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Notre Dame Journal of Formal Logic
Volume V, Number 1, January 1964

FURTHER AXIOMATIZATIONS OF THE ŁUKASIEWICZ THREE-VALUED CALCULUS

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A propositional calculus for three-valued logic was first constructed by J. Łukasiewicz (1920) and subsequently communicated in a lecture before the Polish Philosophical Society. His results were published later [2]. In 1931 M. Wajsberg [4] formalized the three-valued logic of Łukasiewicz by means of two primitive connectives, implication (denoted by C) and negation (denoted by N), and the following axioms stated in the Łukasiewicz convention:

W_1. $CpCqp$

W_2. $CCpqCCqrCpr$

W_3. $CCNpNqCqp$

W_4. $CCCpNppp$.

Wajsberg also assumed the following rules of inference:

S. Any well-formed formula may be substituted for a propositional variable in all its occurrences in a theorem or axiom.

MP. If P and CPQ are theorems, then Q is also a theorem.

The truth tables for C and N of the Łukasiewicz three-valued logic is given by

Cpq	F	U	T	Np
F	T	T	T	T
U	U	T	T	U
T	F	U	T	F

In 1951 Alan Rose [3] introduced several new other axiomatizations of the same propositional logic by taking disjunction (denoted by A) and negation as primitives and substitution and the following as rules of inference:

MP_1. If P and $ANPQ$ are theorems, then Q is also a theorem.

Received September 8, 1963

The truth table for A is the same as that proposed by Dienes [1]:

Apq	F	U	T
F	F	U	T
U	U	T	T
T	T	T	T

In Rose's systems the connective C of Wajsberg is defined by

(a) $Cpq \equiv ANpq$

while the connective A of Rose is defined in the Wajsberg system by

(b) $Apq \equiv CNpq$.

Actually, A. Rose also utilized the abbreviation:

(c) $Kpq \equiv NANpNq$.

Thus, the truth table for K when computed would be given by

Kpq	F	U	T
F	F	F	F
U	F	F	U
T	F	U	T

We shall propose two formulations of three-valued logic each with conjunction (denoted by K) and negation (denoted by N) as primitive connectives and substitution and the following as rules of inference:

MP$_2$. If $NKPNQ$ and P are theorems, then Q is also a theorem.

Admitting as abbreviations

(d) $Cpq \equiv NKpNq$,

and

(e) $Apq \equiv NKNpNq$

the rule **MP**$_2$ then reduces to rule **MP** and our proposed axiomatizations become:

A_1. $NKNKApppp$
A_2. $CKpqq$
A_3. $CNKNqpCNKqrNKrp$
B_1. $CpKAppp$
B_2. $CKpqq$
B_3. Cpp
B_4. $CCpqCNKqrNKrp$

To show that these two axiom systems are adequate for the three-valued logic of Łukasiewicz, we shall first prove that the axiom system B_1 - B_4

follow from A_1 - A_3 and the axioms of Wajsberg W_1 - W_4 follow from axioms B_1 - B_4.

Rule 1.1. *If NKNQP and CQR are theorems, then NKNRP is a theorem.*

Proof: *CNKNqpCNKqrNKrp*	Axiom A_3
CNKNQPCNKQNRNKNRP	Rule **S** with p/P, q/Q, r/NR
NKNQP	Given
CNKQNRNKNRP	**MP** rule
CQR	Given
NKQNR	Definition (d)
CNKQNRNKNRP	Line 4
NKNRP	**MP** rule

Theorem 1.1. *CKApppp*

Proof. *CKpqq*	Axiom A_2
CKApppp	Rule **S** with p/App, q/p

Theorem 1.2. *NKNpp*

Proof. *NKNKApppp*	Axiom A_1
CKApppp	Theorem 1.1
NKNpp	Rule 1.1

Theorem 1.3. *CNKpqNKqp*

Proof. *CNKNqpCNKqrNKrp*	Axiom A_3
CNKNppCNKpqNKqp	Rule **S** with q/p, r/q
NKNpp	Theorem 1.2
CNKpqNKqp	**MP** rule

Rule 1.2. *If NKNPQ is a theorem, then CQP is also a theorem.*

Proof. *CNKpqNKqp*	Theorem 1.3
CNKNPQNKQNP	Rule **S** with p/NP, q/Q
NKNPQ	Given
NKQNP	**MP** rule
CQP	Definition (d)

Theorem 1.4. *Cpp*

Proof. *NKNpp*	Theorem 1.2
Cpp	Rule 1.2

Rule 1.3. *If CPQ is a theorem, then NKNQP is also a theorem.*

Proof. *CPQ*	Given
NKPNQ	Definition (d)
CNKpqNKqp	Theorem 1.3
CNKPNQNKNQP	Rule **S** with p/P, q/NQ
NKPNQ	Line 2
NKNQP	**MP** rule

Rule 1.4. *If CPQ and CQR are theorems, then CPR is a theorem.*

Proof.	*CPQ*	Given
	NKNQP	Rule 1.3
	CQR	Given
	NKNRP	Rule 1.1 on line 2 and 3
	CNKpqNKqp	Theorem 1.3
	CNKNRPNKPNR	Rule **S** with *p*/*NR*, *q*/*P*
	NKNRP	Line 4
	NKPNR	**MP** rule
	CPR	Definition (d)

Theorem 1.5. *CCpqNKNqp*

Proof.	*CNKpqNKqp*	Theorem 1.3
	CNKpNqNKNqp	Rule **S** with *q*/*Nq*
	CCpqNKNqp	Definition (d)

Theorem 1.6. *CCpqCNKqrNKrp*

Proof.	*CNKNqpCNKqrNKrp*	Axiom A_3
	CCpqNKNqp	Theorem 1.5
	CCpqCNKqrNKrp	Rule 1.4

Theorem 1.7. *CpKAppp*

Proof.	*CNKpqNKqp*	Theorem 1.3
	CNKNKAppppNKpNKAppp	Rule **S** with *p*/*NKAppp*, *q*/*p*
	NKNKAppppp	Axiom A_1
	NKpNKAppp	**MP** rule
	CpKAppp	Definition (d)

Theorems 1.7, 1.4, 1.6, and Axiom A_2 are respectively Axioms B_1, B_3, B_4, and B_2. Whence, Axioms A_1 - A_3 implies Axioms B_1 - B_4.

From hereon, we shall assume Axioms B_1 - B_4 together with the two rules of inference.

Theorem 2.1. *CNKpqNKqp*

Proof.	*CCpqCNKqrNKrp*	Axiom B_4
	CCppCNKpqNKqp	Rule **S** with *q*/*p*, *r*/*q*
	Cpp	Axiom B_3
	CNKpqNKqp	**MP** rule

Rule 2.1. *If CPQ is a theorem, then CNKQRNKRP is a theorem.*

Proof.	*CCpqCNKqrNKrp*	Axiom B_4
	CCPQCNKQRNKRP	Rule **S** with *p*/*P*, *q*/*Q*, *r*/*R*
	CPQ	Given
	CNKQRNKRP	**MP** rule

Rule 2.2. *If $NKQP$ is a theorem, then so is $NKPQ$.*

Proof. Cpp	Axiom B_3
CQQ	Rule **S** with p/Q
$CNKQPNKPQ$	Rule 2.1
$NKQP$	Given
$NKPQ$	**MP** rule

Rule 2.3. *If CPQ and CQR are theorems, then CPR is also a theorem.*

Proof. CPQ	Given
$CNKQNRNKNRP$	Rule 2.1 with R/NR
CQR	Given
$NKQNR$	Definition (d)
$CNKQNRNKNRP$	Line 2
$NKNRP$	**MP** rule
$NKPNR$	Rule 2.2
CPR	Definition (d)

Theorem 2.2. $CCpqCNKrqNKrp$

Proof. $CNKpqNKqp$	Theorem 2.1.
$CNKrqNKqr$	Rule **S** with p/r
$CNKNKqrNNKrpNKNNKrpNKrq$	Rule 2.1. with $P/NKrq$, $Q/NKqr$, $R/NNKrp$
$CNKpqNKqp$	Theorem 2.1.
$CNKNNKrpNKrqNKNKrqNNkrp$	Rule **S** with $p/NNKrp$, $q/NKrq$
$CNKNKqrNNKrpNKNNKrpNKrq$	Line 3
$CNKNKqrNNKrpNKNKrqNNKrp$	Rule 2.3. on line 5 and 6
$CCNKqrNKrpCNKrqNKrp$	Definition (d)
$CCpqCNKqrNKrp$	Axiom B_4
$CCpqCNKrqNKrp$	Rule 2.3. on line 8 and 9

Theorem 2.3. $NKNpp$

Proof. Cpp	Axiom B_3
$NKpNp$	Definition (d)
$NKNpp$	Rule 2.2.

Theorem 2.4. $CNNpp$

Proof. $NKNpp$	Theorem 2.3.
$NKNNpNp$	**S** rule with p/Np
$CNNpp$	Definition (d)

Theorem 2.5. $CpNNp$

Proof. $CNNpp$	Theorem 2.4.
$CNNNpNp$	Rule **S** with p/Np
$CCpqCNKqrNKrp$	Axiom B_4
$CCNNNpNpCNKNppNKpNNNp$	Rule **S** with $p/NNNp$, q/Np, r/p
$CNNNpNp$	Line 2

CNKNppNKpNNNp	MP rule
NKNpp	Theorem 2.3
NKpNNNp	MP rule
CpNNp	Definition (d)

Theorem 2.6. *CCpqCNqNp*

Proof. *CCpqCNKqrNKrp*	Axiom B_4
CCNNppCNKpNqNKNqNNp	Rule S with *p/NNp*, *q/p*, *r/Nq*
CNNpp	Theorem 2.4
CNKpNqNKNqNNp	MP rule
CCpqCNqNp	Definition (d)

Theorem 2.7. *CCNNpqCpq*

Proof. *CCpqCNKrqNKrp*	Theorem 2.2
CCpNNpCNKNqNNpNKNqp	Rule S with *q/NNp*, *r/Nq*
CpNNp	Theorem 2.5
CNKNqNNpNKNqp	MP rule
CCpqCNqNp	Theorem 2.6
CCKpqKqpCNKqpNKpq	Rule S with *p/Kpq*, *q/Kqp*
CNKqpNKpq	Theorem 2.1 with *p/q*, *q/p*
CNKNNpNqNKNqNNp	Rule S with *q/NNp*, *p/Nq*
CNKNqNNpNKNqp	Line 4
CNKNNpNqNKNqp	Rule 2.3 on last two lines
CNKpqNKqp	Theorem 2.1
CNKNqpNKpNq	Rule S with *p/Nq*, *q/p*
CNKNNpNqNKNqp	Line 10
CNKNNpNqNKpNq	Rule 2.3 on last two lines
CCNNpqCpq	Definition (d)

Theorem 2.8. *CCNqNpCNNpq*

Proof. *CNKqpNKpq*	Theorem 2.1 with *p/q*, *q/p*
CNKNqNNpNKNNpNq	Rule S with *p/NNp*, *q/Nq*
CCNqNpCNNpq	Definition (d)

Theorem 2.9. *CCNqNpCpq*

Proof. *CCNqNpCNNpq*	Theorem 2.8
CCNNpqCpq	Theorem 2.7
CCNqNpCpq	Rule 2.3 on last two lines

Theorem 2.10. *CCqrCCpqCpr*

Proof. *CCpqCNKrqNKrp*	Theorem 2.2
CCNpNqCNKrNqNKrNp	Rule S with *p/Np*, *q/Nq*
CCNpNqCCrqCrp	Definition (d)
CCqpCNpNq	Theorem 2.6 with *p/q*, *q/p*
CCqpCCrqCrp	Rule 2.3 on last two lines
CCqrCCpqCpr	Rule S with *p/r*, *r/p*

Theorem 2.11. $CCpqCCqrCpr$

Proof. $CNKpqNKqp$	Theorem 2.1
$CNKNrpNKpNr$	Rule **S** with p/Nr, q/p
$CNKNrpCpr$	Definition (d)
$CCqrCCpqCpr$	Theorem 2.10
$CCNKNrpCprCCCqrNKNrpCCqrCpr$	Rule **S** with $q/NKNrp$, r/Cpr, p/Cqr
$CNKNrpCpr$	Line 3
$CCCqrNKNrpCCqrCpr$	**MP** rule
$CCpqCNKqrNKrp$	Axiom B_4
$CCpqCNKqNrNKNrp$	Rule **S** with r/Nr
$CCpqCCqrNKNrp$	Definition (d)
$CCCqrNKNrpCCqrCpr$	Line 7
$CCpqCCqrCpr$	Rule 2.3 on last two lines

Theorem 2.12. $CpCqp$

Proof. $CKpqq$	Axiom B_2
$CKqNpNp$	Rule **S** with p/q, q/Np
$CCpqCNqNp$	Theorem 2.6.
$CCKqNpNpCNNpNKqNp$	Rule **S** with $p/KqNp$, q/Np
$CKqNpNp$	Line 2
$CNNpNKqNp$	**MP** rule
$CNNpCqp$	Definition (d)
$CpNNp$	Theorem 2.5
$CpCqp$	Rule 2.3 on last two lines

Theorem 2.13. $CCNNpNpCpNp$

Proof. $CCpqCCqrCpr$	Theorem 2.11
$CCpNNpCCNNpNpCpNp$	Rule **S** with q/NNp, r/Np
$CpNNp$	Theorem 2.5
$CCNNpNpCpNp$	**MP** rule

Theorem 2.14. $CCCpNppCCNNpNpp$

Proof. $CCpqCCqrCpr$	Theorem 2.11
$CCCNNpNpCpNpCCCpNppCCNNpNpp$	Rule **S** with $p/CNNpNp$, $q/CpNp$, r/p
$CCNNpNpCpNp$	Theorem 2.13
$CCCpNppCCNNpNpp$	**MP** rule

Theorem 2.15. $CCCpNppp$

Proof. $CpKAppp$	Axiom B_1
$CpKNKNpNpp$	Definition (e)
$CCpqCNqNp$	Theorem 2.6
$CCpKNKNpNppCNKNKNpNppNp$	Rule **S** with $q/KNKNpNpp$

CpKNKNpNpp	Line 2
CNKNKNpNppNp	**MP** rule
CNKNKNNpNNpNpNNp	Rule **S** with *p/Np*
CNKCNNpNpNpNNp	Definition (d)
CCCNNpNppNNp	Definition (d)
CNNpp	Theorem 2.4
CCCNNpNppp	Rule 2.3 on last two lines
CCCpNppCCNNpNpp	Theorem 2.14
CCCpNppp	Rule 2.3 on last two lines

Theorems 2.12, 2.11, 2.9 and 2.15 are precisely the four axioms of Wajsberg; hence, it follows that Axioms B_1 - B_4 and therefore A_1 - A_3 imply the axioms of Wajsberg. They are then adequate axiomatizations of the three-valued propositional calculus of Jan Łukasiewicz.

Note. A slight modification of the axiom system B_1 - B_4 gives another axiom system of three-valued logic. This is the following:

C_1. *CpKAppp*
C_2. *CKpqq*
C_3. *CNKpqNKqp*
C_4. *CCpqCNKqrNKrp*

To show that this is a good axiomatization, it suffices to prove *Cpp*.

Rule 3.1. *If CPQ and CQR are theorems, CPR is also a theorem.*

Proof. *CCpqCNKqrNKrp*	Axiom C_4
CCPQCNKQNRNKNRP	**S** rule with *p/P*, *q/Q*, *r/NR*
CPQ	Hypothesis
CNKQNRNKNRP	**MP** rule
CCQRNKNRP	Definition (d)
CQR	Hypothesis
NKNRP	**MP** rule
CNKpqNKqp	Axiom C_3
CNKNRPNKPNR	**S** rule with *p/NR*, *q/P*
NKNRP	Line 7
NKPNR	**MP** rule
CPR	Definition (d)

Theorem 3.1. *Cpp*

Proof. *CKpqq*	Axiom C_2
CKApppp	**S** rule with *p/App*, *q/p*
CpKAppp	Axiom C_1
Cpp	Rule 3.1

The equivalence of Axiom systems B_1 - B_4 and C_1 - C_4 is now clear.

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