Notre Dame Journal of Formal Logic
Volume V, Number 2, April 1964

THREE AXIOM NEGATION-ALTERNATION FORMULATIONS OF THE TRUTH-FUNCTIONAL CALCULUS

GEORGE GOE*

There are numerous formulations of the calculus of truth functions (so-called propositional logic) of the traditional axioms-plus-rules-of-derivations type in current use. Yet each has a combination of features of its own which gives it certain advantages for particular purposes, if only didactical, or simply makes it the favorite of some logicians as a matter of personal taste. It may still be of service, therefore, to record some more such systems which might be profitably used. The purpose of this communication is to note the existence of very simple such logistic formulations of the truth-functional calculus in which alternation is the primitive binary connective, but which, unlike the familiar Hilbert-Ackermann system, have only three axioms (or axiom-schemata).

There are several negation-alternation primitive bases for the truth-functional calculus with three or fewer axioms already recorded in print, but they are hardly, if ever used; apparently, it is felt that objectionable features in them make the reduction of the number of axioms from the Hilbert-Ackermann system, or the retention of alternation as the primitive binary connective, for whatever merit is seen in it, not worthwhile.[1] A diligent search has failed to reveal that any of the systems to be presented here have been proposed before; they are all very similar to each other, and we will hence treat one, which we will refer to as the system Σ_0, as basic and consider the others as variations of it.

We will use familiar vocabulary and formation rules for the object language, with '$\sim$' and '$\vee$' as our primitive connectives; for the abbreviation of wffs, besides the omission of parentheses, we will have occasion to employ only '$\supset$' as a defined connective in the usual manner.

As is the case for all such logistic systems, there are of course two versions of Σ_0 (and of each of its variations), namely with a finite or an infinite axiom set respectively. For the purpose of this presentation we adopt a finite axiom set—there is no intent thereby to express a preference for this approach over the one using axiom schemata in all contexts. Our rules of derivation then are the usual *substitution* and *modus ponens*. We will, of

*The preparation of this paper has been partially supported by a research grant of the Rensselaer Polytechnic Institute.

Received March 29, 1963

course, use simultaneous substitution and other customary abbreviated procedures to indicate how the proofs are constructed rather than record the latter line by line.

The axioms of Σ_0 are:

$A_0 1$: $p \supset (q \vee p)$

$A_0 2$: $(p \vee (p \vee q)) \supset (q \vee p)$

$A_0 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

The first noteworthy point is that '$p \vee \sim p$' can be proved from $A_0 1$ and $A_0 2$ alone, namely by a single application of modus ponens after substituting '$\sim p$' for 'q' in the first, and '$\sim p$' for 'p', 'p' for 'q' in the second. We therefore record as our first proved theorem:

T1: $p \vee \sim p$

In view of $A_0 3$, the rule of hypothetical syllogism clearly holds in our system as a derived rule of derivation, and we may use it in recording the proofs in abbreviated form.

By hypothetical syllogism from $A_0 2$ and the result of substituting '$p \vee q$' for 'p' and 'p' for 'q' in $A_0 1$, we derive:

T2: $(p \vee q) \supset (q \vee p)$

From *T1* and *T2*, by substitution and modus ponens, we derive '$\sim p \vee p$', or, more briefly:

T3: $p \supset p$

By substitution and hypothetical syllogism from $A_0 1$ and *T2* we obtain:

T4: $p \supset (p \vee q)$

Notice the leeway that we have so far had in the order in which we could have recorded the proved theorems. We could now also very simply prove such basic theorems as '$\sim \sim p \supset p$' and '$(p \supset q) \supset (\sim q \supset \sim p)$', but being here primarily interested in establishing the completeness of our system rapidly, we will rather follow another path.

From *T4* and $A_0 3$, after the proper substitutions in the latter, we derive, by modus ponens:

$$(p \vee p) \supset (p \vee (p \vee q)),$$

From this formula and $A_0 2$, by hypothetical syllogism we obtain:

$$(p \vee p) \supset (q \vee p)$$

We combine once more this result with that of the proper substitutions in $A_0 3$ to derive, by modus ponens:

$$(q \vee (p \vee p)) \supset (q \vee (q \vee p))$$

Using hypothetical syllogism again on this last wff and the result of the proper substitutions in $A_0 2$, we next obtain:

$$(q \vee (p \vee p)) \supset (p \vee q)$$

We now substitute '$\sim(p \vee p)$' for 'q' in the conditional just derived, and then drop the antecedent, which is derivable by substitution from *T3*. We are left with '$p \vee \sim(p \vee p)$', and a final application of modus ponens on this last wff and the result of the proper substitutions in *T2* gives us '$\sim(p \vee p) \vee p$', or, more briefly:

T5: $(p \vee p) \supset p$

It is now clear that our system is complete, since *T5, T4, T2* and $A_0 3$ are the axioms of the Hilbert-Ackermann system, which is known to be complete.

The axioms in our system are independent. We can establish the independence of $A_0 1$, $A_0 2$, and $A_0 3$ in Σ_0 by means of the same matrices which serve to show the independence of the second, third, and fourth axioms respectively in the Hilbert-Ackermann system.

We will now consider possible variations of our system Σ_0, keeping the rules of derivation fixed, and modifying only its axioms, really only the first two.

The system Σ_1 has the following axioms:

$A_1 1$: $p \supset (p \vee q)$

$A_1 2$: $(q \vee (p \vee q)) \supset (p \vee q)$

$A_1 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

As in our first system, '$p \vee \sim p$' can be proved in Σ_1 from the first two axioms alone, by modus ponens, after substituting '$\sim p$' for 'q' in both of them, but the proof of a variant of the third axiom of *Principia* is slightly longer than before: we must first apply modus ponens to $A_1 1$ and the result of substitutions in $A_1 3$ to obtain '$(q \vee p) \supset (q \vee (p \vee q))$', and then use hypothetical syllogism on this wff and $A_1 2$. For the rest, the development of Σ_1 closely parallels that of Σ_0, though the *unabbreviated* proof of '$(p \vee p) \supset p$' is yet another couple of lines longer in it than in Σ_0. These are minor differences, and one might conceivably feel that in some respects Σ_1 is the more elegant of two systems.

For the record, we note the following other variations of our original system which preserve its completeness, displaying only their axioms, for the rules of derivation remain the same:

Σ_2:

$A_2 1$: $p \supset (p \vee q)$

$A_2 2$: $(p \vee (p \vee q)) \supset (q \vee p)$

$A_2 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

Σ_3:

$A_3 1$: $p \supset (q \vee p)$

$A_3 2$: $(p \vee (q \vee p)) \supset (p \vee q)$

$A_3 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

Σ_4:

$A_4 1$: $p \supset (p \vee q)$

$A_4 2$: $((q \vee p) \vee q) \supset (p \vee q)$

$A_4 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

Σ_5:

$A_5 1$: $p \supset (p \vee q)$

$A_5 2$: $((p \vee q) \vee q) \supset (q \vee p)$

$A_5 3$: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$

It appears to this writer that the axioms in at least some of these last systems are in themselves somewhat less elegant or simple sounding than those in Σ_0 or Σ_1. All of these last four systems lack the elegant feature, present in Σ_0 and Σ_1, of permitting '$p \vee \sim p$' to be proved from the first two axioms alone, and there is no compensatory feature for this loss in their development: in each of them, though in different degrees, it takes more steps to prove all theorems which are not axioms and that we recorded as proved theorems in Σ_0 that were needed to prove our theorems in that system. It seems therefore that there is no reason why one would prefer Σ_2, Σ_3, Σ_4, or Σ_5 to either Σ_0 or Σ_1.

NOTE

1. In the 2nd edition of A. N. Prior's *Formal Logic* (Oxford: At the Clarendon Press, 1962), p. 305, are listed five such three-axiom negation-alternation systems, respectively labeled as (a), (b), (c), (d), and (e); it appears that these are all those recorded to date. As for the probable reasons for their non-adoption: in (a) and (e) the axioms are intricate, with three distinct statement letters in each; the axioms in (c) and (d), while not actually longer than the first, second, and fourth axiom of the Hilbert-Ackermann system respectively, have a certain artificial complexity which contrasts unfavorably with the latter's naturalness and intuitiveness, or with the magnificent simplicity of the Lukasiewicz three-axiom negation-conditional system to which especially (d) resembles; (b) may possibly be regarded as free of the latter defect, but the early proofs in the development of the calculus in this system are excessively elaborate (cf. H. Rasiowa, "Sur un certain système d'axiomes du calcul des propositions" [*Norsk Mat. Tidskrift*, v. 31, pp. 1-3 (1949)], in which the completeness of the system was first proved, or A. Church, *Introduction to Mathematical Logic* [Princeton, N. J., 1956], p. 138, ex. 25.5).

   Prior also lists two-axiom and one-axiom negation-alternation systems for the truth-functional calculus (*loc. cit.*), all due to C. R. Meredith. The axioms, of course, become even more complex with the further reduction in their number, though substitution and modus ponens are still the rules of derivation.

Rensselaer Polytechnic Institute
Troy, New York