systems of **H** without rule **RIII** depends upon theses *A2-A7* which are also axioms of **B1,** we can, obviously, adopt instead of **B1** the following axiom-system which I call **B2**:

a) It has the rules of procedure **RI** and **RII**

b) The axioms *A1*, *A11*, *A14*, *A15*, *A16* and instead of *A2-A10*, *A12* and *A13*, we have the following ones

P i.e. an arbitrary axiom-system of the positive logic.

B1 *CKpqp*
B2 *CKpqq*
B3 *CpCqKpq*
B4 *CpApq*
B5 *CqApq*
B6 *CApqCCprCCqrr*
B7 *CCNpqCCNpNqNNp*
B8 *CCNNpqCCNNpNqNp*

It is evident that the equivalence of **B1** and **B2** can be proved in an elementary way.

2.2 Positive theses. For our further deductions we need several positive theses which I present here without a proof because **B2** contains **P**:

P1	*CCpqCCqrCpr*	[**P**]
P2	*CCpCqrCqCpr*	[**P**]
P3	*Cpp*	[**P**]
P4	*CCpqCCCprCCpsvCCqrCCqsv*	[**P**]
P5	*CCpqCCCprsCCqrs*	[**P**]
P6	*CCpqCCCrqsCCrps*	[**P**]
P7	*CCqrCCpqCpr*	[**P**]
P8	*CCsCpCqrCCCsrvCCsqCpv*	[**P**]
P9	*CCpCqrCCCsrvCCsqCpv*	[**P**]

2.3 Redundancy of *B7* in **B2**. It will be shown here that in **B2** axiom *B7* is superfluous. Besides, it will be proved that **B2** yields the theses *B9* and *B10*, given below, which we shall need in our further investigations. *Proof*: Let us assume **P**, *B8* and *A11*. Therefore, in virtue of **P** we have *P1-P6* from point **2.2**. Then:

Z1	*CCCpqrCNpr*	[*P1,p/Np,q/Cpq;A11*]
Z2	*CNNNpCCNNpNqNp*	[*Z1,p/NNp,r/CCNNpNqNp;B8*]
Z3	*CNNNpNp*	[*P2,p/NNNp,q/CNNpNNp,r/Np;Z2,q/Np;P3,p/NNp*]
B7	*CCNpqCCNpNqNNp*	[*P4,p/NNNp,q/Np,r/q,s/Nq,v/NNp;Z3;B8,p/Np*]
B9	*CCNNpNpNp*	[*P2,p/CNNpNp,q/CNNpNNp,r/Np;B8;P2,p/NNp*]
Z4	*CpCNpq*	[*P2,p/Np,q/p,r/q;A11*]
Z5	*CCCNpqrCpr*	[*P1,q/CNpq;Z4*]
Z6	*CCNpNNpNNp*	[*P5,p/NNNp,q/Np,r/NNp,s/NNp;Z3;B9,p/Np*]
Z7	*CpNNp*	[*Z5,q/NNp,r/NNp;Z6*]
B10	*CCNppNNp*	[*P6,q/NNp,r/Np,s/NNp;Z7;Z6*]

Thus, *B7* is redundant in **B2**. Since **P** can be obtained in the field of **B1** without the aid of *B7*, an analogous deduction to that given above shows that in that system *B7* is also superfluous.

2.4 Axiom-system **B3**. It will be proved that in **B2** (and also in **B1**) axiom *B8* can be substituted by a shorter thesis, viz. *B9* which is a consequence of **B2**, *cf.* point **2.3**. *Proof*: Assume **P**, *A11* and *B9*. Due to **P** we have *P1*, *P7*, *P8* and *P9*. Then:

Z1	*CCpNqCqCpr*	[*P1,p/CNqCqr,q/CCpNqCpCqr;P7,q/Nq,r/Cqr;P8,s/CpNq*]
Z2	*CCqCNNpNpCqNp*	[*P7,p/q,q/CNNpNp,r/Np;B9*]
Z3	*CCNNpNqCqNp*	[*P9,p/CNNpNq,q/CqCNNpNp,r/CqNp;Z1,p/NNp,r/Np;Z2*]
B8	*CCNNpqCCNNpNqNp*	[*P7,p/CNNpNq,r/Np,s/NNp,v/Np;Z3;B9*]

Thus, $\{$**P**;*A11*;*B9*$\} \rightarrow \{$*B8*$\}$. Therefore this proof together with the deductions given in **2.3** allows us to accept instead of **B2** a simpler axiom-system, viz. **B3** = $\{$**P**;*A1*;*A11*;*A14*;*A15*;*A16*;*B1-B6*;*B9*$\}$. We have to notice that in **B1** we can also substitute axiom *A13* by *B9* obtaining in this way **B1** = $\{$*A1-A11*;*B9*;*A14-A16*$\}$.

2.5 Axiom-system **B4**. Since, as it will be shown, **B3** contains the complete bi-valued $\{C\}$-calculus, we will be able to greatly simplify the axiom-system **B3**. Namely, we can substitute in **B3** the assumptions **P**, *A1* and *B9* by **C**, i.e. an arbitrary axiom-system of the complete, bi-valued $\{C\}$-calculus and the thesis *B10* which is proved already in **2.3**. *Proof*:

2.5.1 Let us assume **P**, *A1* and *B6*. Due to **P** we have *P3* given in **2.2**. Hence,

Z1 *CCCpqpp* [*B6,q/Cpq,r/p;A1;P3*]

Since **P** together with the law of Pierce, i.e. *Z1*, constitutes a complete axiom-system of the bi-valued $\{C\}$-calculus, we have proved that **B3** contains the latter system.

2.5.2 Now, let us assume **C**, *B4*, *B5* and *B10*. Obviously, in virtue of **C** we have the following $\{C\}$-theses

T1 *CCCpqqCCprCCqrr* [**C**]
T2 *CCpCpqCpq* [**C**]

and

T3 *CCCpqrCCrpp* [**C**]

Hence,

A1 *ApCpq* [*T1,q/Cpq,r/ApCpq;T2;B4,q/Cpq;B5,q/Cpq*]

and

B9 *CCNNpNpNp* [*T3,p/Np,q/p,r/NNp;B10*]

Thus, **C**, *B4*, *B5* and *B10* imply *A1* and *B9*. And, therefore, since, obviously, **C** yields the positive logic, the proof is complete.

2.5.3 Hence, in virtue of deductions given in two previous points, instead of **B3** we obtain a more simple axiom-system, viz. **B4** = {**C**;*A11*;*B1*-*B6*;*B10*;*A14*;*A15*;*A16*}.

3. The axiom-system **B4** together with the rules of procedure **RI** and **RII** determines a quite powerful propositional system. Consider, e.g., its subsystem constructed from the axioms **C**, *B1*-*B6*, *B10* and *A11*. In the so defined system, say **D**, we can, obviously, prove the theses

Q1 *CApqCCpqq*

and

Q2 *CCCpqqApq*

which allows us, clearly, to introduce into the system functor *A* by a definition

Df.1 *Apq* = *CCpqq*

and to drop the axioms *B4*, *B5* and *B6* at once. Moreover, in **D** we can define a new kind of negation, viz. N_1, as follows

Df.2 $N_1p = CpNp$

and later easily prove the following formulas

Q3 CCN_1ppp
Q4 $CpCN_1pq$
Q5 CN_1CpN_1qKpq
Q6 $CKpqN_1CpN_1q$

Hence, in virtue of **C**, *Q3* and *Q4*,

(i) we know that there is a subsystem of **D** which is isomorphic to the complete system of bi-valued propositional calculus,

and, due to (i), *Q5* and *Q6*,

(ii) we are allowed to introduce a definition

Df.3 $Kpq = N_1CpN_1q$

and to drop the axioms *B1*, *B2* and *B3*.

Thus, system **D** can be based on the following set of assumptions {**C**;*A11*;*B10*}. It could be of interest because it shows some similarity to modal systems. I shall not here discuss this point. On the other hand, it is evident that **D** is a proper subset of the classical propositional calculus. But the fact that in **D** it is possible to define functors *A* and *K* does not prove that these functors are definable in **B4** or even in the full system **A**. Since in **A** the formula *CCpqCNqNp* is not valid, we are unable to obtain formulas containing functors *A* and *K* under *N* from **D**. Thus, e.g., formula *CNApqNp* is not a thesis of **A**, although $CNApqN_1p$ is provable in **D**. The characteristic matrices of system $\mathcal{A}$ which will be established below explain this situation.

It should be noticed that since **B4** contains **C** and the axioms *B1*, *B2* and *B3*, in virtue of my proof published in [10], every classical {*C*;*K*}-thesis is a consequence of **B4**. Hence, **B4** contains the bi-valued {*C*;*K*}-propositional calculus.

4. An extension of system **A**. An analysis, presented above, of the axiom-system **B1** of **A** allowed us:

1) to simplify greatly this axiom-system, viz. instead of **B1** we were able to accept **B4**,

2) to show that **A** contains the complete, bi-valued {*C*;*K*}-system what is not mentioned in [13].

and

3) to indicate rather peculiar properties of a subsystem of **A**, viz. of system **D**.

On the other hand it is evident and which will be proved later that several propositional formulas verified by the recursive models constructed by Vučković for system **A** are not the consequences of **B4**. In order to clarify this situation I shall present here such possible extension of **A** that it will satisfy the following conditions:

a) it will contain **B4** and every propositional formula verified by recursive model given in [13].
b) it will not contain the formulas clearly rejected in [13].
c) It will be weakly complete partial many-valued propositional calculus.

For this end I have established for functors C, N, K and A the following three-valued matrices:

$\mathfrak{M}1$

	C	1	2	3
	1	3	3	3
	2	3	3	3
*	3	1	2	3

$\mathfrak{M}2$

	p	Np
	1	2
	2	3
*	3	2

$\mathfrak{M}3$

	K	1	2	3
	1	1	2	1
	2	2	2	2
*	3	1	2	3

$\mathfrak{M}4$

	A	1	2	3
	1	2	1	3
	2	1	2	3
*	3	3	3	3

in each of which 3 is the designated value.

It is easy to check that these matrices verify the rules **RI** and **RII**, the axioms of **B4** and the theses $W1$ and $W2$. On the other hand they falsify e.g.: $CCpqCNqNp$, viz. for $p/1$, $q/2$: $CC12CN2N1 = C3C32 = C32 = 2$; $ApNp$, viz. for $p/1$: $A1N1 = A12 = 1$; $CNApqNp$, viz. for $p/1$, $q/1$: $CNA11N1 = C32 = 2$, a.s.o.. Therefore, since the three-valued matrices $\mathfrak{M}1$-$\mathfrak{M}4$ define the bi-valued functors C, N, K and A from which certain properties of the bi-valueness are removed, system $\mathcal{A}$ determined by these matrices is a partial system of the three-valued propositional calculus with one designated value, And, since no set of the bi-valued matrices is sufficient to define such properties of the considered functors which are required in [13], e.g. to verify the axioms of **B4**, and in the same time to falsify for instance $CNNpp$ and $CCpNpNp$, matrices $\mathfrak{M}1$-$\mathfrak{M}4$ are the smallest matrices satisfying the, given above, conditions. It should be here stressed that constructing these matrices I disregarded entirely a problem whether these matrices are verified by recursive model presented in [13].

It will be shown below that system $\mathcal{A}$ defined by matrices $\mathfrak{M}1$-$\mathfrak{M}4$ possesses a finite axiom-system. For this reason we can consider matrices $\mathfrak{M}1$-$\mathfrak{M}4$ as the characteristic matrices of $\mathcal{A}$.

5. Axiomatization of $\mathcal{A}$. I shall prove in **6** that any thesis verified by the characteristic matrices $\mathfrak{M}1$-$\mathfrak{M}4$ of system $\mathcal{A}$ is a consequence of the following mutually independent axioms

(i) $\{C\}$-axiom:

$F1$ $CCCpqrCCrpCsp$ [$F1$ is Łukasiewicz's single axiom of $\{C\}$-system, *cf.* [8]]

(ii) $\{C;N\}$-axioms:

$F2$ $CNpCpq$ [$F2$ is our previous $A11$]
$F3$ $CCNppNNp$ [$F3$ is our previous $B10$]
$F4$ $CpCNqNCpq$ [$\mathfrak{M}1$ and $\mathfrak{M}2$ verify $F4$]
$F5$ $CNCpqNq$ [$\mathfrak{M}1$ and $\mathfrak{M}2$ verify $F5$]

(iii) $\{C;N;K\}$-axioms:

$F6$ $CKpqp$ [$F6$ is our previous $B1$]

F7	*CKpqq*	[*F7* is our previous *B2*]
F8	*CpCqKpq*	[*F8* is our previous *B3*]
F9	*CNpNKpq*	[𝔐1-𝔐3 verify *F9*]
F10	*CNqNKpq*	[𝔐1-𝔐3 verify *F10*]
F11	*CNNpCNNqNNKpq*	[𝔐1-𝔐3 verify *F11*]

(iv) $\{C;N;A\}$-axioms:

F12	*CpApq*	[*F12* is our previous *B4*]
F13	*CqApq*	[*F13* is our previous *B5*]
F14	*CApqCCprCCqrr*	[*F14* is our previous *B6*]
F15	*CNApqCNpNq*	[𝔐1, 𝔐2 and 𝔐4 verify *F15*]
F16	*CNApqCNqNp*	[𝔐1, 𝔐2 and 𝔐4 verify *F16*]
F17	*CNpCNqNApq*	[𝔐1, 𝔐2 and 𝔐4 verify *F17*]
F18	*CCpNpCCqNqCNNpCNNqNApq*	[𝔐1, 𝔐2 and 𝔐4 verify *F18*]

taken together with the rules of procedure **RI** and **RII.**

It can be easily observed that the axioms *F1-F3*, *F6-F8* and *F12-F14* constitute system **D**, and that in the field of **B4** we are able to prove without any difficulty that $\{A14\} \rightleftarrows \{F11\}$, $\{A15\} \rightleftarrows \{F17\}$ and $\{A16\} \rightleftarrows \{F9;F10\}$. On the other hand it has to be noticed that the axioms *F4*, *F5*, *F15*, *F16* and *F18* are not the theses of **B4.**

5.1 The proof of axiomatization of system $\mathcal{A}$ requires several theses which will be deduced here from the axioms *F1-F18*. Since we have *F1*, *F6*, *F7* and *F8*, in virtue of [8] and [10], we can assume any $\{C,K\}$-thesis without a proof. The theses used in the investigations will be marked by asterisk.

**F19*	*Cpp*	[*F1*]
**F20*	*CpCqp*	[*F1*]
**F21*	*CqCpp*	[*F1*]
**F22*	*CpCqCrp*	[*F1*]
F23	*CpCCpCqrCCrsCCsvCqv*	[*F1*]
**F24*	*CCpqCCqrCpr*	[*F1*]
**F25*	*CCqrCCpqCpr*	[*F1*]
F26	*CCprCCqrCCCpsqr*	[*F1*]
F27	*CCpqCCqCrsCrCps*	[*F1*]
**F28*	*CCqrCCpCsqCpCsr*	[*F1*]
F29	*CCpqCCCprsCCqrs*	[*F1*]
**F30*	*CCpqCCprCCqCrsCps*	[*F1*]
**F31*	*CCrsCCCqvCpsCCCpqrs*	[*F1*]
F32	*CCprCCqrCCrsCCCpvqCts*	[*F1*]
**F33*	*CCpqCCprCCpsCCptCCqCrCsCtvCpv*	[*F1*]
**F34*	*CCqpCCqtCCqsCCqvCCpCtCsCvrCqr*	[*F1*]
**F35*	*CCqrCCpCtCsCvqCpCtCsCvr*	[*F1*]
F36	*CCpvCCqvCCCprCCqrsCCvrs*	[*F1*]
**F37*	*CCrsCCCqvsCCCptsCCpCqrs*	[*F1*]
**F38*	*CCpCpqCpq*	[*F1*]
**F39*	*CCpCqrCqCpr*	[*F1*]

F40 CCpCqrCCrsCpCqs [F1]
F41 CCpCqrCsCvCqCpr [F1]
F42 CCpCqrCCspCCsqCsr [F1]
F43 CCpCqrCCsqCCrvCpCsv [F1]
F44 CCpCqrCCCprCsvCqCsv [F1]
F45 CCpCqrCCCprCsvCsCqv [F1]
F46 CCpCqrCCCsvCqrCCspCqr [F1]
F47 CCpCqrCCsCvpCCsCvqCsCvr [F1]
F48 CCpCqrCCsCvCtpCCsCvCtqCsCvCtr [F1]
F49 CCpCqCrsCqCrCps [F1]
F50 CCpCqCrsCrCpCqs [F1]
F51 CCpCqCrsCpCCvqCrCvs [F1]
F52 CCpCqCrsCCsvCCCrvtCpCqt [F1]
F53 CCpCqCrCstCCCqtCvzCvCrCsCpz [F1]
F54 CCCpqpp [F1]
*F55 CCCpqrCpr [F1]
F56 CCCpqrCCrpp [F1]
*F57 CCCpqrCCsrCCpsr [F1]
F58 CCCpCqrsCrs [F1]
*F59 CCCpCqrsCCprs [F1]
F60 CCCprCtsCtCCqrCCpqs [F1]
*F61 CCCCpqrsCCqrCps [F1]
F62 CpCNpq [F39,p/Np,q/p,r/q;F2]
F63 CrCsCNpCpq [F22,p/CNpCpq,q/r,r/s;F2]
F64 CrCsCpCNpq [F22,p/CpCNpq,q/r,r/s;F62]
*F65 CCpNpCpq [F46,p/Np,q/p,r/q,s/p,v/Cpq;F2;F38]
*F66 CCCpqrCCpNpr [F24,p/CpNp,q/Cpq;F65]
F67 CCpNpCrCpq [F66,r/CrCpq;F20,p/Cpq,q/r]
F68 CCpqCNqCpr [F57,q/Np,r/CNqCpr,s/q;F67,q/r,r/Nq;F62,p/q,q/Cpr]
*F69 CCCpqrCNpr [F24,p/Np,q/Cpq;F2]
*F70 CNCpqp [F69,p/Cpq,q/p,r/p;F54]
F71 CCCpNpqCNqp [F52,p/CCpNpq,q/Nq,r/CpNp,s/NCpNp,t/p,v/p; F68,p/CpNp,r/NCpNp;F70,q/Np;F54,q/Np]
*F72 CpNNp [F55,p/Np,q/p,r/NNp;F3]
*F73 NNCpp [F72,p/Cpp;F19]
*F74 CqNNCpp [F20,p/NNCpp;F73]
F75 CCpNpNNCpq [F24,p/CpNp,q/Cpq,r/NNCpq;F65;F72,p/Cpq]
F76 CCNNpNpNp [F56,p/Np,q/p,r/NNp;F3]
*F77 CNNNpNp [F69,p/NNp,q/Np,r/Np;F76]
F78 CCNprCCNNprr [F23,p/CCNNpNpNp,q/CNpr,r/CCNNpNpr, s/CCrNNpNNp,v/CCNNprr;F76;F24,p/CNNpNp,q/Np; F56,p/NNp,q/Np;F56,p/r,q/NNp,r/NNp]
*F79 CCNNprCCqrCCNpqr [F60,p/Np,s/r,t/CNNpr;F78]
F80 CCNprCCqrCCNNpqr [F29,p/NNNp,q/Np,s/CCqrCCNNpqr; F77;F79,p/Np]
F81 CCsCNNprCCsCqrCsCCNpqr [F42,p/CNNpr,q/Cqr,r/CCNpqr;F79]
F82 CCNpqCNqNNp [F51,p/CNNpNNp,q/CqNNp,r/CNpq,s/NNp,v/Nq; F79,r/NNp;F19,p/NNp;F2,p/q,q/NNp]

F83 *CCNNpNNqCNqNp* [*F43,p/CNNpNNq,q/NNNq,r/NNNp,s/Nq,v/Np; F82,p/Np,q/NNq;F72,p/Nq;F77*]

**F84* *CNNCpqCpNNq* [*F27,q/CNqNCpq,r/NNCpq,s/NNq;F4;F82,p/q,q/NCpq*]

F85 *CNNqNNCpq* [*F82,p/Cpq,q/Nq;F5*]

**F86* *CCpNNqNNCpq* [*F57,q/Np,r/NNCpq,s/NNq;F75;F85*]

F87 *NNCNNpp* [*F86,p/NNp,q/p;F19,p/NNp*]

**F88* *CCpCqrCKpqr* [*F1;F6;F7*]

**F89* *CCKpqrCpCqr* [*F1;F8*]

**F90* *CCCKpqrsCCpCqrs* [*F1;F6*]

**F91* *CCCpCqrsCCKpqrs* [*F1;F8*]

**F92* *CCpKqrCpq* [*F1;F6*]

**F93* *CCpKqrCpr* [*F1;F7*]

**F94* *CCpqCCprCpKqr* [*F1;F8*]

**F95* *CCNNpNqNKpq* [*F80,q/Nq,r/NKpq;F9;F10*]

**F96* *CNKpqCNNpNq* [*F27,p/NNp,q/CNNqNNKpq,r/NKpq,s/Nq; F11;F83;p/q,q/Kpq*]

**F97* *CNNKpqNNp* [*F82,q/NKpq;F9*]

**F98* *CNNKpqNNq* [*F82,p/q,q/NKpq;F10*]

**F99* *CNNKpqKNNpNNq* [*F94,p/NNKpq,q/NNp,r/NNq;F97;F98*]

**F100* *CKNNpNNqNNKpq* [*F88,p/NNp,q/NNq,r/NNKpq;F11*]

F101 *CNKpqCCNpNqNq* [*F27,p/NKpq,q/CNNpNq,r/CNqNq,s/CCNpNqNq; F96;F79,q/Nq,r/Nq;F19,p/Nq*]

**F102* *CNApqCpr* [*F68,q/Apq;F12*]

**F103* *CCApqrCpr* [*F24,p/Apq;F12*]

**F104* *CNApqCqr* [*F68,p/q,q/Apq;F13*]

**F105* *CCApqrCqr* [*F24,p/q,q/Apq;F13*]

**F106* *CCCpsqApq* [*F26,r/Apq;F12;F13*]

**F107* *CCCprCCqrsCCApqrs* [*F36,v/Apq;F12;F13*]

F108 *CCCpvqCtNNApq* [*F32,r/Apq,s/NNApq;F12;F13;F72,p/NNApq*]

**F109* *CApqCCpqq* [*F50,p/Apq,q/Cpq,r/Cqq,s/q;F14,r/q;F19,p/q*]

**F110* *CCprCCqrCApqr* [*F49,p/Apq,q/Cpr,r/Cqr,s/r;F14*]

**F111* *CCCApqrsCCprCCqrs* [*F40,p/Cpr,q/Cqr,r/CApqr;F110*]

F112 *CNpCNNqNNApq* [*F44,p/NApq,q/Np,r/Nq,s/NNq,v/NNApq; F15;F82,p/Apq,q/Nq*]

F113 *CNqCNNpNNApq* [*F44,p/NApq,q/Nq,r/Np,s/NNp,v/NNApq; F16;F82,p/Apq,q/Np*]

F114 *CNqCCNqNNpNNApq* [*F81,p/q,q/NNp,r/NNApq,s/Nq; F62,p/Nq,q/NNApq;F113*]

F115 *CCNNqCCpvqCCNqNNpNNApq* [*F80,p/q,q/CCpvq,r/CCNqNNpNNApq; F114;F108,t/CNqNNp*]

F116 *CrCsCNqCNpNApq* [*F41,p/Np,q/Nq,r/NApq,s/r,v/s;F17*]

F117 *CrCsCCNpNqCNpNApq* [*F47,p/CNNpCNpNApq,q/CNqCNpNApq, r/CCNpNqCNpNApq,s/r,v/s;F79,q/Nq,r/CNpNApq; F63,p/Np,q/NApq;F116*]

**F118* *CNNApqCNqNNp* [*F45,p/Np,q/Nq,r/NApq,s/NNApq,v/NNp; F17;F82,q/NApq*]

F119 CNpCCNqNNpNNApq [*F81,p/q,q/NNp,r/NNApq,s/Np; F112;F62,p/Np,q/NNApq*]

**F120 CCNNpCNNqCCpvqCCNqNNpNNApq* [*F80,q/CNNqCCpvq, r/CCNqNNpNNApq;F119;F115*]

**F121 CNNApqCNNpCNNqCCpNpq* [*F53,p/CpNp,q/CqNq,r/NNp,s/NNq, t/NApq,v/NNApq,z/q;F18;F71,p/q,q/NApq*]

F122 CCpNpCCqNqCCNpNqCNNqNApq [*F47,p/CNNpCNNqNApq, q/CNqCNNqNApq,r/CCNpNqCNNqNApq,s/CpNp,v/CqNq;F79,q/Nq, r/CNNqNApq;F18;F64,p/Nq,q/NApq,r/CpNp,s/CqNq*]

**F123 CCpNpCCqNqCCNpNqCCNqNpNApq* [*F48,p/CNNqNApq,q/CNpNApq, r/CCNqNpNApq,s/CpNp,v/CqNq,t/CNpNq;F78,p/q,q/Np,r/NApq; F122;F117,r/CpNp,s/CqNq*]

F124 CNNApqCNNNpNNq [*F24,p/NNApq,q/CNqNNp,r/CNNNpNNq; F118;F82,p/q,q/NNp*]

F125 CNNApqCCNNpNNqNNq [*F81,p/Np,q/NNq,r/NNq,s/NNApq; F124;F21,p/NNq,q/NNApq*]

F126 CNNApqANNpNNq [*F24,p/NNApq,q/CCNNpNNqNNq,r/ANNpNNq; F125;F106,p/NNp,q/NNq*]

F127 NNApNp [*F120,q/Np;F62,p/NNp,q/CCpvNp;F19,p/NNp*]

F128 CNKpqANpNq [*F24,p/NKpq,q/CCNpNqNq, r/ANpNq; F101;F106,p/Np,q/Nq*]

It should be noticed that it has shown here that the axioms *F1-F18* imply not only the theses marked by asterisks, but also *F128*, *F126*, *F99*, *F127* and *F87*, i.e. Vučković's axioms *A14*, *A15*, *A16* and the theses *W1* and *W2* respectively. Thus, we have proved that $\mathcal{A}$ contains **B4.**

5.2 Besides the theses which are proved in **5.1** and are there marked by the asterisks we have to establish four simple metarules which will be used in our further discussion. Namely, the inductive reasonings show without any difficulty that in $\mathcal{A}$, i.e. in the system generated by the axioms *F1-F18* taken together with the rules **RI** and **RII**, for any formulas which are accepted as the theses and have the structures indicated in the contents of the, given below, metatheorems, the following forms of deduction:

MRI *For any* $n \geq 1$, $\{C\alpha_1 C\alpha_2 C\ldots C\alpha_{n-1} C\alpha_n \beta\} \rightleftarrows \{C\alpha_n C\alpha_1 C\alpha_2 C\ldots C\alpha_{n-1}\beta\}$

[Proof by *F39*, *F25* and induction]

MRII *If* $\vdash C\beta\gamma$ *and* $\vdash C\gamma\beta$, *then, for any* $n \geq 1$, $\{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \beta\} \rightleftarrows \{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \gamma\}$ [Proof by *F24*, *F25* and induction]

MRIII *If* $\vdash C\beta\gamma$, $\vdash C\beta\delta$ *and* $\vdash C\gamma C\delta\beta$, *then, for any* $n \geq 1$, $\{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \beta\} \rightleftarrows \{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \gamma; C\alpha_1 C\alpha_2 C\ldots C\alpha_n \delta\}$

[Proof by *F24, F30, F25* and induction]

MRIV *If* $\vdash C\beta\gamma$, $\vdash C\beta\delta$, $\vdash C\beta\epsilon$, $\vdash C\beta\zeta$ *and* $\vdash C\gamma C\delta C\epsilon C\zeta\beta$ *then, for any* $n \geq 1$, $\{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \beta\} \rightleftarrows \{C\alpha_1 C\alpha_2 C\ldots C\alpha_n \gamma;\ C\alpha_1 C\alpha_2 C\ldots C\alpha_n \delta;\ C\alpha_1 C\alpha_2 C\ldots C\alpha_n \epsilon\};\ C\alpha_1 C\alpha_2 C\ldots C\alpha_n \zeta\}$ [Proof by *F24, F33, F24* and induction]

hold.

NOTES

1. A notion "the degree of completeness" is defined in [11], definition 5, p. 35.

2. Concerning a provability of *W1* and *W2* in Heyting's system, *cf.* [1] and [2]. Also, [9], p. 58.

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To be continued.

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