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TWO REMARKS CONCERNING MENGER'S AND SCHULTZ' POSTULATES FOR THE SUBSTITUTIVE ALGEBRA OF THE 2-PLACE FUNCTORS IN THE 2-VALUED CALCULUS OF PROPOSITIONS.

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We denote the sixteen 2-place functors of the 2-valued calculus of propositions by the symbols introduced by Menger:[1]

$$1, A, B, C, D, E, I, J, 1' \ (=0),\ A', B', C', D', E', I', J'.$$

A is incompatibility, B is disjunction, C is implication, D is the converse implication, E is equivalence, I and J are the selectors, 1 is the constant functor of value 1. Primes indicate negation. The two hundred fifty-six 2-place transformations (i.e., ordered pairs of 2-place functors) constitute a semigroup which is isomorphic to the semigroup of all functions mapping $\{1,2,3,4\}$ into $\{1,2,3,4\}$.

Menger and Schultz show that this semigroup can be generated by the three transformations

$$h = (J', I'),\ e = (I', E),\ a = (A, I')$$

The first two transformations, by themselves, generate a subgroup isomorphic to the symmetric group on four elements.

Our first remark is that while h, e, and a are in terms of four functors (viz., I', J', E, A) the semigroup of the 2-place transformations in terms of only three 2-place functors, e.g. by either one of the following triples of transformations in terms of I', E, A.

$$e = (I',E),\ c = (E,I'),\ a = (A,I') \ \text{or}\ e = (I',E),\ c = (E,I'),\ b = (A,E).$$

All that has to be shown in order to prove the first contention is that h can be expressed in terms of e and c. But this is the case since $h = e\ e\ c\ e$. The second contention can be established by the following expression of a in terms of e, c, and b:

$$a = e\ c\ b\ c\ c$$

It should be noted that

$$h\ h\ h = h,\ c\ c\ c\ c = c,\ e\ e\ e\ e\ e = e.$$

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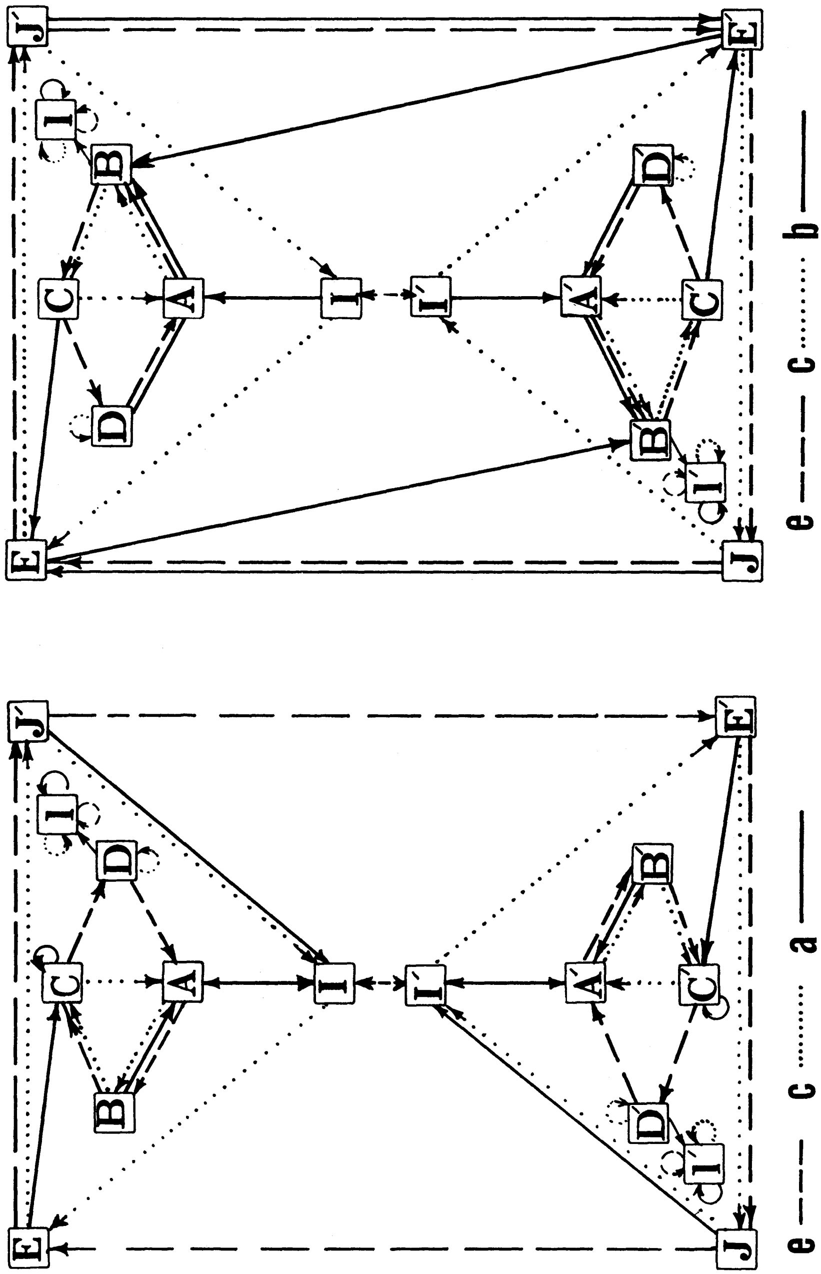

Figure 1

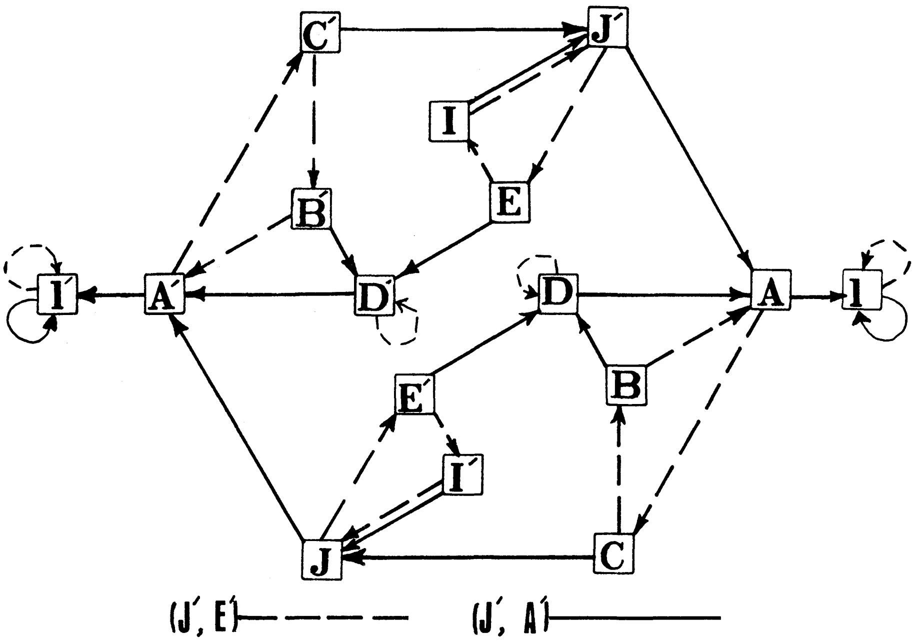

Figure 2

The result of substituting e, c, a, b, and two other transformations to be used later into the sixteen functors are tabulated below:

	e	c	a	b		
	(I',E)	(E,I')	(A,I')	(A,E)	(J',E')	(J',A')
A	B	B	I	B	C	1
B	C	C	A	1	A	D
C	D	A	C	E	B	J
D	A	D	1	A	D	A
E	J'	J'	C	B'	I	D'
I	I'	E	A	A	J'	J'
J	E	I'	I'	E	E'	A'
1	1	1	1	1	1	1

The remaining entries of the above table are given by the formula:

$$X'(Y,Z) = (X(Y,Z)').$$

The table may be represented by the following graphs (see p. 126) in which dashed and dotted arrows indicate the substitution of e and c, while the solid arrows indicate substitution of a in the first graph, and of b in the second.

Note that the first graph is drawn completely in the plane while the second cannot be drawn in the plane. This can be seen by considering the points E, I, E' and J, B, J'. Each point in the first group is joined to each point in the second group by broken non-intersecting lines. This cannot happen in the plane.

Our second remark is that it is possible to obtain every functor from every nonconstant functor by a chain formed of two transformations which do not generate the full semigroup of all transformations. An example is the pair of transformations:

$$(J', E') \qquad \text{and} \qquad (J', A')$$

Which is in terms of the three functors J', E', A'. As can be seen from the preceding table, the result may be represented by the graph on page 127.

Note: (J', E') and (J', A') generate 145 out of the 256 transformations in the semigroup.

BIBLIOGRAPHY

[1] K. Menger and M. Schultz, Postulates for the Substitutive Algebra of the 2-place Functors in the 2-valued Calculus of Propositions. *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 188-192. Professor Menger has asked me to point out that an error was made in the diagram on p. 190. Since $Ja = I'$ (and not E) and $J'a = I$ (and not E') the solid arrows from J to E and from J' to E' should be deleted, and solid arrows from J to I', and from J' to I should be added.

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