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A NOTE ON THE GENERALIZED CONTINUUM HYPOTHESIS. III.

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§5*

In [11], p. 72, point ($\ddot{\text{v}}$ii), and p. 76, point ($\ddot{\text{x}}$i), it is proved that the formulas *C1*, *B2* and *B3* which are the particular instances of formulas *C* and *B*, cf. [7], p. 274, are such that *C1* is a consequence of $\mathbf{E}_1$, *B2* follows from $\mathfrak{C}$ and *B3* is provable in the general set theory. Now we shall show:

1) that the following particular instances of *D*

D2 For any cardinal number $\mathfrak{m}$ *and any aleph* $\mathfrak{a}$, *if* $2^{\mathfrak{m}} = 2^{\mathfrak{a}}$, *then* $\mathfrak{m} = \mathfrak{a}$.

and

D3 For any cardinal number $\mathfrak{m}$ *and any aleph* $\mathfrak{a}$, *if* $2^{\mathfrak{m}} = 2^{2^{\mathfrak{a}}}$, *then* $\mathfrak{m} = 2^{\mathfrak{a}}$.

are consequences of Cantor's hypothesis on alephs.

2) that the following particular instance of *C*

C2 For any cardinal number $\mathfrak{m}$ *and any aleph* $\mathfrak{a}$, *if* $\mathfrak{a} < \mathfrak{m}$, *then* $2^{\mathfrak{a}} < 2^{\mathfrak{m}}$.

is a consequence of *D2*;

3) that the formulas *D1* and *C1*, which are, obviously, the instances of *D2* and *C2* respectively, are equivalent in the field of general set theory; and

4) that the following formula

$\mathbf{E}_3$ *For any cardinal number* $\mathfrak{m}$ *and any aleph* $\mathfrak{a}$, *if* $\mathfrak{m} < 2^{2^{\mathfrak{a}}}$, *then* $\mathfrak{m} \leqslant 2^{\mathfrak{a}}$

and which is such that $\mathbf{E}_2$ is its substitution follows from $\mathfrak{C}$.

We prove it as follows:

(xii) *Cantor's hypothesis on alephs implies formulas D2, D3 and* $\mathbf{E}_3$.

($\mathfrak{m}$) *Proof of D2.* Let us assume the conditions of *D2*, viz. that

*The first and the second parts of this paper appeared in *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 274-278, and v. IV (1963), pp. 67-79. They will be referred to throughout this third part as [7] and [11] respectively. See the additional Bibliography given at the end of this part. An acquaintance with [7] and [11] is presupposed.

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(81) $\mathfrak{m}$ *is an arbitrary cardinal number,* $\mathfrak{a}$ *is an arbitrary aleph and* $2^{\mathfrak{m}} = 2^{\mathfrak{a}}$.

Since $\mathfrak{a}$ is an aleph, $2^{\mathfrak{a}}$ is an infinite cardinal. Hence, by (81),

(82) $\mathfrak{m}$ *is a cardinal which is not finite*

and, moreover, *there exists an ordinal number* α *such that*

(83) $\mathfrak{a} = \aleph_\alpha$

Hence, in virtue of $\mathfrak{C}$, (81) and (83) we have

(84) $2^{\mathfrak{a}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1} = 2^{\mathfrak{m}} > \mathfrak{m}$

Therefore, by (82) and (84),

(85) *our arbitrary cardinal* $\mathfrak{m}$ *and cardinal* $2^{\mathfrak{m}}$ *are alephs*

Hence, due to (85) we can establish that

there exists an ordinal number β *such that*

(86) $\mathfrak{m} = \aleph_\beta$

which, by $\mathfrak{C}$, implies

(87) $2^{\mathfrak{m}} = 2^{\aleph_\beta} = \aleph_{\beta+1}$

Hence, by (84) and (87),

(88) $\aleph_{\beta+1} = \aleph_{\alpha+1}$

which, gives at once

(89) $\beta + 1 = \alpha + 1$

Since the ordinal numbers $\beta + 1$ and $\alpha + 1$ are of the first kind, we can conclude from (89) that

(90) $\beta = \alpha$

which due to (83) and (86) shows that

(91) $\mathfrak{m} = \mathfrak{a}$

Thus, formula *D2* follows from $\mathfrak{C}$.

(n) *Proof of D3*. Assume the conditions of *D3*, viz. that

(92) m *is an arbitrary cardinal,* $\mathfrak{a}$ *is an arbitrary aleph and* $2^{\mathfrak{m}} = 2^{2^{\mathfrak{a}}}$.

Since due to (92) $\mathfrak{a}$ is an aleph, in virtue of $\mathfrak{C}$ we have

there exists an ordinal number α *such that*

(93) $\mathfrak{a} = \aleph_\alpha$ and $2^{\mathfrak{a}} = \aleph_{\alpha+1}$

Hence, by (93),

(94) $2^{\mathfrak{a}}$ is an aleph

and, therefore, (92) and (94) together with *D2* imply

(95) $\mathfrak{m} = 2^{\mathfrak{a}}$

which shows that *D3* is a consequence of $\mathfrak{C}$.

($\mathfrak{n}$) *Proof of* $\mathbf{E}_3$. Assume the conditions of $\mathbf{E}_3$, viz. that

(96) $\mathfrak{m}$ *is an arbitrary cardinal number,* $\mathfrak{a}$ *is an arbitrary aleph and* $\mathfrak{m} < 2^{2^{\mathfrak{a}}}$

Then these conditions together with $\mathfrak{C}$ imply

there exists an ordinal number α *such that*

(97) $\mathfrak{a} = \aleph_\alpha$ and $2^{\mathfrak{a}} = \aleph_{\alpha+1}$ and $2^{2^{\mathfrak{a}}} = \aleph_{\alpha+2}$

Hence it follows from (96) and (97) immediately that

(98) *either* $\mathfrak{m}$ *is a finite cardinal or* $\mathfrak{m}$ *is an aleph*

But, both cases of (98) imply the desired conclusion, viz. that

(99) $\mathfrak{m} \leqslant 2^{\mathfrak{a}}$

because: 1) if $\mathfrak{m}$ *is finite cardinal* and $\mathfrak{a}$ is an aleph by assumption, then, obviously (99) holds, and 2) if on the other hand $\mathfrak{m}$ *is an aleph,* then (99) follows from (96) and $\mathbf{E}_2$ which, *cf.* [6], is a consequence of $\mathfrak{C}$ alone. Thus, Cantor's hypothesis on alephs implies $\mathbf{E}_3$.

(xiii) *Formula D2 implies C2.* Let us assume *D2* and the conditions of *C2*, viz. that

(100) $\mathfrak{m}$ *is an arbitrary cardinal,* $\mathfrak{a}$ *is an arbitrary aleph and* $\mathfrak{a} < \mathfrak{m}$

Hence (100) together with general set theory implies at once

(101) *either* $2^{\mathfrak{a}} = 2^{\mathfrak{m}}$ or $2^{\mathfrak{a}} < 2^{\mathfrak{m}}$

Since the first case of (101), viz. $2^{\mathfrak{a}} = 2^{\mathfrak{m}}$, together with (100) and *D2* gives $\mathfrak{a} = \mathfrak{m}$ which is inconsistent with our assumption (100), the second case of (101), namely

(102) $2^{\mathfrak{a}} < 2^{\mathfrak{m}}$

holds. Therefore, *C2* follows from *D2.*

(xiv) *Formula D1 is equivalent to C1.* Since the formulas *D1* and *C1* are the instances of *D2* and *C2* respectively, it is evident that they follow from $\mathfrak{C}$.

($\mathfrak{p}$) *Formula D1 implies C1.* Assume the conditions of *C1*, viz. that

(103) $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs* and $\mathfrak{a} < \mathfrak{b}$

Hence, it follows from general set theory and (103) that

(104) *either* $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$ *or* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$

Since the first case of (104), viz. $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$, together with (103) and *D1* gives $\mathfrak{a} = \mathfrak{b}$ which is incompatible with our assumption (103), the second case of (104), viz.

(105) $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$

holds and, therefore the proof is completed.

($\mathfrak{q}$) *Formula C1 implies D1.* Assume the conditions of *D1*, viz. that

(106) $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs and* $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$

Hence, by (106) and the law of trichotomy for alephs,

(107) *either* $\mathfrak{a} = \mathfrak{b}$ *or* $\mathfrak{a} < \mathfrak{b}$ *or* $\mathfrak{b} < \mathfrak{a}$

Since in virtue of (106) and *C1* the second and the third cases of (107), viz. $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{a}$ imply $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$ and $2^{\mathfrak{b}} < 2^{\mathfrak{a}}$ respectively which contradicts our assumption (106), the first case of (107), viz.

(108) $\mathfrak{a} = \mathfrak{b}$

holds which shows that *D1* follows from *C1*. Thus, we can establish that $\{D1\} \rightleftarrows \{C1\}$. On the other hand, I note that I was unable to prove that *C2* implies *D2*.

§6

In [6], pp. 60-63, I have proved that $\{\mathsf{E}_1;\mathsf{E}_2\} \rightleftarrows \{\mathfrak{C}\}$. In this and the subsequent paragraphs I shall present other sets of formulas such that each of these sets is equivalent to Cantor's hypothesis on alephs.

(xv) *The set of the formulas* E_3 *and C2 is equivalent to* $\mathfrak{C}$. It is evident that it sufficies to prove that formulas E_3 and *C2* imply $\mathfrak{C}$. Moreover, since E_2 follows, obviously, from E_3 by substitution, we have to prove only E_1. Hence, let us assume the conditions of E_1, viz. that

(109) $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs and* $\mathfrak{b} < 2^{\mathfrak{a}}$

Then, by (109) and *C2*,

(110) $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$

which together with (109) and E_3 implies

(111) *either* $2^{\mathfrak{b}} = 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{b}} < 2^{\mathfrak{a}}$

Since *C2* implies *C1* and, therefore, *D1*, cf. (xiv), and since *B3* is a consequence of general set theory, cf. (ix) in [7], we can apply *D1* and *B3* to (109) and (111) giving

(112) $\mathfrak{b} \leq \mathfrak{a}$

at once. And, therefore, E_1 follows from E_3 and *C2*. Thus, $\{\mathsf{E}_3;C2\} \rightleftarrows \{\mathfrak{C}\}$.

(xvi) Since formula *D2* implies *C2*, cf. (xiii), point (xv) allows us to establish that also $\{\mathsf{E}_3;D2\} \rightleftarrows \{\mathfrak{C}\}$.

(xvii) *The set of formulas* E_3, *D3 and D1 is equivalent to* $\mathfrak{C}$. Obviously, it is sufficient to prove that the former formulas imply E_1. Hence, assume the conditions of E_1, i.e. point (109) which implies at once

(112) *either* $2^{\mathfrak{b}} = 2^{2^{\mathfrak{a}}}$ *or* $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$

Since the first case of (112), viz. $2^{\mathfrak{b}} = 2^{2^{\mathfrak{a}}}$, together with (109) and *D3* gives $\mathfrak{b} = 2^{\mathfrak{a}}$ which contradicts our assumption (109), the second case of (112), viz.

(113) $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$

holds, and, therefore, by (109) and $\mathbf{E}_3$.

(114) *either* $2^{\mathfrak{b}} = 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{b}} < 2^{\mathfrak{a}}$

Since we have *D1* and *B3*, cf. (ix) in [11], these two formulas together with (109) and (114) allow us to conclude that

(115) $\mathfrak{b} \leqslant \mathfrak{a}$

which shows that $\mathbf{E}_1$ follows from $\mathbf{E}_3$, *D3* and *D1*. Thus, since $\mathbf{E}_2$ is a consequence of $\mathbf{E}_3$ by substitution, we know that $\{\mathbf{E}_3; D3; D1\} \rightleftarrows \{\mathfrak{C}\}$, and, moreover, since $\{D1\} \rightleftarrows \{C1\}$, that $\{\mathbf{E}_3; D3; C1\} \rightleftarrows \{\mathfrak{C}\}$.

§7

The following two formulas

$\mathbf{E}_4$ *For any alephs* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$, *then* $2^{\mathfrak{b}} \leqslant 2^{\mathfrak{a}}$

and

$\mathbf{E}_5$ *For any alephs* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$, *then* $2^{\mathfrak{a}} \leqslant \mathfrak{b}$

are, obviously, consequences of Cantor's hypothesis on alephs, because $\mathbf{E}_4$ and $\mathbf{E}_5$ are the particular substitutions of $\mathbf{E}_3$ and $\mathbf{C}$, *cf.* [6], p. 58 and [11], p. 71, respectively. I shall show here that there are several sets of formulas such that each of these sets is equivalent to $\mathfrak{C}$ and, moreover, each of them contains either $\mathbf{E}_4$ or $\mathbf{E}_5$. We proceed as follows:

(xviii) *Formulas C2 and* $\mathbf{E}_4$ *imply* $\mathbf{E}_1$. Assume the conditions of $\mathbf{E}_1$, viz. that

(116) $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs and* $\mathfrak{b} < 2^{\mathfrak{a}}$

Then, it follows from (116) and *C2* that

(117) $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$

which together with (116) and $\mathbf{E}_4$ implies

(118) *either* $2^{\mathfrak{b}} = 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{b}} < 2^{\mathfrak{a}}$

Since, as we know, *C2* implies *D1* and *B3* is a consequence of general set theory, (116), (118), *C2* and *B3* yield

(119) $\mathfrak{b} \leqslant \mathfrak{a}$

which shows that $\mathbf{E}_1$ is a consequence of *C2* and $\mathbf{E}_4$.

(xix) *Formulas* $\mathbf{E}_5$ *and C1 imply* $\mathbf{E}_4$. Assume the conditions of $\mathbf{E}_4$, viz. that

(120) $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs and* $2^{\mathfrak{b}} < 2^{2^{\mathfrak{a}}}$

Then, by (120), *C1*, formula *A* (which is provable in general set theory, *cf.* [7], p. 74), and the general properties of alephs,

(121) *either* $2^{\mathfrak{b}} = 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{b}} < 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$

Hence, in virtue of E_5, (120) and (121) we know that

(122) *either* $2^{\mathfrak{b}} \leqslant 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{a}} \leqslant \mathfrak{b}$

Since due to (120) $\mathfrak{b}$ is an aleph, formula $2^{\mathfrak{a}} < \mathfrak{b}$ says that $2^{\mathfrak{a}}$ is also an aleph. Hence, by (120), (122) and *C1*,

(123) *either* $2^{\mathfrak{b}} \leqslant 2^{\mathfrak{a}}$ *or* $2^{\mathfrak{a}} = \mathfrak{b}$ *or* $2^{2^{\mathfrak{a}}} < 2^{\mathfrak{b}}$

But, the second and the third cases of (123) contradict our assumption (120), because they, together with (120), give an impossible conclusion viz. that $2^{\mathfrak{b}} < 2^{\mathfrak{b}}$. Hence, the first case of (123), viz.

(124) $2^{\mathfrak{b}} \leqslant 2^{\mathfrak{a}}$

holds which shows that E_4 follows from E_5 and *C1*. I do not know whether E_4 and *C1* imply E_5.

(xx) *The set of formulas* E_4 *and D2 is equivalent to* $\mathfrak{C}$. It is evident that it is sufficient to prove that the former formulas imply $\mathfrak{C}$. Since, as we know, *C2* follows from *D2*, we have, by (xviii), E_1 at our disposal. Now, let us assume the condition of $\mathfrak{C}$, i.e. of Cantor's hypothesis on alephs, viz. that

(125) α *is an arbitrary ordinal number*

In virtue of the known theorem, which says that

T3 *For any ordinal number* α, $2^{\aleph_{\alpha+1}} \leqslant 2^{2^{\aleph_\alpha}}$

and which is provable without the use of the axiom of choice and Cantor's hypothesis on alephs[8], and point (125) we can establish that

(126) *either* $2^{\aleph_{\alpha+1}} = 2^{2^{\aleph_\alpha}}$ *or* $2^{\aleph_{\alpha+1}} < 2^{2^{\aleph_\alpha}}$

which together with *D2* and E_4 implies at once

(127) *either* $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ *or* $2^{\aleph_{\alpha+1}} \leqslant 2^{\aleph_\alpha}$

i.e., obviously, that

(128) *either* $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ *or* $\aleph_{\alpha+1} < 2^{\aleph_\alpha}$

Since in virtue of E_1 the second case of (128), viz. $\aleph_{\alpha+1} < 2^{\aleph_\alpha}$, gives an impossible conclusion, namely that $\aleph_{\alpha+1} < \aleph_\alpha$, the first case of (128), viz.

(129) $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$

holds which shows that $\mathfrak{C}$ is a consequence of E_4 and *D2*.

(xxi) *The set of formulas* E_5 *and D2 is equivalent to* $\mathfrak{C}$. It follows obviously from points (xix) and (xx). Thus, we can establish that $\{\mathsf{E}_4; D2\} \rightleftarrows \{\mathsf{E}_5; D2\} \rightleftarrows \{\mathfrak{C}\}$. I do not know whether in the discussed sets, $D2$ can be substituted by $C2$.

(xxii) *The set of formulas* E_4, E_5 *and D1 is equivalent to* $\mathfrak{C}$. It is sufficient to prove that the former formulas imply $\mathfrak{C}$. Therefore, assume the condition of $\mathfrak{C}$, i.e. point (125). Hence in virtue of **T3** we have also point (126) which together with E_4 yields

(130) *either* $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$ *or* $2^{\aleph_{\alpha+1}} \leqslant 2^{\aleph_\alpha}$

Since the second case of (130), viz. *either* $2^{\aleph_{\alpha+1}} = 2^{\aleph_\alpha}$ *or* $2^{\aleph_{\alpha+1}} < 2^{\aleph_\alpha}$ together with $D2$ and $B3$, *cf.* point (1) in [11], p. 76, implies an impossible condition, namely $\aleph_{\alpha+1} \leqslant \aleph_\alpha$, the first case of (130), viz.

(131) $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$

holds which in virtue of E_4 yields that

(132) *either* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *or* $2^{\aleph_\alpha} < \aleph_{\alpha+1}$

But, the second case of 132 , viz. $2^{\aleph_\alpha} < \aleph_{\alpha+1}$, is obviously false. Hence, the first case of (132), viz.

(133) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

holds and, therefore, we know that $\{D2; \mathsf{E}_4; \mathsf{E}_5\} \rightleftarrows \{\mathfrak{C}\}$. Since, as it was proved above, $\{C1\} \rightleftarrows \{D1\}$, and E_1 implies $C1$, we can conclude that $\{\mathfrak{C}\} \rightleftarrows \{D1; \mathsf{E}_4; \mathsf{E}_5\} \rightleftarrows \{C1; \mathsf{E}_4; \mathsf{E}_5\} \rightleftarrows \{\mathsf{E}_1; \mathsf{E}_4; \mathsf{E}_5\}$. It is unknown whether the formulas belonging to each of the last three sets are mutually independent.

§8

The following two formulas

K1 For any aleph $\mathfrak{a}$, $2^{\mathfrak{a}}$ *is an aleph*

and

K2 For any cardinal number $\mathfrak{m}$ *which is not finite and any aleph* $\mathfrak{a}$, *if* $\mathfrak{m} < 2^{\mathfrak{a}}$, *then* $\mathfrak{m}$ *is an aleph*

are obvious and rather banal consequences of Cantor's hypothesis on alephs. But, each of the following sets $\{\mathsf{E}_1; K1\}$ and $\{\mathsf{E}_1; K2\}$ is equivalent to $\mathfrak{C}$.

Proof: Assume the conditions of **C**, viz. that

(134) $\mathfrak{n}$ *is an arbitrary cardinal number which is not finite*, $\mathfrak{a}$ *is an arbitrary aleph and* $\mathfrak{n} < 2^{\mathfrak{a}}$

Then in virtue of *K1* or *K2*, point (134) and the general set theory we can establish that

(135) $\mathfrak{n}$ *is an aleph*

Hence, by E_1, (134) and (135),

(136) $\mathfrak{n} \leqslant \mathfrak{a}$

which proves that C follows from E_1 and *K1* or *K2*. Therefore, we have $\{\mathfrak{C}\} \rightleftarrows \{\mathsf{E}_1; K1\} \rightleftarrows \{\mathsf{E}_1; K2\}$.

NOTES

8. This theorem is due to Tarski and it was announced without a proof in [2], p. 311, theorem **81**. *Cf.* also [3], p. 397.

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To be continued

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