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A THEOREM OF SIERPIŃSKI ON TRIADS AND THE AXIOM OF CHOICE

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In [4] Sierpiński has proved with the aid of the axiom of choice the following[1]

Theorem **S**. *If E is an arbitrary set which is not finite, then there exists a family F of such subsets of the set E that* 1) *each set belonging to F contains three and only three elements of the set E, and* 2) *each subset of E constructed from two and only two elements of the set E is a part of one and only one set of the family F.*

Sierpiński points out ([4], p. 15) that even for the set of all real numbers he was unable to prove this theorem without the aid of the axiom of choice.

In this note I shall show that in the field of general set theory[2] this particular theorem of Sierpiński is equivalent to the axiom of choice. Let us accept theorem **S** and let us assume that

(1) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

It is well known[3] that without the aid of the axiom of choice we can associate with $\mathfrak{m}$ a certain so-called Hartogs' aleph, viz. $\aleph(\mathfrak{m})$, which possesses the following property:

(2) $\aleph(\mathfrak{m})$ is the least aleph such that $\aleph(\mathfrak{m})$ is not $\leq \mathfrak{m}$.

Since $\aleph(\mathfrak{m})$ is an aleph, we also know that

(3) *there exists an ordinal number* λ *such that* $\aleph(\mathfrak{m}) = \aleph_\lambda$

and, therefore,

(4) ω_λ *is an initial number of the class of ordinal numbers* $Z(\aleph_\lambda)$[4]

Hence, by (1) and (2), there is a cardinal number

(5) $\mathfrak{m} + \aleph(\mathfrak{m})$

which, obviously, is not finite, and which together with (3) and (4) allows us to conclude that *there are sets E, M, and N such that each of them is not finite and such that*

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(6) $\overline{\overline{N}} = \aleph(\mathfrak{m}) = \aleph_\lambda$

(7) *N is a set of all ordinal numbers* $< \omega_\lambda$

(8) $\overline{\overline{M}} = \mathfrak{m}$

(9) $M \cap N = 0$[5]

(10) $E = M \cup N$

(11) $\overline{\overline{E}} = \overline{\overline{M}} + \overline{\overline{N}} = \mathfrak{m} + \aleph(\mathfrak{m})$[6]

Since, by (1) and (10), set E is not finite, in virtue of theorem **S** we obtain

(12) *There exists a family F of such subsets of the set E that*

a) *each set belonging to F contains three and only three elements of the set E and*

b) *each subset of E constructed from two and only two elements of the set E is a part of one and only one set of the family F.*

We define now a family of sets F_m as follows

D1 *For any x and m*, $x \in F_m$ *if and only if*

a) $x \in F$

b) $m \in M$

c) *there is such* α *that* $\alpha \in N$ *and* $\{m, \alpha\} \subset x$[7]

Obviously, *D1* implies that

(13) *for any m*, $F_m \subset F$

We shall now prove the following lemmas:

Lemma 1. For any m, if $m \in M$, *then there is such x that* $x \in F_m$

Proof: Assume that $m \in M$. Hence, by (10), $m \in E$, and if $\alpha \in N$, then $\alpha \in E$. Therefore, by (9) and (10), because set N is unempty, we know that for any α, if $\alpha \in N$, then $m \neq \alpha$, $\{m, \alpha\}$ exists and $\{m, \alpha\} \subset E$. Hence, by (12), there is such x that $x \in F$ and $\{m, \alpha\} \subset x$. Whence, by *D1*, $x \in F_m$. Thus, lemma 1 is proved.

Lemma 2. *For any* m, x, y, a, b, c, *and* d, *if* $m \in M$, $a \in E$, $b \in E$, $c \in E$, $d \in E$, $x \in F_m$, $y \in F_m$, $x = \{m,a,b\}$, $y = \{m,c,d\}$, $x \neq y$, *then* $a \neq c$, $a \neq d$, $b \neq c$, $b \neq d$ *and at least one element of x and at least one element of y belong to N.*

Proof. Lemma 2 follows immediately from *D1*, (13), (12) and (10).

Lemma 3. *For any* m, α, β, x *and* z, *if* $m \in M$, $\alpha \in N$, $\beta \in N$, $x \in F_m$, $z \in F$, $x = \{m,\alpha,\beta\}$ *and* $\{\alpha,\beta\} \subset z$, *then* $x = z$.

Proof: The assumptions of lemma 3, (9) and (10) imply that $m \in E$, $\alpha \in E$, $\beta \in E$, and that $m \neq \alpha$, and $m \neq \beta$. Moreover, by (13), $x \in F$. Hence, in virtue of (12) we know that $\{m,\alpha\} \subset E$, $\{m,\beta\} \subset E$ and $\{\alpha,\beta\} \subset E$. Therefore, there is one and only one set belonging to F such that $\{m,\alpha\}$, $\{m,\beta\}$and $\{\alpha,\beta\}$ are included as its subsets. Whence, it follows from the assumptions of lemma 3 that $x = z$. Thus, lemma 3 is proved.

Lemma 4. *For any m, if $m \in M$, then there exist such x, α and β that $x \in F_m$, $x = \{m, \alpha, \beta\}$, $\alpha \in N$ and $\beta \in N$.*

Proof: For this end, first, we define a set M_r as follows

D2 For any n and r, $n \in M_r$ if and only if

a) $n \in M$

b) $r \in M$

c) $n \neq r$

d) *there is such x that $x \in F_r$ and $n \in x$.*

and, second, we assume that:

For any m,

($\mathfrak{a}$) $m \in M$

and

($\mathfrak{b}$) *for any x, α and n, if $x \in F_m$, $x = \{m, \alpha, n\}$, $\alpha \in N$, then $n \in M$*

It will be shown that a conjunction of the assumptions ($\mathfrak{a}$) and ($\mathfrak{b}$) leads to a contradiction. Namely, we can deduce without any difficulty that:

i) it follows from ($\mathfrak{a}$), ($\mathfrak{b}$), (9), (10), (13), *D1* and lemma 1 that

(14) *for any α, if $\alpha \in N$, then there exists one and only one such x that $x \in F_m$ and $\alpha \in x$.*

and that

(15) *for any x, if $x \in F_m$, then there exists one and only one such α that $\alpha \in N$ and $\alpha \in x$.*

and, moreover, that

ii) it follows from ($\mathfrak{a}$), ($\mathfrak{b}$), (12), *D1* and *D2* that

(16) *for any n, if $n \in M_m$, then there exists one and only one such x that $x \in F_m$ and $n \in x$.*

and that

(17) *for any x, if $x \in F_m$, then there exists one and only one such n that $n \in M_m$ and $n \in x$.*

It is clear that points (14), (15), (16) and (17) allow us to establish that

(18) *the sets N, F_m and M_m possess the same number of elements,*

i.e. that

(19) $\overline{\overline{N}} = \overline{\overline{F}}_m = \overline{\overline{M}}_m$

But, *D2*, for $r = m$, implies that

(20) $M_m \subset M$

which, in virtue of an elementary consequence of the Schröder--Bernstein theorem, immediately gives

(21) $\bar{\bar{\bar{M}}}_m \leq \bar{\bar{\bar{M}}}$

Hence, by (19), (21), (6) and (8), we obtain

(22) $\aleph(\mathfrak{m}) = \bar{\bar{\bar{N}}} = \bar{\bar{\bar{M}}}_m \leq \bar{\bar{\bar{M}}} = \mathfrak{m}$

i.e. that

(23) $\aleph(\mathfrak{m}) \leq \mathfrak{m}$

Since (23) contradicts (2) and since (23) is a consequence of ($\mathfrak{a}$) and ($\mathfrak{b}$), the conjunction of these two assumptions is not true. Therefore, we have:

(24) *For any m, if $m \in M$, then there exist such x, α and β that $x \in F_m$, $x = \{m, \alpha, \beta\}$, $\alpha \in N$ and $\sim(\beta \in M)$*

Since in (24) $x \in F_m$, by (13), $x \in F$. Therefore, due to (24) and (12), $\beta \in E$. Since $\beta \in E$ and $\sim(\beta \in M)$, by (10), $\beta \in N$. Thus, the proof of lemma 4 is complete.

Then, it follows from lemma 3, (13) and (12) at once that

(25) *for any $m, n, \alpha, \beta, \gamma, \delta, x$ and y, if $m \in M$, $n \in M$, $m \neq n$, $\alpha \in N$, $\beta \in N$, $\gamma \in N$, $\delta \in N$, $x \in F_m$, $x = \{m, \alpha, \beta\}$, $y \in F_n$, $y = \{n, \delta, \gamma\}$, then $\{\alpha, \beta\} \neq \{\gamma, \delta\}$* [8]

We now define a family of sets F_m^* as follows

D3 *For any x and m, $x \in F_m^*$ if and only if*
a) $x \in F_m$
b) *there are such α and β that $\alpha \in N$, $\beta \in N$ and $x = \{m, \alpha, \beta\}$*

Hence, by *D3*,

(26) *for any m, $F_m^* \subset F_m$*

and, due to *D3*, *D1*, (26) and lemma 4 we know that

(27) *for any m, if $m \in M$, then there is such x that $x \in F_m^*$*

and

(28) *for any m, x and y, if $x \in F_m$, $y \in F_m$, $x \neq y$, then there are such α, β, γ and δ that*

$\alpha \in N$, $\beta \in N$, $\gamma \in N$, $\delta \in N$, $x = \{m, \alpha, \beta\}$, $y = \{m, \gamma, \delta\}$ and $\{\alpha, \beta\} \neq \{\gamma, \delta\}$

We now introduce the following two definitions of the sets P and P_m respectively:

D4 *For any x, $x \in P$ if and only if*
a) *x is an element of the set of all subsets of the set N*
b) *there are such α and β that $\alpha \in N$, $\beta \in N$ and $x = \{\alpha, \beta\}$*

and

D5 *For any x and m, $x \in P_m$ if and only if*
a) $x \in P$
b) *there are such α, β and y that $\alpha \in N$, $\beta \in N$, $x = \{\alpha, \beta\}$, $y \in F_m^*$ and $\{\alpha, \beta\} \subset y$*

Hence, by *D5*.

(29) *for any* m, $P_m \subset P$

by (27), *D3*, *D1*, *D4* and *D5*,

(30) *for any* m, *if* $m \in M$, *then there is such* x *that* $x \in P_m$

and, by *D5* and (28),

(31) *for any* x *and* m, *if* $x \in P_m$, *then there exists one and only one such* y *that* $y \in F^*_m$ *and* $x \subset y$.

Since, in virtue of (7), every element of the set N is an ordinal number $< \omega_\lambda$, we can accept that

(32) *every pair* $\{\alpha,\beta\}$, *for* $\alpha \in N$ *and* $\beta \in N$, *is an ordered pair of two ordinal numbers ordered according their magnitude.*

(Thus, e.g., if $\alpha < \beta < \omega_\lambda$, then $\{\alpha,\beta\} = <\alpha,\beta>$. And, due to (7), *D4*, *D5* and (32), the sets P and P_m (defined above) can be considered respectively as the set of all and as the set of some ordered pairs of ordinal numbers $< \omega_\lambda$.

We introduce now the following ordering of the elements belonging to the set P:[9]

For any $\alpha,\beta,\gamma,\delta$, *if* $\alpha < \beta < \omega_\lambda$ *and* $\gamma < \delta < \omega_\lambda$, *then* $<\alpha,\beta> \prec <\gamma,\delta>$ *if and only if either* $\alpha + \beta < \gamma + \delta$ *or* $\alpha + \beta = \gamma + \delta$ *and* $\alpha < \gamma$

and, due to this ordering, we can, obviously, consider the set P as a well-ordered set of the type ω_λ, i.e.:

(33) $P = \left\{<\alpha_\xi, \beta_\xi>\right\}_{\xi < \omega_\lambda}$

Therefore, in virtue of (29), (30), (32) and (33), we obtain:

(34) *for any* m, *if* $m \in M$, *then the unempty set* P_m *is a well-ordered set*

which yields at once that

(35) *for any* m, *if* $m \in M$, *then there exists a first element of the unempty well-ordered set* P_m

Hence, (35), *D5*, *D3*, *D1* and (10) imply that

(36) *for any* m, $m \in M$ *if and only if there are two and only two ordinal numbers* α *and* β *and one and only one* x *such that:*
- a) $\alpha < \beta < \omega_\lambda$
- b) $<\alpha,\beta>$ *is the first element of the unempty well-ordered set* P_m
- c) $x \in F^*_m$
- d) $x = \{m,\alpha,\beta\}$

which together with (25) allows us to conclude that

(37) *there exists a function such that*
- a) *it associates with each element* m *of the set* M *a certain ordered pair and only one such pair of ordinal numbers* $<\alpha,\beta>$ *such that* $\alpha < \beta < \omega_\lambda$

b) *for any* $m, n, \alpha, \beta, \gamma, \delta$, *if* $m \in M$, $n \in M$, $m \neq n$, $\alpha < \beta < \omega_\lambda$, $\gamma < \delta < \omega_\lambda$ *and the ordered pairs* $\langle \alpha, \beta \rangle$ *and* $\langle \alpha, \beta \rangle$ *are associated by this function with m and n respectively, then* $\langle \alpha, \beta \rangle \neq \langle \gamma, \delta \rangle$

Therefore, in virtue of two known theorems about the properties of ordinal numbers[10], which say that

T1 *There exists a function of two variables* $\phi(\xi, \eta)$ *satisfying the following conditions:*

a) *for any two ordinal numbers* ξ *and* η, $\zeta = \phi(\xi, \eta)$ *is a well-defined ordinal number*

b) *for every ordinal number* ζ *there exists one and only one ordered pair of ordinal numbers* ξ *and* η *such that* $\zeta = \phi(\xi, \eta)$

c) *for every ordinal number* α *the inequality* $\phi(\xi, \eta) < \omega^\alpha$ *is equivalent to the system of inequalities* $\xi < \omega^\alpha$ *and* $\eta < \omega^\alpha$.

and

T2 *Every initial number is a power of number* ω

and which are, obviously, provable without the aid of the axiom of choice, point (37) allows us to establish without any difficulty that

(38) *there exists a function such that*

a) *it associates with each element m of the set M one and only one ordinal number*

b) *for any* m, n, ϕ, *and* ψ, *if* $m \in M$, $n \in M$, $m \neq n$, $\phi < \omega_\lambda$, $\psi < \omega_\lambda$ *and the ordinal numbers* ϕ *and* ψ *are associated by this function with m and n respectively, then* $\phi \neq \psi$

Now, we define a set of ordinal numbers N^* as follows

D6 *For any* α, $\alpha \in N^*$ *if and only if*

a) $\alpha \in N$

b) *there exists such m that* $m \in M$ *and the ordinal number* α *is associated by the function given in* (38) *with m.*

Hence, by *D6*,

(39) $N^* \subset N$

and, therefore, in an elementary way we obtain

(40) $\overline{\overline{N}}{}^* \leq \overline{\overline{N}}$

On the other hand, it follows clearly from (38) and *D6* that

(41) $\overline{\overline{M}} = \overline{\overline{N}}{}^*$

which, by (8), (41), (40) and (6), implies at once

(42) $\mathfrak{m} = \overline{\overline{M}} = \overline{\overline{N}}{}^* \leq \overline{\overline{N}} = \aleph(\mathfrak{m})$

Hence, in virtue of (42) and (2), we conclude that

(43) $\mathfrak{m} < \aleph(\mathfrak{m})$

which together with (1) says that an arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph. Hence, theorem **S** implies the axiom of choice. Thus, since Sierpiński has shown that the said axiom yields theorem **S**, we have a proof that this theorem is equivalent to the axiom of choice.

NOTES

1. Sierpiński mentions in [4], p. 13, that theorem **S** is related to the following combinatorial problem put by J. Steiner in 1852 (*Jour. f. r. u. a. Math.*, v. 45 (1853), p. 181), viz.: Let E be a finite set containing n elements. What number n must be so that a family F, defined exactly as in theorem **S**, could exist. An answer (given e.g. in E. Netto: *Lehrbuch der Combinatorik,* Leipzig, 1901, pp. 206-211) is: *either 6k + 1 or 6k + 3, for any k.*

2. It means that the proofs which are given below are established within the general set theory, i.e. the set theory from which the axiom of choice and all its consequences otherwise unprovable have been removed. It is well-known that if we base so defined a general set theory on an axiomatic system in which the notions of the cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system. by means of special axioms. It has to be noted that a proof presented below can be carried out without the use of the notion of cardinal numbers, but in such a case the deductions would be longer.

3. *Cf,* [1], and, e.g., [3], pp. 407-409

4. Symbol "$Z(\aleph_\alpha)$" was introduced by Hausdorff, *Cf.*, e.g., [3], p. 389.

5. I.e. that the sets M and N do not have the common elements.

6. Point (11) will not be used below. It is given here in order to present the addition of two cardinal numbers completely.

7. In this paper $\{a,b,c, \ldots\}$ means a set composed from the distinct elements $a, b, c, \ldots$, and $<a,b>$ symbolizes an ordered pair. Thus, if a formula has $\{a, b\}$ as its part, we know that the elements a and b are not identical.

8. We have to notice that if the conditions of (25) are satisfied, then $F_m \cap F_n = 0$. But, without these assumptions the families of sets F_m and F_n, for $m \in M$, $n \in M$ and $m \neq n$, can possess the common elements. E.g.: $x = \{m, \alpha, n\}$, for any $\alpha \in N$.

9. An ordering of the ordered pairs of ordinal numbers used here is the same as in Sierpiński's [4], p. 14.

10. The theorems **T1** and **T2** are announced without proofs by Tarski in [2], pp. 308-309. The proofs are given by Sierpiński in [3], p. 330 and p. 394 for **T1** and **T2** respectively. It is worth while to notice that function given in **T1** can be defined effectively, *cf.* [3], 330.

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